*Math 245 - Mathematics of Physics and Engineering I*

# Lecture 3. First Order Differential Equations: Separable Equations

January 13, 2012

# Agenda

- First order ODEs
- Separable Equations
- Implicit Solutions
- Summary and Homework

# First Order ODEs

In Lecture 2, we considered first order linear ODEs: $y'(x) + p(x)y = g(x)$ and learned how to solve them using the method of integrating factors.

Our next goal is to consider general (not necessarily linear) **first order ODE**:

$$\boxed{\frac{dy}{dx} = f(x, y)} \qquad (1)$$

where $f$ is a given function of two variables. We aim to develop methods for finding solutions of these equations.

For an arbitrary function $f$, there is no general method for solving Eq. (1).Our strategy will be to describe several methods, each of which is applicable to a certain subclass of first order ODEs.

Important classes of first order ODEs:

- Linear equations (Lecture 2)
- Separable equations (Lecture 3)
- Exact equation (later)

# Separable Equations

The general first order ODE is

$$y' = f(x, y) \tag{2}$$

## Definition

This equation is called **separable** if $f(x, y) = \frac{g(x)}{h(y)}$. In other words, the equation is separable if it can be written as follows:

$$h(y)y' = g(x), \tag{3}$$

or, equivalently, in the differential form,

$$\boxed{h(y)dy = g(x)dx} \tag{4}$$

Remark:

- It is called separable because in (4), terms involving each variable are separated by the equals sign.

# How to solve separable equation $h(y)y' = g(x)$?

Let $H$ and $G$ be any antiderivatives of $h$ and $g$ respectively:

$$\frac{dH(y)}{dy} = h(y), \qquad \frac{dG(x)}{dx} = g(x)$$

Then (3) becomes

$$\frac{dH(y)}{dy}\frac{dy}{dx} = \frac{dG(x)}{dx} \tag{5}$$

Since $y$ is a function of $x$, according to the chain rule, we have:

$$\frac{dH(y)}{dx} = \frac{dH(y)}{dy}\frac{dy}{dx}$$

Consequently, we can write Eq. (5) as

$$\frac{dH(y)}{dx} = \frac{dG(x)}{dx} \tag{6}$$

By integrating Eq. (6), we obtain

$$\boxed{H(y) = G(x) + C} \qquad C \text{ is a constant} \tag{7}$$

# Solution of separable equations

## Main Result

*Any differentiable function $y = \phi(x)$ that satisfies equation (7),*

$$\boxed{H(y) = G(x) + C}$$

*is a solution of the separable equation (3),*

$$h(y)y' = g(x)$$

Important Remarks:

- Eq. (7) defines the solution implicitly rather than explicitly.
- In practice, Eq. (7) is usually obtained directly from Eq. (4),

$$h(y)dy = g(x)dx,$$

by integrating the right hand side with respect to $y$ and the left hand side with respect to $x$.

# Example

- Solve the equation

$$y'(x) = \frac{x^2}{1 - y^2}$$

# Initial value problem

If, in addition to the differential equation, an initial condition

$$y(x_0) = y_0$$

is given, then the solution of the initial value problem

$$\begin{cases} h(y)y' = g(x), \\ y(x_0) = y_0. \end{cases}$$

is obtained by setting $x = x_0$ and $y = y_0$ in Eq. (7) and finding the value of a constant:

$$C_0 = H(y_0) - G(x_0)$$

The solution of the initial value problem is then

$$\boxed{H(y) = G(x) + C_0}$$

# Example

- Find the solution of the initial value problem in explicit form:

$$\begin{cases} y' = \dfrac{3x^2 + 4x + 2}{2(y-1)}, \\ y(0) = -1. \end{cases}$$

# Test Questions and a Useful Observation

Q1: Find at least one solution of the following ODE:

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$$

Q2: Is this equation separable?

Observation:
Sometimes a first order ODE

$$\frac{dy}{dx} = f(x,y)$$

has a constant solution $y = y_0$. Such a solution is usually easy to find because if $f(x, y_0) = 0$ for all $x$, then the constant function $y = y_0$ is a solution of the differential equation.

# Example

- Solve the differential equation (Problem 3, page 50)

$$y' + y^2 \sin x = 0$$

# Summary

- We started to study first order ODEs

$$\frac{dy}{dx} = f(x, y)$$

- Separable equations

$$h(y)dy = g(x)dx$$

is an important subclass of first order ODEs

- Solution of the separable equation is (implicitly) given by

$$H(y) = G(x) + C,$$

where $H$ and $G$ are antiderivatives of $h$ and $g$ respectively.

Homework:

- Section 2.1
  - 2, 7, 11(a), 18(a)