# Modified Metropolis-Hastings algorithm with Delayed Rejection for High-Dimensional Reliability Analysis

Konstantin Zuev & Lambros Katafygiotis

Department of Civil Engineering
Hong Kong University of Science and Technology

zuev@ust.hk, lambros@ust.hk

23 June, 2009

# Outline

- Introduction
- Modifications of the Metropolis-Hastings algorithm
  - Standard Metropolis-Hastings algorithm
  - Modified Metropolis-Hastings algorithm
  - Metropolis-Hastings algorithm with Delayed Rejection
  - Modified Metropolis-Hastings algorithm with Delayed Rejection
- Example
- Summary

# Introduction

Problem: To find the probability of failure $p_F$

$$p_F = P(x \in F) = \int\limits_{\mathbb{R}^N} I_F(x)\pi(x)dx$$

# Introduction

Problem: To find the probability of failure $p_F$

$$p_F = P(x \in F) = \int_{\mathbb{R}^N} I_F(x)\pi(x)dx$$

Notation

- $x \in \mathbb{R}^N$ is a random vector
- $\pi$ is the joint PDF of $x$
- $F \subset \mathbb{R}^N$ is a failure domain
- $I_F(x) = \begin{cases} 1, & x \in F; \\ 0, & x \notin F. \end{cases}$

# Introduction

Problem: To find the probability of failure $p_F$

$$p_F = P(x \in F) = \int\limits_{\mathbb{R}^N} I_F(x)\pi(x)dx$$

Notation

- $x \in \mathbb{R}^N$ is a random vector
- $\pi$ is the joint PDF of $x$
- $F \subset \mathbb{R}^N$ is a failure domain
- $I_F(x) = \begin{cases} 1, & x \in F; \\ 0, & x \notin F. \end{cases}$

Assumptions

- $N$ is very large, $N \sim 1000$
- $p_F$ is very small, $p_F \sim 10^{-3} - 10^{-6}$

# Introduction

Problem: To find the probability of failure $p_F$

$$p_F = P(x \in F) = \int_{\mathbb{R}^N} I_F(x)\pi(x)dx$$

Notation

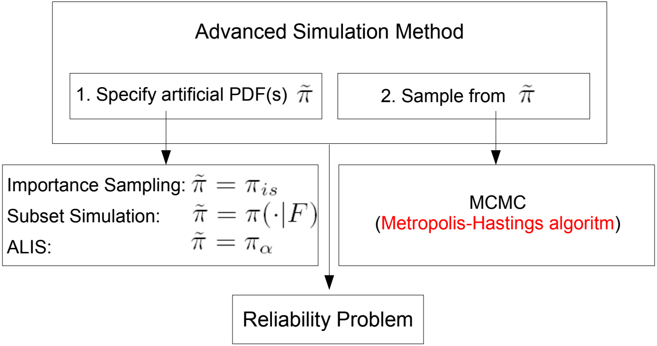
- $x \in \mathbb{R}^N$ is a random vector
- $\pi$ is the joint PDF of $x$
- $F \subset \mathbb{R}^N$ is a failure domain
- $I_F(x) = \begin{cases} 1, & x \in F; \\ 0, & x \notin F. \end{cases}$

Assumptions

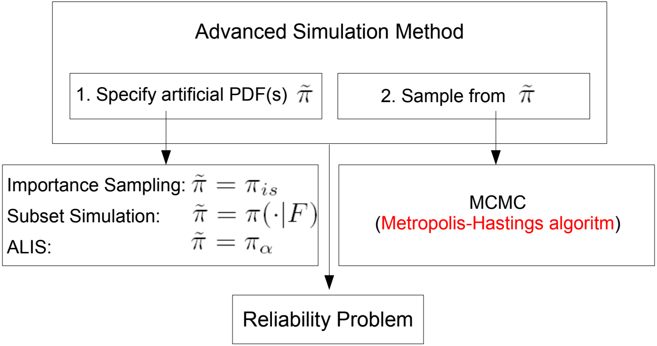
- $N$ is very large, $N \sim 1000$
- $p_F$ is very small, $p_F \sim 10^{-3} - 10^{-6}$

We need to use Advanced Simulation Methods

# Introduction

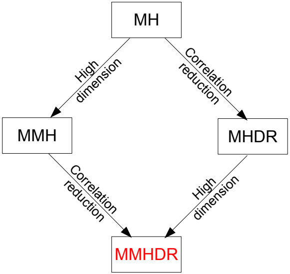


# Introduction

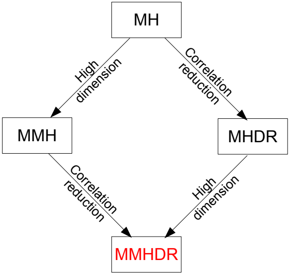



<u>Main goal</u>: to develop a novel effective MCMC algorithm for sampling from complex high-dimensional distributions

