Math 245 - Mathematics of Physics and Engineering I

Lecture 7. Exact Equations and Integrating Factors

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Exact Equations

Let us rewrite a first order ODE $\frac{dy}{dx} = f(x, y)$ in the following form:

$$M(x, y) + N(x, y)y' = 0 \tag{1}$$

Suppose that we can identify a function $\psi(x, y)$ such that

$$\boxed{\frac{\partial \psi}{\partial x} = M(x, y)} \qquad \boxed{\frac{\partial \psi}{\partial y} = N(x, y)}$$

Then

$$M(x, y) + N(x, y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x, y(x))$$

Therefore ODE (1) becomes

$$\frac{d}{dx}\psi(x, y(x)) = 0 \tag{2}$$

In this case, (1) is said to be **exact** differential equation and its solutions are given implicitly by

$$\boxed{\psi(x, y) = C} \qquad C = \text{const}$$

Criterion of Exactness

In Lecture 6, we considered an example where it was relatively easy to see that the equation was exact and easy to find ψ and solve the equation.

Q: How to systematically determine whether a given ODE is exact?

$$M(x,y) + N(x,y)y' = 0 \tag{3}$$

Theorem

Let $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ be continuous in the region $R : x \in (\alpha, \beta)$, $y \in (\gamma, \delta)$. Then equation (3) is an exact differential equation in R if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \tag{4}$$

In other words, a function ψ satisfying $\frac{\partial \psi}{\partial x} = M(x,y)$ and $\frac{\partial \psi}{\partial y} = N(x,y)$ exists **if and only if** *M and N satisfy (4).*

Corollary and Examples

Corollary

Any **separable** *differential equation is also* **exact**.

Examples:

- Solve the differential equation

$$y \cos x + 2xe^y + (\sin x + x^2 e^y - 1)y' = 0$$

- Is the following equation exact?

$$3xy + y^2 + (x^2 + xy)y' = 0$$

Integrating Factors

It is sometimes possible to convert a differential equation that is not exact into an exact equation by a suitable integrating factor.

Let us multiply the original equation $M(x, y) + N(x, y)y' = 0$ by a function $\mu(x, y)$ and then try to choose $\mu(x, y)$ so that the resulting equation

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)y' = 0 \tag{5}$$

is exact. By the Theorem, the above equation is exact if and only if

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \tag{6}$$

Equation (6) is the first order partial differential equation for μ:

$$\boxed{M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = 0} \tag{7}$$

If we can find μ that satisfies (7), then (5) is exact and we can solve it. Unfortunately, equation (7) is usually hard to solve.

Example

- Find an integrating factor for the equation

$$3xy + y^2 + (x^2 + xy)y' = 0$$

and then solve the equation.

Summary and Homework

- Criterion of exactness: The equation

$$M(x,y) + N(x,y)y' = 0$$

is an exact differential equation **if and only if**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- All separable equations are exact.
- It is sometimes possible to convert a differential equation that is not exact into an exact equation by a suitable integrating factor μ. Equation for μ is

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = 0$$

Homework:

- Section 2.5
  - 11(a), 13 (just solve the initial value problem), 26(a)