\[ Z(t+1) = H(Z(t), x(t), t) \]
\[ F = \{ x : \max_{t \in [1,T]} G(z(t)) \geq b^* \} \]

Dynamic reliability problem

\[ \int_{\mathbb{R}^T} \pi(x) f(x) dx \]

To estimate \[ P_F = \int_{\mathbb{R}^T} \pi(x) f(x) dx \]

We can calculate \[ F(x) \] but very expensive

\[ N = nT \text{ very large. } N \sim 10^{60} \Rightarrow \text{not suitable} \]

\[ P_F = \int_F \pi(x) dx \text{ very small } \sim 10^{-3} - 10^{-6} \]

Monte Carlo very suitable


How to sample from \[ \pi^*(\cdot | F_K) ? \]

\[ \pi^* \sim \pi_0 \]
M. Diaconis 2007 MCMC revolution
"Markov chain Monte Carlo Method"

"Modeling from inference of data"

Markov chains

\[ K(x) : \mathcal{X} \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d \quad \text{satisfies} \]

- \( K(x) = 1 \)
- \( K(x, x) = 0 \) not necessarily zero

Transitional Kernel

\[ K(x, A) = \mathbb{P}(x \rightarrow A) = \mathbb{P}(A \mid x) \]

Kernel not PDF:

\[ K(x, x) = \mathbb{P}(x \mid x) = 0 \]

Markov chain is completely defined by the initial state & conditional kernel.

Markov chain: \( x_0, K \)

To find the distribution of \( x_n \) when \( n \to \infty \)

\[ x_0 \xrightarrow{K} x_1 \xrightarrow{K} x_2 \xrightarrow{K} \cdots \]

Invariant distr.
\[ \pi(A) = \int K(x, A) \pi(x) dx, \text{ for all measurable sets } A. \]

The distribution of input at this time is the distribution of output at that time.

\[ x_n \sim \pi \Rightarrow x_{n+1} \sim \pi \]

The central result:
(Nurmelin 1984)

\( K \) satisfies certain ergodic conditions.

- \( \pi \) is the unique invariant distribution
- \( x_n \sim \pi \) when \( n \to \infty \)

1) Irreducibility

\( K \) is irreducible if

\[ \forall x, \forall A \text{ s.t. } \pi(A) > 0 \]

\[ \exists x_0 \text{ s.t. } P(x \to A) > 0 \]

2) Aperiodicity (there is no any order, any period)

E.g. a periodic Markov chain

\[ x_0 = a_1, \ldots, 4, 6, 8, 10 \]

\[ x_2 \to a_1, \quad 4, 6, 8, 10 \quad \text{etc.} \]
Lemma:
Let $K$ be irreducible, and
\[ \pi \left( \{ x : K(x|x) = 0 \} \right) = 0 \]
\[ \Rightarrow \text{intuitive ergodic} \]
a periodic

\[ \text{E.g. } K(x,A) = I_x(A) \]
\[ \Rightarrow \pi_n = \pi_{n+1} \quad \text{state indicator fcn.} \]
\[ \int K(x,A) \pi(A) \, dx \]
\[ = \int \pi(x) \, dx = \pi(A) \quad \text{not periodic} \]

\[ \text{MCMC \quad (Given } \pi \text{ to find } K \]

The invariant distribution $\pi$ is given.
It is the target distribution.
To find an appropriate transition kernel $K$

\[ \text{MCMC} \quad ( \text{Given } K \text{ to find } \pi ) \]
\[ K(x,dy) = K(x,y) \, dy \]
\[ r(x) = \begin{cases} K(x,y) & \text{if } x \neq y \\ 1 - \int K(x,y) \, dy & \text{if } x = y \end{cases} \]
probably to remain in $\mathbb{X}$
How to understand

\[ K(x, dy) = h(x, y) \, dy + r(x) \delta_x(dy) \]

\[ \pi(x, dy) = \int_A \pi(x, dy) \]

\[ P(x | x) = r(x) \]

\[ P(y_0 - \frac{dy}{2} < y < y_0 + \frac{dy}{2} | x) = \int \pi(x, y_0) \, dy \]

\[ \pi(x) k(x, y) = \pi(y) k(y, x) \]

It is the invariant distribution for \( K \).