# Modifications of the Metropolis-Hastings algorithm

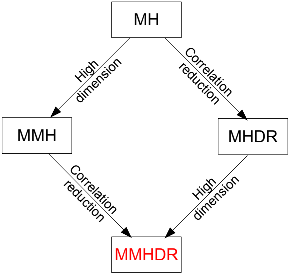


# Modifications of the Metropolis-Hastings algorithm



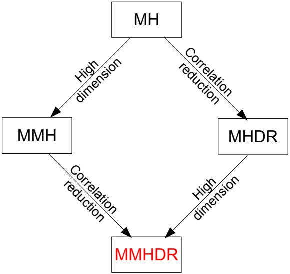

- Metropolis-Hastings (MH) algorithm, Metropolis et al. 1953, Hastings 1970

# Modifications of the Metropolis-Hastings algorithm



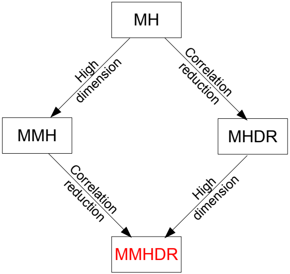

- Metropolis-Hastings (MH) algorithm, Metropolis et al. 1953, Hastings 1970
- Modified MH (MMH) algorithm, Au & Beck 2001

# Modifications of the Metropolis-Hastings algorithm




- Metropolis-Hastings (MH) algorithm, Metropolis et al. 1953, Hastings 1970
- Modified MH (MMH) algorithm, Au & Beck 2001
- MH algorithm with delayed rejection (MHDR), Tierney & Mira 1999

# Modifications of the Metropolis-Hastings algorithm



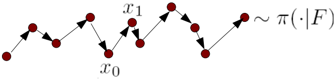

- Metropolis-Hastings (MH) algorithm, Metropolis et al. 1953, Hastings 1970
- Modified MH (MMH) algorithm, Au & Beck 2001
- MH algorithm with delayed rejection (MHDR), Tierney & Mira 1999
- Modified Metropolis-Hastings algorithm with delayed rejection (MMHDR)

# Standard Metropolis-Hastings algorithm

How to sample from $\pi(\cdot|F)$?

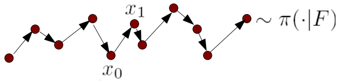
# Standard Metropolis-Hastings algorithm

How to sample from $\pi(\cdot|F)$?



# Standard Metropolis-Hastings algorithm

How to sample from $\pi(\cdot|F)$?

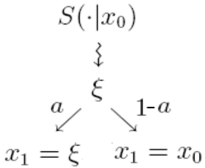



1. Simulate $\xi \sim S(\cdot|x_0)$
2. Compute the acceptance probability

$$a(x_0, \xi) = \min\left\{1, \frac{\pi(\xi)S(x_0|\xi)}{\pi(x_0)S(\xi|x_0)} I_F(\xi)\right\}$$

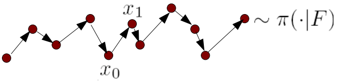
3. Accept/Reject $\xi$

$$x_1 = \begin{cases} \xi, & \text{with prob. } a(x_0, \xi); \\ x_0, & \text{with prob. } 1 - a(x_0, \xi). \end{cases}$$



# Standard Metropolis-Hastings algorithm

How to sample from $\pi(\cdot|F)$?

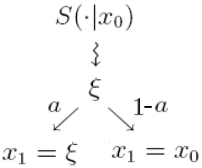



1. Simulate $\xi \sim S(\cdot|x_0)$
2. Compute the acceptance probability

$$a(x_0, \xi) = \min\left\{1, \frac{\pi(\xi)S(x_0|\xi)}{\pi(x_0)S(\xi|x_0)} I_F(\xi)\right\}$$

3. Accept/Reject $\xi$

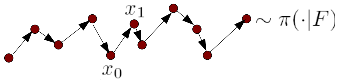
$$x_1 = \begin{cases} \xi, & \text{with prob. } a(x_0, \xi); \\ x_0, & \text{with prob. } 1 - a(x_0, \xi). \end{cases}$$




In high dimensions: $a(x_0, \xi) \approx 0 \quad \Rightarrow \quad x_1 = x_0$

# Standard Metropolis-Hastings algorithm

How to sample from $\pi(\cdot|F)$?

