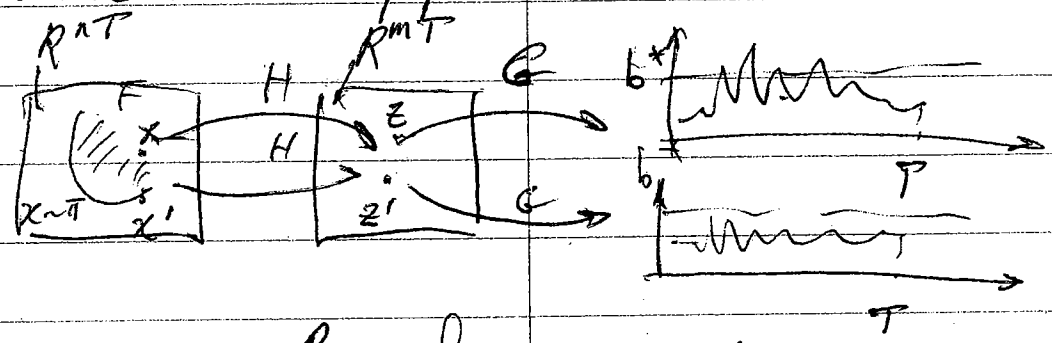
Kostia's Presentation Mar 19, 2010

$$z(t+1) = H(z(t), x(t), t)$$

$$F = \{ x : \max_{t \in [1,T]} G(z(t)) \geq b^* \}$$

Dynamic reliability problem



To estimate $P_F = \int_{R^{nT}} \pi(x) I_F(x) dx$

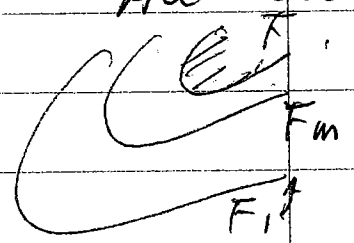
We can calculate $I_F(x)$ but very expensive

$N = nT$ very large. $N \sim 1000 \Rightarrow$ not suitable

$P_F = \int_F \pi(x) dx$. very small $\sim 10^{-3} \sim 10^{-6}$

Monte Carlo very suitable

Subset simulation. Au & Beck 2001.



How to sample from $\pi(\cdot | F_k)$?

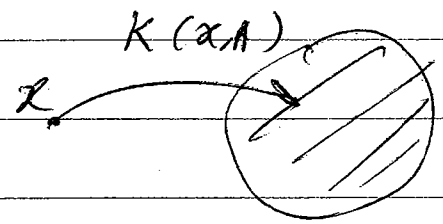
use M-H!

P. Diaconis 2009 MCMC revolution

"Markov chain Monte Carlo Method"

"Modeling from inference of data"

Markov Chains



$A \subset \mathbb{R}^d$

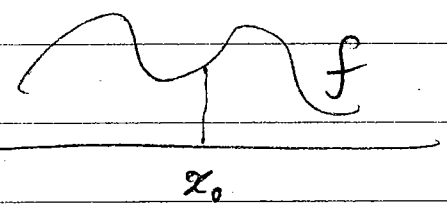
- $K(x, \mathbb{R}^d) = 1$
- $K(x, \{x\})$ not necessarily zero

Transitional Kernel

$$K(x,A) = P(x \to A) = P(A \mid x)$$

Kernel not PDF.

$$K(x \to x) \equiv P(x \mid x) \neq 0.$$

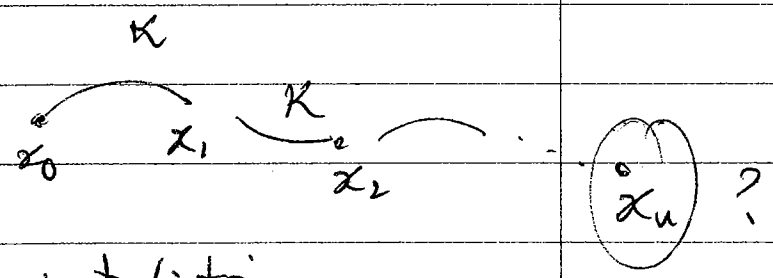


$x \sim f \quad P(x = x_0) = 0.$

Markov chain is completely defined by the initial state & conditional kernel.

