Problem 1. Calculate the following limits. If the limit is infinite, indicate whether it is +∞ or -∞.

$$(a)\ \lim_{x\to 3} \frac{x^2-5x+6}{x^2-2x-3} = \lim_{x\to 3} \frac{(x-3)(x-2)}{(x-3)(x+1)} = \lim_{x\to 3} \frac{x-2}{x+1} = \frac{1}{4}$$

$\boxed{\text{(a)}\quad \frac{1}{4}}$

$$(b)\ \lim_{x\to 5+} \sqrt{\frac{x-1}{2x-10}}$$

When $x \to 5+$, $(x-1) \to 4+$

$(2x-10) \to 0+$

Therefore, $\lim_{x\to 5+} \sqrt{\frac{x-1}{2x-10}} = +\infty$

$\boxed{\text{(b)}\quad +\infty}$

$$(c)\ \lim_{x\to +\infty} \sqrt{\frac{9x^2+1000x-1}{25x^2-100x+1}} = \lim_{x\to +\infty} \sqrt{\frac{9+\frac{1000}{x}-\frac{1}{x^2}}{25-\frac{100}{x}+\frac{1}{x^2}}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$\boxed{\text{(c)}\quad \frac{3}{5}}$

$$(d)\ \lim_{x\to -1} \left(\frac{1}{x+1}+\frac{1}{x^2+x}\right) = \lim_{x\to -1} \frac{x+1}{x(x+1)} =$$

$$= \lim_{x\to -1} \frac{1}{x} = -1$$

$\boxed{\text{(d)}\quad -1}$

Problem 2. Find the first and the second derivatives of the given function

$$f(x) = \left(x + \frac{1}{x}\right)^2 - \frac{1}{\sqrt{x}}$$

$f'(x) = 2\left(x + \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right) + \frac{1}{2}x^{-\frac{3}{2}} = 2\left(x + \frac{1}{x} - \frac{1}{x} - \frac{1}{x^3}\right) + \frac{x^{-\frac{3}{2}}}{2} = 2\left(x - \frac{1}{x^3}\right) + \frac{x^{-3/2}}{2}$

$f''(x) = 2\left(1 + \frac{3}{x^4}\right) - \frac{3}{4}x^{-5/2}$

(a) The first derivative

$2\left(x - \frac{1}{x^3}\right) + \frac{x^{-3/2}}{2}$

(a) The second derivative

$2 + \frac{6}{x^4} - \frac{3}{4}x^{-5/2}$

Problem 3. Find an equation of the tangent line to $y^{\frac{3}{2}} + x^{-\frac{3}{2}} = 9$ at the point where $x = 1$.

Let us use the point-slope form of the line equation.

$y - y_0 = m(x - x_0)$

In our case, $\underline{\underline{x_0 = 1}}$. Let us find y_0:

$$y^{\frac{3}{2}} + (1)^{-\frac{3}{2}} = 9$$

$$y^{\frac{3}{2}} = 8$$

$$y = 8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 4.$$

So, $\underline{\underline{y_0 = 4}}$

To find the slope $m = y'$,

use the implicit differentiation:

$$\frac{3}{2} y^{\frac{1}{2}} y' - \frac{3}{2} x^{-\frac{5}{2}} = 0$$

$$y' = \frac{x^{-5/2}}{y^{1/2}} \Rightarrow \text{ therefore } m = \left.\frac{dy}{dx}\right|_{\substack{x = x_0 \\ y = y_0}} = \frac{1}{4^{1/2}} = \underline{\underline{\frac{1}{2}}}$$

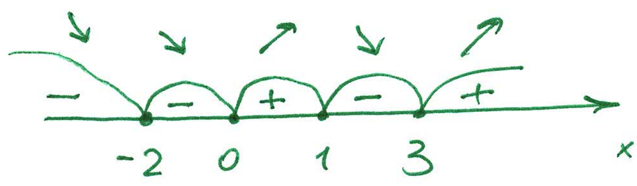
Thus, $y - 4 = \frac{1}{2}(x - 1)$

Equation of the tangent line:

$y - 4 = \frac{1}{2}(x - 1)$

Problem 4. The derivative of a function $f(x)$ is given. Find all critical numbers of $f(x)$, and classify each critical number as a relative maximum, a relative minimum, or neither.

$$f'(x) = \sqrt[5]{x}(1-x)(x+2)^2(3-x)^3$$



$$f'(x) = 0 \iff \begin{cases} x = 0 \\ x = 1 \\ x = -2 \\ x = 3 \end{cases}$$

$f'(-3) < 0$, $f'(-1) < 0$, $f'(\frac{1}{2}) > 0$

$f'(2) < 0$, $f'(4) > 0$

You may not need all of the lines.

	Critical number	Max/Min/Neither
1)	−2	neither
2)	0	Min
3)	1	MAX
4)	3	MIN
5)		

Problem 5. Consider the function f below. Its first and second derivatives are also given.

$$f(x) = \frac{x^2 - 4}{x^2 - 1} \qquad f'(x) = 6\frac{x}{(x^2-1)^2} \qquad f''(x) = -6\frac{3x^2+1}{(x^2-1)^3}$$

(a) Find the domain of f

$x^2 - 1 \neq 0 \iff x \neq 1,\ x \neq -1$

(a) $x \neq 1$, $x \neq -1$

(b) Find all intercepts of f. Write "none" if there are none.

If $x=0 \Rightarrow f(0) = 4 \Rightarrow (0,4)$ is y-intercept

$f(x) = 0 \iff x = 2$ or $x = -2 \Rightarrow (-2,0), (2,0)$ are x-int.

(b)
x-intercepts: $(-2,0)\ (2,0)$
y-intercept: $(0,4)$

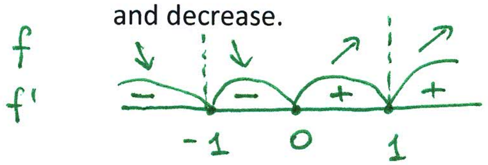
(c) Determine all vertical and horizontal asymptotes of the graph of f. Write "none" if there are none.

$\lim_{x\to+\infty} f(x) = 1$, $\lim_{x\to-\infty} f(x) = 1 \Rightarrow y = 1$ is hor. asympt.

$\lim_{x\to x_0} f(x) = \infty$ only if $x_0 = 1$ or $x_0 = -1$

(c)
Vertical: $x = 1$, $x = -1$
Horizontal: $y = 1$

(d) Find the critical points of the graph of f (write "none" if there are none) and intervals of increase and decrease.



$x = 0$ is the critical number

if $x = 0 \Rightarrow y = 4$

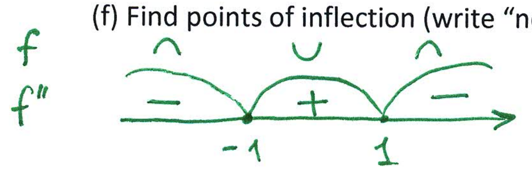
(d)
Critical points: $(0,4)$
Increasing: $(0,1), (1,+\infty)$
Decreasing: $(-\infty,-1)\ (-1,0)$

(e) Find all relative extrema (both coordinates). Write "none" if none.

$\begin{cases} x = 0 \\ y = 4 \end{cases}$ is a rel. min.

(e)
Rel. max: none
Rel. min: $(0,4)$

(f) Find points of inflection (write "none" if none) and intervals of concavity.



There are no points of inflection since $x = \pm 1$ are not in the domain

(f)
Conc. UP: $(-1, 1)$
Conc. Down: $(-\infty,-1)\ (1,+\infty)$
Inf. points: none

(g) Sketch the graph of f. Your graph should clearly show all the information you found in parts (b)-(f).