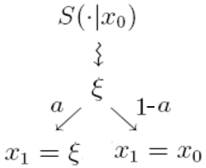



1. Simulate $\xi \sim S(\cdot|x_0)$
2. Compute the acceptance probability

$$a(x_0,\xi) = \min\left\{1, \frac{\pi(\xi)S(x_0|\xi)}{\pi(x_0)S(\xi|x_0)} I_F(\xi)\right\}$$

3. Accept/Reject $\xi$

$$x_1 = \begin{cases} \xi, & \text{with prob. } a(x_0,\xi); \\ x_0, & \text{with prob. } 1 - a(x_0,\xi). \end{cases}$$




In high dimensions: $a(x_0,\xi) \approx 0 \quad \Rightarrow \quad x_1 = x_0$

Reason: $S(\cdot|x_0)$ is $N$-dimensional PDF

# Modified Metropolis-Hastings algorithm

1. Generate candidate state $\xi$

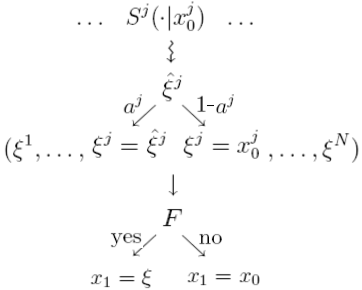
   For each $j = 1 \dots N$:

   1. Simulate $\hat{\xi}^j \sim S^j(\cdot|x_0^j)$
   2. Compute the acceptance probability

   $$a^j(x_0^j, \hat{\xi}^j) = \min\left\{1, \frac{\pi_j(\hat{\xi}^j) S^j(x_0^j|\hat{\xi}^j)}{\pi_j(x_0^j) S^j(\hat{\xi}^j|x_0^j)}\right\}$$

   3. Accept/Reject $\hat{\xi}^j$

   $$\xi^j = \begin{cases} \hat{\xi}^j, & \text{with prob. } a^j(x_0^j, \hat{\xi}^j); \\ x_0^j, & \text{with prob. } 1 - a^j(x_0^j, \hat{\xi}^j). \end{cases}$$

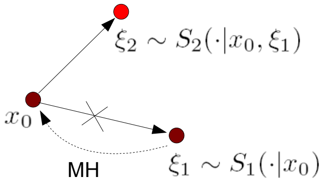



2. Accept/Reject $\xi$

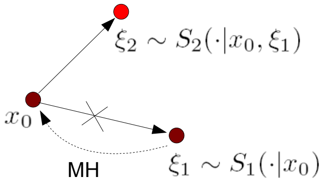
$$x_1 = \begin{cases} \xi, & \text{if } \xi \in F; \\ x_0, & \text{if } \xi \notin F. \end{cases}$$

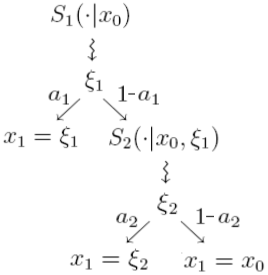
# Metropolis-Hastings algorithm with Delayed Rejection

# Metropolis-Hastings algorithm with Delayed Rejection

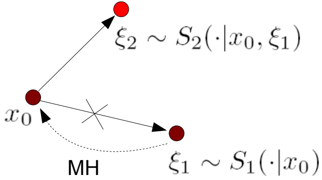


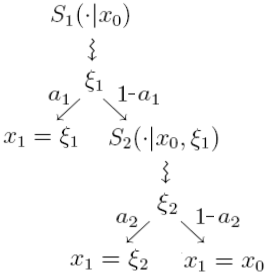
# Metropolis-Hastings algorithm with Delayed Rejection



# Metropolis-Hastings algorithm with Delayed Rejection

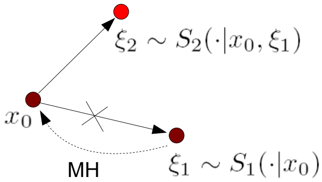


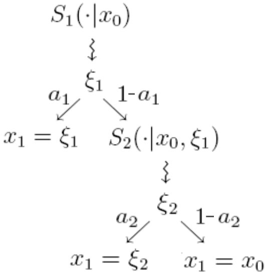




$$a_1(x_0, \xi_1) = \min\left\{1, \frac{\pi(\xi_1)S_1(x_0|\xi_1)}{\pi(x_0)S_1(\xi_1|x_0)} I_F(\xi_1)\right\}$$

$$a_2(x_0, \xi_1, \xi_2) = \min\left\{1, \frac{\pi(\xi_2)S_1(\xi_1|\xi_2)S_2(x_0|\xi_2, \xi_1)(1 - a_1(\xi_2, \xi_1))}{\pi(x_0)S_1(\xi_1|x_0)S_2(\xi_2|x_0, \xi_1)(1 - a_1(x_0, \xi_1))} I_F(\xi_2)\right\}$$

