Risk Estimation and Uncertainty Quantification by Markov Chain Monte Carlo Methods

Konstantin Zuev

Institute for Risk and Uncertainty
University of Liverpool

http://www.liv.ac.uk/risk-and-uncertainty/staff/k-zuev/

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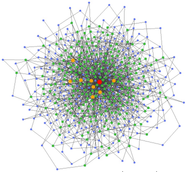
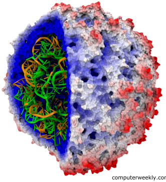
Workshop on Risk and Uncertainty

MCMC Revolution

P. Diaconis (2009), "The Markov chain Monte Carlo revolution":

> *...asking about applications of Markov chain Monte Carlo (MCMC) is a little like asking about applications of the quadratic formula... you can take any area of science, from hard to social, and find a burgeoning MCMC literature specifically tailored to that area.*





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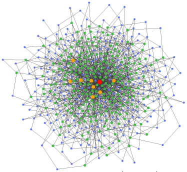
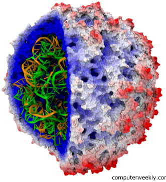


slowmuse.wordpress.com



computerweekly.com

Main message: MCMC algorithms can be efficiently used for solving engineering problems involving risk and uncertainty

Outline

1. What problems is MCMC meant to solve?
2. Why is MCMC useful in Engineering?
3. How does MCMC work?

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Purpose of MCMC

Problem: To estimate

$$I = \int_{\Theta} h(\theta)\pi(\theta)d\theta$$

- $\Theta \subseteq \mathbb{R}^d$ parameter space
- $h : \Theta \to \mathbb{R}$ function of interest
- $\pi(\theta)$ "target" PDF on Θ

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- d is small $(d = 1, 2, 3) \Rightarrow$ numerical integration
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Solution: Use an appropriate MCMC method

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1. What problems is MCMC meant to solve?
2. Why is MCMC useful in Engineering?
   - Bayesian Inference
   - Performance-Based Design Optimization
   - Reliability Problem
3. How does MCMC work?

Bayesian Inference

- $\mathcal{M}$ the assumed model class for the target dynamic system:
  - set of I/O probability models $p(y|\theta, u)$ $\quad \xrightarrow{u}$ $\boxed{\text{System}}$ $\xrightarrow{y}$
  - $\theta \in \Theta$ the uncertain model parameters
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Bayesian approach:

- Update $\pi_0(\theta)$ to posterior PDF $\pi(\theta|\mathcal{D})$ via Bayes' theorem:

$$\pi(\theta|\mathcal{D}) = \frac{L(\mathcal{D}|\theta)\pi_0(\theta)}{\mathcal{Z}}$$

- $\mathcal{D}$ the measured data from the system
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Problems:

- Posterior expectations

$$\mathbb{E}_\pi[h] = \int_\Theta h(\theta)\pi(\theta|\mathcal{D})d\theta$$

- Evidence

$$\mathcal{Z} = \int_\Theta L(\mathcal{D}|\theta)\pi_0(\theta)d\theta$$

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- The objective function for a robust-to-uncertainties design:

$$\mathbb{E}_\pi[h(\varphi, \theta)] = \int_\Theta h(\varphi, \theta)\pi(\theta|\varphi)d\theta$$

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Problem:

$$\varphi^* = \arg\min_{\varphi\in\Phi} \mathbb{E}_\pi[h(\varphi,\theta)]$$

Reliability Problem

Reliability Problem: To estimate the probability of failure p_F

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Notation:

- $\theta \in \mathbb{R}^d$ represents the uncertain input load
  - θ is a random vector and has joint PDF π
- $F \subset \mathbb{R}^d$ a failure domain (unacceptable performance)

$$F = \{\theta : g(\theta) \geq b^*\}$$

- $g(\theta)$ a performance function
- b^* a critical threshold for performance
- $I_F(\theta) = 1$ if $\theta \in F$ and $I_F(\theta) = 0$ if $\theta \notin F$

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MCMC: The Main Idea

- Monte Carlo method

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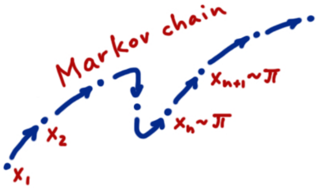
- MCMC samples from π and computes integrals using Markov chains:



$$\int_\Theta h(\theta)\pi(\theta)d\theta \approx \frac{1}{N-N_0}\sum_{i=N_0+1}^{N} h(x_i)$$

Papers

Applications of MCMC to Engineering Problems:

- Bayesian Inference
  - J.L. Beck and K.M. Zuev (2013), "Asymptotically independent Markov sampling: a new Markov chain Monte Carlo scheme for Bayesian inference," *Int. J. for Uncertainty Quant.*, 3(5), 445-474.
- Performance-Based Design Optimization
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- Reliability Problem
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  - K.M. Zuev and L.S. Katafygiotis (2011), "Modified Metropolis-Hastings algorithm with delayed rejection," *Probabilistic Engineering Mechanics*, 26(3), 405-412.
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