Math 245 - Mathematics of Physics and Engineering I

Lecture 4. Modeling with First Order Equations

January 18, 2012

Agenda

- Modeling: General Framework
  - Construction of the Model
  - Analysis of the Model
  - Comparison with Experiment or Observation
- Examples
- Summary and Homework

Modeling

For mathematicians, differential equations are interesting in their own right.

For nonmathematicians, differential equations are important because mathematical models of real-life processes often lead to differential equations.

Mathematical modeling and experiment/observation are both critically important in scientific investigations, and they play complementary roles:

- Models are validated by comparison of their predictions with experimental results
- Models may suggest the most promising directions for experimental exploration.

Regardless of the scientific filed of application, there are 3 main stages that are always present in the process of mathematical modeling:

1. Construction of the Model
2. Analysis of the Model
3. Comparison with Experiment or Observation

Construction of the Model

In the first stage, we translate the physical situation into mathematical terms

Most critical at this stage:

to state clearly the physical principe that is believed to govern the process:

- for example, it has been observed that in come circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference (Newton's Law of Cooling)

Based on this physical principle we can write a differential equation, and this equation is the model of the process.

Remark:

It is important to realize that the mathematical equations are almost always only an approximate description of the actual process.

Analysis of the Model

Once the problem has been formulated mathematically, we are often faced with the problem of solving one or more differential equations

It may happen that these equations are quite difficult, and if so, further approximations (of the reality) may be required to make the problem more susceptible to mathematical investigation:

- for example, a nonlinear equation may be approximated by a linear one
- slowly varying coefficient may be replaced by a constant

Remark:

Any such approximation must also be examined from the physical point of view to make sure that the simplified mathematical problem still reflects the essential features of the physical process.

Comparison with Experiment or Observation

Finally, having obtained the solution (or at least some information about it), we must interpret this information in the original physical context

- in particular, we should always check that the mathematical solution appears physically reasonable
- if possible, we can calculate the values of the solution at selected points and compare them with experimentally observed values
- we can ask whether behavior of the solution after a long time is consistent with observations.

Example 1: Escape Velocity

Problem

A body of mass m is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0. Assuming there is no air resistance, but taking into account the variation of the earth's gravitational field with distance, find

1. *an expression for the velocity of the body*
2. *the initial velocity v_0 that is required to lift the body to a given maximum altitude z above the surface of the earth*
3. *the least initial velocity for which the body will not return to the earth (**escape velocity**)*

Example 2: Mixing

Problem

At time $t = 0$ a tank contains Q_0 lb of salt dissolved in V gal of water. Assume that water containing q lb of salt/gal is entering the tank at a rate of r gal/min and that the well-stirred mixture is draining from the tank at the same rate.

1. *Set up the initial value problem that describes this flow process*
2. *Find the quantity of salt $Q(t)$ in the tank at time t*
3. *Find the limiting quantity Q_L that is present after a very long time*

Summary

- We discussed modeling of physical processes with first order differential equations
- Three stages of modeling:
  - Construction of the Model
  - Analysis of the Model
  - Comparison with Experiment or Observation
- Two examples:
  - Escape velocity (physics)
  - Mixing (chemistry)

<u>Homework:</u>

- Section 2.2
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