Math 245 - Mathematics of Physics and Engineering I

Lecture 9. Systems of Two First Order Linear ODEs

January 30, 2012

Agenda

- General Form and Matrix Notation
- Trajectories and Phase Portraits
- Existence and Uniqueness of Solutions
- Autonomous Systems
- Equilibrium Solutions
- Transformation of a 2^{nd} order ODE to a system of two 1^{st} order ODEs
- Summary and Homework

The **general** system of two first order linear ODEs has the following form:

$$\begin{cases} \dfrac{dx_1}{dt} = p_{11}(t)x_1 + p_{12}(t)x_2 + g_1(t) \\ \dfrac{dx_2}{dt} = p_{21}(t)x_1 + p_{22}(t)x_2 + g_2(t) \end{cases} \quad (1)$$

- x_1 and x_2 are unknown functions, we refer to x_1 and x_2 as state variables, and to the x_1x_2-plane as the phase plane.
- $p_{11}(t), p_{12}(t), p_{21}(t), p_{22}(t), g_1(t)$, and $g_2(t)$ are given.

The equations (1) cannot be solved separately, but must be investigated together. In dealing with systems of equations, it is most advantageous to use matrix notation: this facilitates calculations and saves space. Using matrix notation, we can rewrite (1) as

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t) \quad (2)$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \qquad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \quad (3)$$

- We refer to $\mathbf{x} = (x_1, x_2)^T$ as the state vector.

Example

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -x_1 + 2\sin t \end{cases}$$

- Write this system in matrix notation
- Show that

$$\mathbf{x} = \begin{pmatrix} \sin t - t\cos t \\ t\sin t \end{pmatrix}$$

  is a solution of the system

Trajectories and Phase Portraits

If $x_1(t)$ and $x_2(t)$ are the components of a solution to

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

then the parametric equations

$$x_1 = x_1(t) \qquad x_2 = x_2(t)$$

give the coordinates x_1 and x_2 of a point in the phase plane as a function of time. Each value of the parameter t determines a point $(x_1(t), x_2(t))$, and the set of all such point is a curve in the phase plane. This curve is called a **trajectory** or **orbit**, that graphically displays the path of the state of the system in the state plane.

A plot of a representative sample of the trajectories is called a **phase portrait** of the system of equations.

Initial Value Problem

Frequently, there will also be given initial conditions:

$$x_1(t_0) = x_1^0 \qquad x_2(t_0) = x_2^0 \tag{4}$$

We can write (4) in matrix form:

$$\mathbf{x}(t_0) = \mathbf{x}_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} \tag{5}$$

Then equations

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{6}$$

form an initial value problem.

Existence and Uniqueness of Solutions

Theorem

Let

- *each of the functions* $p_{11}(t), p_{12}(t), p_{21}(t), p_{22}(t), g_1(t)$, *and* $g_2(t)$ *be continuous on an open interval* $I = (\alpha, \beta)$
- $t_0 \in I$
- x_1^0 *and* x_2^0 *be any given numbers*

Then there **exists a unique solution** *of the initial value problem*

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

on the interval I.

Importance of the theorem:

- it ensures that the problem we are trying to solve actually has a solution
- if we are successful in finding a solution, we can be sure that it is the only one
- it promotes confidence in using numerical approximation methods when we are sure that a solution exists

Autonomous Systems

Definition

If the right hand side of

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

does not depend explicitly on t, the system is said to be **autonomous**.

For the system to be autonomous, the elements of the coefficient matrix $\mathbf{P}$ and the components of the vector $\mathbf{g}$ must be constants. In this case we will usually use the notation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{7}$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{b}$ is a constant vector.

It follows from the theorem, that the solution of the initial value problem for an autonomous system exists and is unique on the entire t-axis

Equilibrium Solutions of Autonomous Systems

Constant solutions are called **equilibrium solutions**. For autonomous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

we find the equilibrium solutions by setting $d\mathbf{x}/dt = 0$. Hence any solution of

$$\mathbf{A}\mathbf{x} = -\mathbf{b} \tag{8}$$

is an equilibrium solution.

- If $\mathbf{A}$ is nonsingular, then (8) has a single solution $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$
- If A is singular. then (8) has either no solution or infinitely many.

Remark:

It is important to understand that equilibrium solutions are found by solving algebraic, rather than differential equations.

Example

- Find all equilibrium solutions of the following system of ODEs

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1 - x_2 - 8 \\ \dfrac{dx_2}{dt} = x_1 + 2x_2 - 5 \end{cases}$$

2^{nd} order ODE $\rightarrow$ system of two 1^{st} order ODEs

Consider the second order linear ODE

$$y'' + p(t)y' + q(t)y = g(t) \tag{9}$$

This equation can be transformed into a system of two first order linear ODEs.

First, let us introduce new variables x_1 and x_2:

$$x_1 = y, \qquad x_2 = y' \tag{10}$$

Next, by differentiation, we obtain

$$x_1' = y', \qquad x_2' = y'' \tag{11}$$

Now we can express the right hand sides of Eqs (11) in terms of x_1 and x_2 using (9) and (10):

$$\begin{cases} x_1' = x_2 \\ x_2' = -q(t)x_1 - p(t)x_2 + g(t) \end{cases} \tag{12}$$

2^{nd} order ODE $\rightarrow$ system of two 1^{st} order ODEs

Using matrix notation, we can write (12) as

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \tag{13}$$

Initial conditions for

$$y'' + p(t)y' + q(t)y = g(t)$$

are of the form

$$y(t_0) = y_0, \qquad y'(t_0) = y_1$$

These initial conditions are then transferred to the state variables x_1 and x_2:

$$\mathbf{x}(t_0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

Example

- Consider the differential equation

$$u'' + 0.25u' + 2u = 3\sin t$$

together with the initial conditions

$$u(0) = 2, \qquad u'(0) = -2$$

Write this initial value problem in the form of a system of two first order linear ODEs

Summary

- General system of two first order linear ODEs:

$$\boxed{\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

  - $\mathbf{x} = (x_1, x_2)^T$ is the state vector
  - $\{\mathbf{x}(t), t \in I\}$ is the trajectory
  - $\{\text{trajectories}\}$ is the phase portrait

- If $\mathbf{P}$ and $\mathbf{g}$ are continuous, then there exists a unique solution of the initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

- Autonomous systems: $\mathbf{P}(t) = \mathbf{A}$, $\mathbf{g}(t) = \mathbf{b}$
- If A is nonsingular, then the equilibrium solution of autonomous system is

$$\mathbf{x}_{eq} = -\mathbf{A}^{-1}\mathbf{b}$$

- Any second order linear ODE can be transformed into a system of two first order linear ODEs

Homework

Homework:

- Section 3.2
  - 5, 10(a), 15(a), 26