*Math 245 - Mathematics of Physics and Engineering I*

# Lecture 2. Classification of Differential Equations and Method of Integrating Factors

January 11, 2012

# Agenda

- Classification of Differential Equations
- Linear Equations
  - Method of Integrating Factors
  - Examples
- Summary and Homework

# Classification of Differential Equations

In Lecture 1, we were able to find an explicit analytic solution of the differential equation that models heat exchange between an object and its constant temperature surroundings

$$\frac{du}{dt} = -k(u - T_0)$$

Important message: There is no general method for finding analytical solutions to all differential equations.

- Why? Because a differential equation is simply an equation containing one or more derivatives of the unknown function and, therefore, there are too many different kinds of differential equations.

Strategy: Identify a class of equations and a corresponding method that can be used to solve all equations in the class. This approach gives us a collection of important classes of equations with corresponding solution methods.

Let us start with a very general classification of differential equations.

# Ordinary and Partial Differential Equations

Q: Does the unknown function depend on a single independent variable or on several independent variables?

- If the unknown function depends only on one independent variables, then only ordinary derivatives appear in the differential equation. In this case, the equation is called an **ordinary differential equation**.
  Example:

$$\frac{du}{dt} = -k(u - T_0)$$

- If the unknown function depends on several independent variables, then the derivatives are partial derivatives. In this case, the equation is called an **partial differential equation**.
  Example:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Laplace Equation

# Order

## Definition

The order of a differential equation is the order of the highest derivative that appears in the equation.

Examples:

- First order

$$\frac{du}{dt} = -k(u - T_0)$$

- Second order

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- Second order if $a \neq 0$; and first order if $a = 0$ and $b \neq 0$

$$ay'' + by' + cy = f(t)$$

An ordinary differential equation of order $n$ can be written as follows:

$$\boxed{y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})}$$

# Linear and Nonlinear Equations

## Definition

The ordinary differential equation

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0$$

is said to be linear if $F$ is a linear function of the variables $y, y', y'', \ldots, y^{(n)}$.

The general **linear** differential equation of order $n$ is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_{n-1}(t)y' + a_n(t)y = g(t), \quad a_0(t) \neq 0$$

An equation that is not of this form is a **nonlinear** equation.

Examples:

- Linear equation:

$$\frac{du}{dt} = -k(u - T_0)$$

- Nonlinear equation:

$$\frac{d^2\theta}{d\theta^2} + \frac{g}{L}\sin\theta = 0$$

# Solutions

In this course we will study ordinary differential equations (ODEs)

$$\boxed{y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})}$$

There are three fundamental questions with respect to solutions of ODEs. Before we discuss these questions, let us define more precisely what we mean by solution of an ODE.

## Definition

A solution of the equation

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$$

on the interval $\alpha < t < \beta$ is a function $\phi$ such that

- $\phi', \phi'', \ldots, \phi^{(n)}$ exist and
- they satisfy $\phi^{(n)} = f(t, \phi, \phi', \phi'', \ldots, \phi^{(n-1)})$ for every $t \in (\alpha, \beta)$.

# Fundamental Questions

$$\boxed{y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})}$$

1. Existence

   Does an ODE always have a solution? The answer is "No". How can we tell whether some particular ODE has a solution? This is the question of **existence** of a solution, and it is answered by theorems stating that under certain conditions on the function $f$, the equation always has a solution.

2. Uniqueness

   Assume that a given ODE has at least one solution. Then the following question arises naturally: how many solutions does it have, and what additional conditions must be specified to single out a particular solution? This is the question of **uniqueness**.

3. Given an ODE, can we actually determine a solution, and if yes, then how? Note, the questions 1 and 3 are different:
   - Without knowledge of existence theory, we might use a computer to find a numerical approximation to a "solution" that does not exist.

# First Order Linear ODEs

## Definition

A differential equation that can be written in the form

$$\boxed{\frac{dy}{dt} + p(t)y = g(t)} \qquad (1)$$

is called a first order linear ODE.

Q1: Is $\frac{du}{dt} = -k(u - T_0)$ a first order linear ODE?

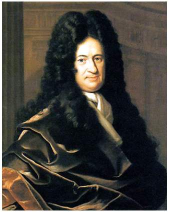
- Eq. (1) is referred to as the **standard form** for a first order linear ODE. The more general form is

$$a_0(t)\frac{dy}{dt} + a_1(t)y = b(t)$$

- If $g(t) \equiv 0$, it is said to be **homogeneous**; otherwise the equation is **nonhomogeneous**.

Q2: Linear, Homogeneous? $y' = y\cos t$, $y' + 1/t = ty$, $y' + y^2 = t$

# Method of Integrating Factors



The method that is used to solve $y' + p(t)y = g(t)$ is due to Leibniz. It involves multiplying the equation by a certain function $\mu(t)$, chosen so that the resulting equation is readily integrable. The function $\mu(t)$ is called an **integrating factor**. The main difficulty is to determine how to find $\mu(t)$.

$$y'(t) + p(t)y = g(t)$$

1) Multiply this equation by an (as yet undetermined) function $\mu(t)$

$$\mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)g(t)$$

2) Let $\mu(t)$ be such that

$$\mu(t)y'(t) + \underbrace{p(t)\mu(t)}_{\mu'(t)}y(t) = \mu(t)g(t)$$

3) In other words, let $\mu(t)$ be a solution of the following ODE:

$$\mu'(t) = p(t)\mu(t)$$

# Method of Integrating Factors

Q3: How to find a solution of the homogeneous equation $\mu'(t) = p(t)\mu(t)$ ?

4) Then the equation from step 2) can be written as

$$[\mu(t)y(t)]' = \mu(t)g(t)$$

Hence

$$\mu(t)y(t) = \int \mu(t)g(t)dt = \int_{t_0}^{t} \mu(t)g(t)dt + C,$$

where $C$ is an arbitrary constant.

## Main Result

*The general solution of the first order linear ODE*

$$y'(t) + p(t)y(t) = g(t)$$

*is*

$$y(t) = \frac{1}{\mu(t)} \left( \int_{t_0}^{t} \mu(t)g(t)dt + C \right),$$

*where $C$ is a constant and $\mu(t) = e^{\int p(t)dt}$.*

# Examples

- Solve the ODE $y'(t) - 2y = 4 - t$

- Solve the initial value problem $ty(t)' + 2y = 4t^2$, $y(1) = 2$.

# Summary

- There are many different types of differential equations: ordinary and partial, linear and nonlinear, homogeneous and nonhomogeneous.
- There is no general method for finding analytical solutions to all differential equations. Thus, we try to identify a class of equations and a corresponding method that can be used to solve all equations in the class.
- First order linear ODEs can be solved by the Method of Integrating Factors.

Homework:

- Section 1.4
  - 5, 6.
- Section 1.2
  - 17, 30.