Math 245 - Mathematics of Physics and Engineering I

Lecture 6. Autonomous and Exact Equations

January 23, 2012

Agenda

- Definition of Autonomous ODEs
- Applications: Population Dynamics
  - Logistic equation and its solutions
- Exact Equations: first example
- Summary and Homework

Autonomous Equations

A general first order ODE has the following form:

$$\frac{dy}{dt} = f(t, y)$$

An important class of first order ODEs are those in which the independent variable does not appear explicitly:

Definition

An equation of the form

$$\boxed{\frac{dy}{dt} = f(y)} \tag{1}$$

is called **autonomous**.

Example: Heat exchange equation $\frac{du}{dt} = -k(u - T_0)$ is autonomous.

Autonomous ODEs are often used to describe the population dynamics.

Population Dynamics

Let $y(t)$ be the population size at time t.

Simple assumption: The rate of change of the population size is proportional to the current population size. This assumtion leads to the following equation

$$\frac{dy}{dt} = ry, \tag{2}$$

where r is called the rate of growth (if $r > 0$) or the rate of decline (if $r < 0$). Solving this simple autonomous ODE subject to the initial condition $y(0) = y_0$, we obtain

$$y(t) = y_0 e^{rt} \tag{3}$$

Thus, the mathematical model (2) predicts that the population will grow exponentially for all time (here we assume $r > 0$)

This model may be very accurate for many populations for short periods of time. In general, for long periods of time, it is clear that this model can't be realistic: limitations of space and food supply will reduce the growth rate.

Population Dynamics

To improve the model, we replace the constant r by a function $h(y)$:

$$\frac{dy}{dt} = ry \qquad \rightsquigarrow \qquad \frac{dy}{dt} = h(y)y \tag{4}$$

Q: How to reasonably choose $h(y)$? What do we want from $h(y)$?

- If y is small, $h(y) \approx r > 0$
- As y gets larger, $h(y)$ decreases
- If y is sufficiently large, then $h(y) < 0$

The simplest function that has these properties is

$$h(y) = r - ay, \qquad a > 0 \tag{5}$$

Using (5), we obtain:

$$\frac{dy}{dt} = (r - ay)y = r\left(1 - \frac{y}{K}\right)y, \qquad K = \frac{r}{a} \tag{6}$$

This autonomous ODE is called the **logistic equation**.

Solutions of the logistic equation

Important observation: Autonomous equations are separable

$$\frac{dy}{dt} = f(y)$$

Therefore, we know how to solve them (see Lecture 3).

In particular, the logistic equation is separable.

Main Result

The solution of the initial value problem

$$\begin{cases} \frac{dy}{dt} = r\left(1 - \frac{y}{K}\right) y, \\ y(0) = y_0. \end{cases}$$

is given by

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

Exact Equations

Let us rewrite a first order ODE $\frac{dy}{dx} = f(x, y)$ in the following form:

$$M(x, y) + N(x, y)y' = 0 \tag{7}$$

Suppose that we can identify a function $\psi(x, y)$ such that

$$\boxed{\frac{\partial \psi}{\partial x} = M(x, y)} \qquad \boxed{\frac{\partial \psi}{\partial y} = N(x, y)}$$

Then

$$M(x, y) + N(x, y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x, y(x))$$

Therefore ODE (7) becomes

$$\frac{d}{dx}\psi(x, y(x)) = 0 \tag{8}$$

In this case, (7) is said to be **exact** differential equation and its solutions are given implicitly by

$$\boxed{\psi(x, y) = C} \qquad C = \text{const}$$

Example

- Solve the differential equation:

$$2x + y^2 + 2xyy' = 0$$

- Show that any separable equation is also exact

Summary

- An equation of the following form is called autonomous.

$$\frac{dy}{dt} = f(y)$$

- An important example of autonomous equation is the logistic equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right) y$$

- Autonomous equations are separable $\Rightarrow$ we can solve them.
- The solution of the logistic equation subject to $y(0) = y_0$ is

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

- We defined exact equations and considered one example.
- Any separable equation is exact.

Homework

<u>Homework:</u>

- Section 2.5
  - 1(b), 3(b), 5(b), 9(b)