How to construct a new MCMC?

1. \( \pi \to \pi^* \)
2. \( r(x) = 1 - \int K(x, y) \, dy \)
3. *Check that \( K \) is ergodic.

If \( \pi \) is *T*, \( \pi^* \) is its inv distr, however, \( K \) is not a soln of DB.

\[ \pi(x) k(x, y) = \pi(y) k(y, x) \Rightarrow \]

\[ \pi(A) = \int_A \pi(x) k(x, y) \, dy \]
Proof. \[ K(x, y) = \int_A k(x, dy) \]
\[ = \int \int k(x, y) \pi(x) dx \, dy + \int_A \pi(x) \int_{dy} k(x, dy) \pi(x) dx \]
\[ = \int \int k(x, y) \pi(x) dx \, dy \]
\[ = \int \int k(y, x) \pi(y) dx \, dy \]
\[ = \int A \int k(y, x) dx \, \pi(y) dy \]
\[ = \int \pi(y) dy \]

Metropolis-Hasting Method

Is nothing but a particular case of the detailed balance equation.

\[ K_{\text{MH}}(x, y) = S(y|x) \alpha(x, y) \]

\( S(y|x) \) proposal distr.

\( \alpha(x, y) \) acceptance probability.

\[ \alpha(x, y) = \min \left\{ 1, \frac{\pi(y) S(y|x)}{\pi(x) S(x|y)} \right\} \]
\[ \pi(x) \propto_{\text{MH}} (x ; y) = \pi(x) \cdot s(y | x) \cdot \min \left\{ 1, \frac{\pi(y) \cdot s(y | x)}{\pi(x) \cdot s(y | x)} \right\} \]

- Simple fact
  
  \[ a \cdot \min \left\{ 1, \frac{b}{a} \right\} = b \cdot \min \left\{ 1, \frac{a}{b} \right\} \]

- N. Metropolis et al (1953)
  
  # Statistical mechanics
  # Boltzmann distribution
  # \[ s(x | y) = s(y | x) \]
  # Absence of Markov chains

- W. K. Hastings (1970)
  
  # Generalization
  # Markov chain
  # Non-symmetric proposals
  # Different acceptance prob.

1) \[ a(x, y) \quad \text{Don't need to know } \pi \]

\[ \pi(x) = \frac{p(x)}{Z}, \quad Z = \int p(x) \, dx \]

\[ \text{Posterior distribution} \]

2) \[ s(x | y) = s(y | x) \]

\[ a(x, y) = \min \left( 1, \frac{\pi(y)}{\pi(x)} \right) \]
always accepted from x to y

\[ a(x, y) = \frac{\pi(y)}{\pi(x)} \]

Jim: Its important to get the 2nd peak.

Barker (1965)

\[ a(x, y) = \frac{\pi(y)}{\pi(x) + \pi(y)} \]

quoted by Hastings (1970)

Talk about Hastings being a "sloane guy".

Peckun '73

Dr. Liu's book on Monte Carlo Method?

Question: Can I use any S or not?

Answer: No, almost any.

\[ \forall S, \pi \text{ is invariant hist.} \]

Check K is ergodic.

"Take jan S not too stupid."

bad s
4) The choice of proposal PDF $S_y$

Different ways

Random Walk is a popular way,

$S_y(x|y) = S_1(x-y)$

$Y_t = x_t + \xi_t$, $\xi_t \sim S_1$

$\#S_2(y)$ "independent walk"

Namely, $\text{Th. (Deborah, 84)}$??

1) If $\text{supp } S_1 = \mathbb{R}^d \Rightarrow K$ is ergodic.

2) If $\text{supp } S_1 \neq \mathbb{R}^d$, but $\exists u(0) \subset \text{supp } S_1$

Then ergodic $\iff \text{supp } T$

is connected and open.

Useful for Random Walk.