Math 245 - Mathematics of Physics and Engineering I

Lecture 1. Differential Equations: An Introduction

January 9, 2012

Agenda

- Differential Equations
- Example: Heat Transfer
  - Basic terminology
  - Solutions and Integral Curves
  - Initial Value Problems
  - Direction Fields
- Summary

Differential Equations

Definition

A differential equation (DE) is an equation for an unknown function that contains derivatives of that function.

DEs are used in all fields of science and engineering as well as in economics and social sciences. DEs are used to study problems such as

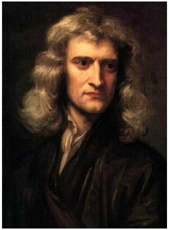
- wave propagation
- heat transfer
- weather forecasting
- controlling the flight of airplanes
- determining the price of financial derivatives

We often refer to a differential equation that describes some physical process as a mathematical model of the process. DEs are often used to model dynamic systems that change *continuously* with time.

Example: Heat Transfer

Suppose a material object is placed in some environment. If the object is hotter or colder than the surrounding environment, its temperature will *approach the temperature of the environment*:

- if the object is hotter ⇒ its temperature will decrease
- if the object is colder ⇒ its temperature will increase



Newton's Law of Cooling

The rate of change of the temperature of the object is negatively proportional to the difference between its temperature and the temperature of the surroundings.

Suppose

- $u(t)$ is the temperature of the object
- T_0 is the temperature of the surroundings (the ambient temperature)
- du/dt is then the rate at which the temperature of the object changes

Example: Heat Transfer

Newton's Law of Cooling:

$$\frac{du}{dt} \propto -(u - T_0)$$

- Let k be a positive constant of proportionality (the transmission coefficient)

$$\boxed{\frac{du}{dt} = -k(u - T_0)} \qquad (1)$$

Remarks:

1. Equation (1) is a differential equation
2. If $u(t) > T_0$, then the sign of du/dt is negative: the object cools down.
3. If $u(t) < T_0$, then the sign of du/dt is positive: the object warms up.
4. The **transmission coefficient measures the rate of heat exchange** between the object and surroundings:
   - if k is large, then the rate of heat exchange is rapid
   - if k is small, then the rate of heat exchange is slow

Example: Heat Transfer

$$\boxed{\frac{du}{dt} = -k(u - T_0)}$$

Terminology:

- Time t is an **independent variable**
- Temperature u is a **dependent variable** (it depends on t)
- T_0 and k are **parameters** of the model
- The equation is an ordinary differential equation of the first order

Definition

A solution of equation $\frac{du}{dt} = -k(u - T_0)$ is a differentiable function $u = \phi(t)$ that satisfies that equation.

One solution is $u = T_0$. Assume $u \neq T_0$.
Question: Is there any other solutions?

Example: Heat Transfer

$$\frac{du}{dt} = -k(u - T_0), \qquad u \neq T_0$$

$$\frac{du/dt}{u - T_0} = -k \qquad \Rightarrow \qquad \frac{d}{dt} \ln|u - T_0| = -k$$

$$\ln|u - T_0| = -kt + C \qquad \Rightarrow \qquad |u - T_0| = e^{-kt+C} = e^C e^{-kt}$$

$$u - T_0 = \pm e^C e^{-kt} \qquad \Rightarrow \qquad u = T_0 \pm e^C e^{-kt} = T_0 + \widehat{C} e^{-kt}$$

Thus, $u = T_0 + \widehat{C} e^{-kt}$ is a solution of the equation, where $\widehat{C} \neq 0$. Note that if $\widehat{C} = 0$, then $u = T_0$ is also a solution. Therefore, the expression

$$\boxed{u = T_0 + ce^{-kt}} \tag{2}$$

where c is any constant, contains all possible solutions of the equation (1). It is called the **general solution** of the equation.

Remark:

- Given (2), it is easy to verify that it is indeed a solution of (1)

Example: Heat Transfer

<u>Question:</u> How can we represent the general solution geometrically?

Definition

The geometrical representation of the general solution

$$\boxed{u = T_0 + ce^{-kt}}$$

is an infinite family of curves in the (tu)-plane. This family of curves is called **integral curves**.

<u>Remarks:</u>

- Each integral curve is associated with a particular value of constant c; it is the graph of the solution corresponding to that value of c.
- Integral curves $u = T_0 + ce^{-kt}$ can be easily sketch by hand

Example: Heat Transfer

Sometimes we want to focus our attention on a single member of the infinite family of solutions. Most often, we do this by specifying a point that must lie on the graph of the solution.For example, we could require that the temperature u have a given value u_0 at a certain time t_0. In other words, the graph of the solution must pass through the point (t_0, u_0).From this additional condition we can find the value of c that corresponds to this specific solution:

$$c = (u_0 - T_0)e^{kt_0}$$

Definition

The additional condition $u(t_0) = u_0$ is called the **initial condition**. The differential equation with the initial condition forms an **initial value problem**.

Example: Heat Transfer

Suppose that we don't know the general solution of the equation (1),

$$\frac{du}{dt} = -k(u - T_0)$$

Q: Is it still possible to determine the qualitative behavior of its solutions?
A: Yes! By examining the geometric meaning of the statement $u' = -k(u - T_0)$

Let us fix some value of u, say, $u = u^*$. Then, by evaluating the right hand side of (1), we can find the corresponding value of u', that is $u' = -k(u^* - T_0)$. This means that the slope of a solution has the value $-k(u^* - T_0)$ at any point where $u = u^*$. We can display this information graphically in the tu-plane by drawing short line segments with slope $-k(u^* - T_0)$ at several points on the line $u = u^*$. By proceeding in the same way with other values of u, we obtain an example of what is called a **direction filed**.

A direction field is important because each line segment is a tangent line to the graph of a solution. By looking at the direction field we can visualize how solutions vary with time.

Summary

- Differential Equations are very important since they are used to model dynamical systems.
- Differential Equation is a mathematical model of a physical (economical, biological, etc) process.
- Newton's Law of Cooling $\Rightarrow \frac{du}{dt} = -k(u - T_0)$
- The general solution of this equation is $u = T_0 + ce^{-kt}$, where c is an arbitrary constant.
- Integral Curves, Initial Value Problems, Direction Fields.

<u>Homework:</u>

- Section 1.1
    1. 3
    2. 17
    3. 33 (first, solve the equation)
    4. 38* (evaporation rate is dV/dt)