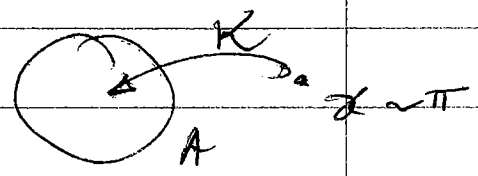
Markov chain: $x_0 K$

To find the distribution of x_n when $n \to \infty$



Invariant distri

$\pi(A) = \int K(x, A)\, \pi(x)\, dx$, for all measurable sets A.



the distri of input at this time
= the distri of output at that time

$$x_n \sim \pi \Rightarrow x_{n+1} \sim \pi$$

<u>The central result:</u>
(Nummelin 1984)

<u>K satisfies certain "ergodic conditions".</u>

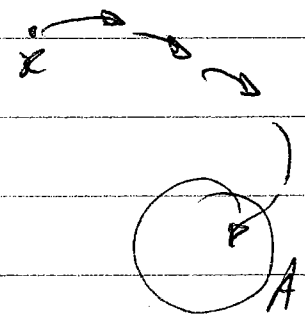
- π is the unique invariant distribution
- $x_n \sim \pi$ when $n \to \pi$

1) Irreducibility

K is irr if

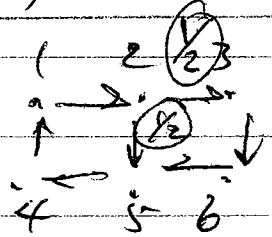
$\forall x, \forall A$ s.t. $\pi(A) > 0$ $(\neq 0)$

$p(x \to \to \to A) \neq 0$



2) Aperiodicity (there is no any order, any period).

e.g. a periodic Markov chain



$x_0 = a$,
$x_1 \neq a$, $\underline{4, 6, 8, 10 \ldots}$
$x_2 \neq a$,

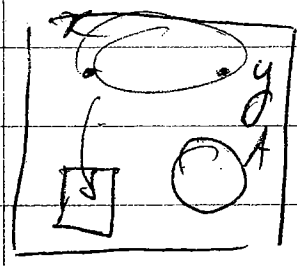
Lemma:

Let K be irreducible, and

$\pi(\{x : K(x|x) \neq 0\}) \neq 0$ ← intuitive proof

$\Downarrow$ $\not\Uparrow$

a periodic



e.g. $K(x, A) = I_x(A)$

$\cdot\ x_n = x_{n+1}$ ← state indicator fcn.

$\int K(x, A)\, \pi(x)\, dx$

$= \int_A \pi(x)\, dx = \pi(A)$ not periodic ???

MCMC → Given π to find K

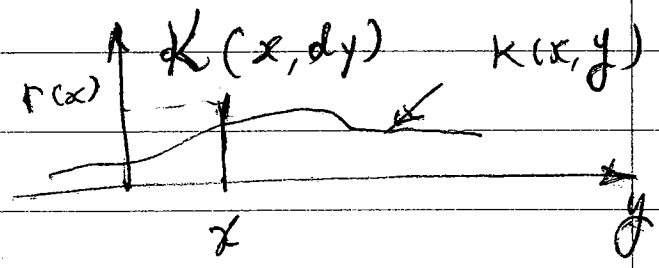
The invariant distribution π is given.

It is the target distribution.

To find an appropriate transition kernel K.

MC theory

Given K to find π.



