Math 245 - Mathematics of Physics and Engineering I

Lecture 5. Existence and Uniqueness of Solutions: Linear and Nonlinear first order ODEs

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Existence and Uniqueness of Solutions

In Lecture 4, we discussed two initial value problems, Escape Velocity and Mixing, each of which had a solution and apparently only one solution.

Question: Does every initial value problem have exactly one solution?

Q: Why is this question important?

- we might want to know that the problem has a solution before spending time and effort in trying to find it
- if we find one solution, we might be interested in knowing whether other solutions exist

Theorem

Consider the following first order **linear** *ODE:*

$$y' + p(t)y = g(t)$$

If $p(t)$ and $g(t)$ are continuous on an open interval (α, β) containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies this ODE for each $t \in (\alpha, \beta)$, and that also satisfies the initial condition $y(t_0) = y_0$ where y_0 is an arbitrary prescribed initial value.

Existence and Uniqueness of Solutions

Using the Method of Integrating Factors, we can obtain the unique solution of the initial value problem

$$\begin{cases} y' + p(t)y = g(t), \\ y(t_0) = y_0. \end{cases}$$

The unique solution is

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} \mu(s)g(s)ds + y_0 \right),$$

where

$$\mu(t) = \exp \int_{t_0}^{t} p(s)ds$$

Existence and Uniqueness of Solutions

Q: What about nonlinear equations?

Theorem

Consider the following first order **nonlinear** *ODE:*

$$y' = f(t, y)$$

Let the functions f *and* $\partial f/\partial y$ *be continuous in some open rectangle* $t \in (\alpha, \beta)$, $y \in (y_1, y_2)$ *containing the point* (t_0, y_0)*. Then, in some interval* $t \in (t_0 - h, t_0 + h) \subset (\alpha, \beta)$*, there is a unique solution* $y = \phi(t)$ *of the initial value problem*

$$y' = f(t, y), \qquad y(t_0) = y_0$$

Remarks:

- The proof of this theorem is relatively complicated.
- Conditions stated are sufficient to guarantee the existence of a unique solution, but they are not necessary. In fact, the existence of a solution (but not uniqueness!) can be proved on the basis of the continuity of f alone.

Example 1

Problem

Find an interval in which the initial value problem

$$ty' + 2y = 4t^2 \qquad y(1) = 2$$

has a unique solution

Example 2

Problem

Prove that the initial value problem

$$y' = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1$$

has a unique solution in some interval about $x = 0$.

Example 3

Problem

Consider the following initial value problem

$$y' = y^{1/3}, \qquad y(0) = 0$$

1. *Is Theorem 2 applicable?*
2. *Does the initial problem have a solution?*
3. *Is the solution unique?*

Remark: The nonuniqueness of the solution does not contradict the existence and uniqueness theorem. The theorem is just not applicable!

Summary and Homework

- We discussed the existence and uniqueness of the first order ODEs
- The **linear** ODEs $y' + p(t)y = g(t)$ has several nice properties:
  - If coefficient p and g are continuous, then there is a general solution that includes all solutions of the equation. A particular solution that satisfies a given initial condition can be picked by choosing the proper value for the constant.
  - An expression for the solution is

$$\boxed{y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)ds + y_0\right)} \qquad \mu(t) = \exp\int_{t_0}^{t} p(s)ds$$

  - The points of discontinuity, or singularities, of the solution can be identified without solving the problem (!) by finding the points of discontinuity of the coefficients.
- Careful! None of this properties is true, in general, for **nonlinear** ODEs.

Homework:

- Section 2.3
  - 1, 9, 15