Math 245 - Mathematics of Physics and Engineering I

Lecture 8. Systems of Two Linear Algebraic Equations: A Review

January 27, 2012

Agenda

- Systems of Two Linear Algebraic Equations
- Geometric interpretation
- Solutions, Cramer's rule, Determinants
- Identity matrix, Inverse Matrix
- Singular and Nonsingular Matrices
- Homogeneous Systems
- Eigenvalues and Eigenvectors
- Important Theorem
- Homework

Question: Why do we need to review systems of algebraic equations?

Answer: We studied first order ODEs. Our next goal is to study systems of two linear ODEs. It turns out that the solution of a system of two linear ODEs is directly related to the solutions of an associated system of two linear algebraic equations.

In this lecture we will review the properties of such linear algebraic systems.

Consider the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

where $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are given coefficients and x_1 and x_2 are to be determined. Geometrically, each equation defines a straight line in the x_1x_2-plane.

- If the two lines intersect at a single point $(x_1^*, x_2^*) \Rightarrow (x_1^*, x_2^*)$ is the single solution of the system.
- If the two lines are parallel $\Rightarrow$ the system has no solution.
- If the two lines are coincide $\Rightarrow$ the system has infinitely many solutions.

Our system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \qquad (1)$$

can be rewritten in matrix form:

$$Ax = b \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Cramer's rule:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\det A} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\det A} \qquad (2)$$

where

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Theorem

- *The system (1) has a unique solution* $\Leftrightarrow \det A \neq 0$
- *In this case the solution is given by (2)*
- *If* $\det A = 0$, *then (1) has either no solution or infinitely many*

Let us introduce two important matrices.

Definition

The 2×2 **identity matrix** is denoted by I and is defined to be

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

- For any 2×2 matrix A, $AI = IA = A$ (hence the name)

Definition

Let A be a 2×2 matrix. Matrix B is called the **inverse** of A if

$$AB = BA = I \tag{4}$$

- The inverse matrix is denoted by $B = A^{-1}$
- If A^{-1} exists, then A is called **nonsingular** or **invertible**.
- If A^{-1} does not exist, then A is called **singular** or **noninvertible**.

Theorem

Matrix A is nonsingular $\Leftrightarrow \det A \neq 0$.
If A is nonsingular, then the inverse matrix is

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \tag{5}$$

Recall, that the system $Ax = b$ has a unique solution $\Leftrightarrow \det A \neq 0$ (i.e. A is nonsingular). In this case, this unique solution can be written in the following form:

$$x = A^{-1} b \tag{6}$$

Definition

The system $Ax = b$ is called **homogeneous** is $b = 0$ (i.e. $b_1 = b_2 = 0$); otherwise, it is called **nonhomogeneous**.

- The homogeneous system always has the **trivial solution** $x_1 = x_2 = 0$.
- The **trivial solution is the only solution** of the system $\Leftrightarrow \det A \neq 0$
- **Nontrivial solution** exists $\Leftrightarrow \det A = 0$
  - If $A = 0$, then every point (x_1, x_2) is a solution of the system.
  - If $A \neq 0$, $\det A = 0$, then all solutions lie on a line through the origin.

Characteristic Equation

The equation $y = Ax$ can be considered as a transformation of vector x to a new vector y. In many applications it is of particular importance to find those vectors x that are transformed into λx, where λ is a scalar factor. These vectors satisfy

$$Ax = \lambda x \tag{7}$$

- $x = 0$ is always ($\Rightarrow$ "not interesting") a solution of (7). So we require $x \neq 0$.

System (7) can be written in the following homogeneous form:

$$(A - \lambda I)x = 0 \tag{8}$$

As we already know, (8) has nontrivial ($x \neq 0$) solutions if and only if

$$\det(A - \lambda I) \equiv \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \tag{9}$$

Definition

Equation (9) is called the **characteristic equation** of the matrix A.

Eigenvalues and Eigenvectors

It can be shown that the characteristic equation

$$\det(A - \lambda I) = 0$$

can be written in the following form:

$$\boxed{\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0} \qquad (10)$$

where $\mathrm{tr}(A) = a_{11} + a_{22}$. Characteristic equation is a quadratic equation in λ, so it has two roots λ_1 and λ_2.

Definition

The values λ_1 and λ_2 are called **eigenvalues** of A.
The corresponding vectors x_1 and x_2 are called the **eigenvectors** of A.

There are 3 possible options for eigenvalues:

- λ_1 and λ_2 are real and different, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$
- λ_1 and λ_2 are real and equal, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 = \lambda_2$
- λ_1 and λ_2 are complex and conjugate, $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 = a + ib$, $\lambda_2 = a - ib$.

Examples

- Find the eigenvalues and eigenvectors of the matrices
  - $$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$
  - $$A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$$

Important Theorem

Theorem

Let A have two distinct eigenvalues $\lambda_1 \neq \lambda_2$, and let the corresponding eigenvectors be

$$x_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \qquad x_2 = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

If X is the matrix with first and second columns taken to be x_1 and x_2, respectively,

$$X = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

then

$$\boxed{\det X \neq 0}$$

Homework

<u>Homework:</u>

- Section 3.1
  - 13, 15, 17, 33.