$\cdot\ K(x, dy) = k(x, y)\, dy + r(x)\, \delta_x(dy)$

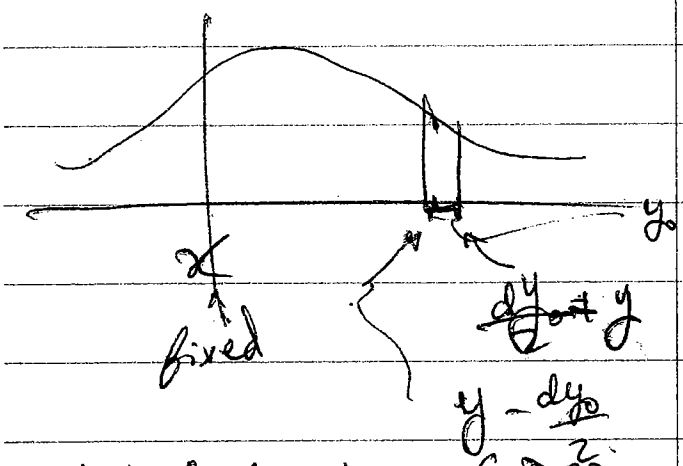
$\cdot\ r(x) = 1 - \int k(x, y)\, dy$

probably to remain in x

How to understand

$$K(x,dy) = k(x,y)\,dy + r(x)\,\delta_x(dy)$$

$$K(x,dy) = \int_A K(x,dy)$$



$$P(x|x) = r(x)$$

$$P\left(y_0 - \frac{dy}{2} < y < y_0 + \frac{dy}{2} \,\middle|\, x\right)$$

$$= k(x, y_0)\,dy$$

<u>detailed balance.</u> (DB)

$$\pi(x)\,k(x,y) = \pi(y)\,k(y,x)$$

π is the invariant distribution for K.

How to construct a new MCMC?

(1) $k \to$ DB.

(2) $r(x) = 1 - \int K(x,y)\,dx$.

(3)* Check that K is ergodic.

$\exists\, k$ s.t. π is its inv distr, however, k is not a soln of DB.

<u>Th</u> $\pi(x)k(x,y) = \pi(y)k(y,x) \Rightarrow$

$$\pi(A) = \int_A K(x,dy)\,\pi(x)\,dx$$

Proof. $\longrightarrow$ use $K(x,A) = \int_A k(x,dy)$

$$= \underbrace{\int\int_A k(x,y)\,dy\,\pi(x)\,dx}_{} + \underbrace{\int\int_A r(x)\,\delta_x(dy)\,\pi(x)\,dx}_{= \int_A r(x)\pi(x)\,dx}$$

$$= \iint_A k(x,y)\,\pi(x)\,dx\,dy$$

$$= \iint_A k(y,x)\,\pi(y)\,dx\,dy$$

$$= \underbrace{\int_A \underbrace{\int k(y,x)\,dx}_{1-r(y)}\,\pi(y)\,dy}_{= \int_A \pi(y)\,dy}$$

Metropolis-Hastings Method

is nothing but a particular case (soln) to detailed balance equation.

$$k_{MH}(x,y) = S(y|x)\,a(x,y)$$

- $S(\cdot|x)$ proposal distr.
- $a(x,y)$ acceptance probability

$$a(x,y) = \min\left\{1, \frac{\pi(y)\,S(x|y)}{\pi(x)\,S(y|x)}\right\}$$

$$\pi(x)\, k_{MH}(x,y) = \pi(x)\, S(y|x) \min\left\{1, \frac{\pi(y)\, S(x|y)}{\pi(x)\, S(y|x)}\right\}$$

$$= \frac{\text{Simple fact}}{a \min \left\{1, \frac{b}{a}\right\} = b \min\left\{1, \frac{a}{b}\right\}}$$

$$\forall\, a, b > 0.$$

- N. Metropolis et al. (1953)
  - # statistical mechanics
  - # Boltzman distribution
  - # $S(x|y) = S(y|x)$
  - # Absence of Markov chains.

$$X_n \xrightarrow{MH} X_{n+1}$$

$$S(\cdot|x_n)$$

$$\downarrow$$

$$y$$

$$\xrightarrow{a} X_{n+1} = y \qquad \xrightarrow{1-a} X_{n+1} = x_n$$

- W.K. Hastings (1970)
  - # Generalization
  - # Markov chain
  - # non symmetric proposals
  - # Different acceptance prob.

1) $a(x,y)$ — Don't need to know the normalizing constant of π.

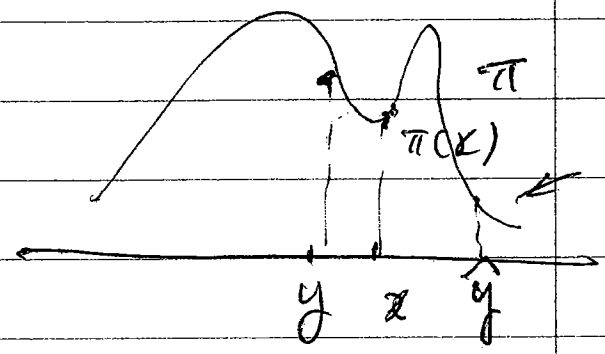
$$\pi(x) = \frac{p(x)}{Z}, \quad Z = \int p(x)\,dx$$

$\leftarrow$ the normalization in Baye's theorem.

post $\propto$ pr. L.

2) $S(x|y) = S(y|x)$

$$a(x,y) = \min\left(1, \frac{\pi(y)}{\pi(x)}\right)$$



always accepted from x to y

$$a(x, \hat{y}) = \frac{\pi(\hat{y})}{\pi(x)}$$

Jim: It's important to get the 2nd peak.

3) Barker (1965)

$$a(x, y) = \frac{\pi(y)}{\pi(x) + \pi(y)}$$

quoted by Hastings (1970)

Talk about Hastings being a "strange guy".

Peskun '73

~~X~~ Liu's book on Monte Carlo Method!

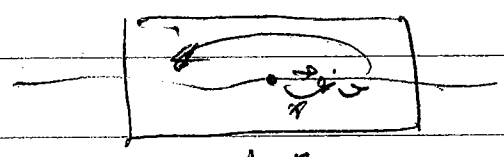
<u>Question</u>: Can I use any S or not?

<u>Answer</u>: No, almost any.

$\forall S$, π is invariant dist.

Check K is ergodic

"Take an S not too stupid."



~~bad~~ bad S

4) The choice of proposal PDF S,

Different ways.

Random Walk is a popular way.

$$\left.\begin{aligned} S(y|x) &= S_1(x-y) \\ y &= x_n + \xi, \quad \xi \sim S_1 \end{aligned}\right\} \text{random walk}$$

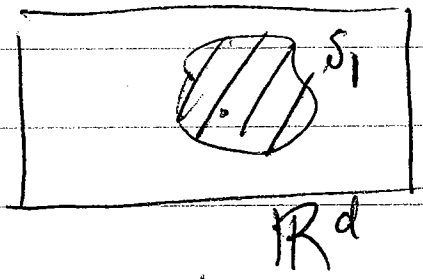
$S_2(y)$ "independence walk".

not whole space

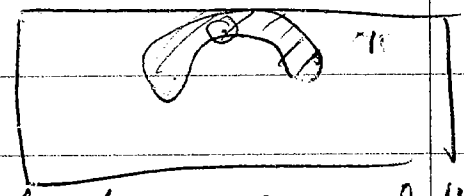
Th (~~Neumann~~ Nummelin, 84) ???

1) If $\operatorname{supp} S_1 = \mathbb{R}^d \Rightarrow K$ is ergodic.

2) If $\operatorname{supp} S_1 \neq \mathbb{R}^d$, but $\exists\, u(0) \subset \operatorname{supp} S_1$



then ergodic $\Longleftrightarrow$ supp T is connected and open.



useful for Random Walk.