# Metropolis-Hastings algorithm with Delayed Rejection



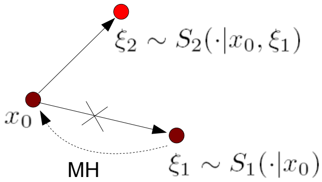



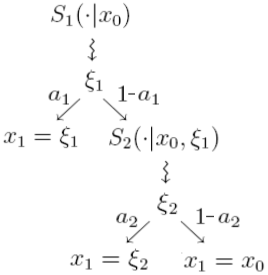

$$a_1(x_0, \xi_1) = \min\left\{1, \frac{\pi(\xi_1)S_1(x_0|\xi_1)}{\pi(x_0)S_1(\xi_1|x_0)} I_F(\xi_1)\right\}$$

$$a_2(x_0, \xi_1, \xi_2) = \min\left\{1, \frac{\pi(\xi_2)S_1(\xi_1|\xi_2)S_2(x_0|\xi_2, \xi_1)(1 - a_1(\xi_2, \xi_1))}{\pi(x_0)S_1(\xi_1|x_0)S_2(\xi_2|x_0, \xi_1)(1 - a_1(x_0, \xi_1))} I_F(\xi_2)\right\}$$

In high dimensions: $a_1(x_0, \xi_1) \approx a_2(x_0, \xi_1, \xi_2) \approx 0 \quad \Rightarrow \quad x_1 = x_0$

# Metropolis-Hastings algorithm with Delayed Rejection




$$a_1(x_0,\xi_1)=\min\left\{1,\frac{\pi(\xi_1)S_1(x_0|\xi_1)}{\pi(x_0)S_1(\xi_1|x_0)}I_F(\xi_1)\right\}$$

$$a_2(x_0,\xi_1,\xi_2)=\min\left\{1,\frac{\pi(\xi_2)S_1(\xi_1|\xi_2)S_2(x_0|\xi_2,\xi_1)(1-a_1(\xi_2,\xi_1))}{\pi(x_0)S_1(\xi_1|x_0)S_2(\xi_2|x_0,\xi_1)(1-a_1(x_0,\xi_1))}I_F(\xi_2)\right\}$$

In high dimensions: $a_1(x_0,\xi_1)\approx a_2(x_0,\xi_1,\xi_2)\approx 0 \quad\Rightarrow\quad x_1=x_0$

Reason: $S_1(\cdot|x_0)$ and $S_2(\cdot|x_0,\xi_1)$ are $N$-dimensional PDFs

# Modified MH algorithm with Delayed Rejection

$$x_0 \longrightarrow x_1$$

$$\cdots \quad S_1^j(\cdot|x_0^j) \quad \cdots$$

$$\downarrow$$

$$\hat{\xi}_1^j$$

$$a_1^j \swarrow \quad \searrow 1\text{-}a_1^j$$

$$\xi_1 = (\xi_1^1, \ldots, \xi_1^j = \hat{\xi}_1^j \quad \xi_1^j = x_0^j, \ldots, \xi_1^N)$$

$$\downarrow$$

$$x_1 = \xi_1 \xleftarrow{\text{yes}} F \xrightarrow{\text{no}}$$

$$\downarrow$$

$$\cdots \quad S_2^j(\cdot|x_0^j, \xi_1^j) \quad \cdots$$

$$\downarrow$$

$$\hat{\xi}_2^j$$

$$a_2^j \swarrow \quad \searrow 1\text{-}a_2^j$$

$$\xi_2 = (\xi_2^1, \ldots, \xi_2^j = \hat{\xi}_2^j \quad \xi_2^j = x_0^j, \ldots, \xi_2^N)$$

$$\downarrow$$

$$x_1 = \xi_2 \xleftarrow{\text{yes}} F \xrightarrow{\text{no}} x_1 = x_0$$

# Modified MH algorithm with Delayed Rejection

Features of the Algorithm

# Modified MH algorithm with Delayed Rejection

## Features of the Algorithm

- $S_2^j = S_2^j(\cdot|x_0^j, \xi_1^j)$

# Modified MH algorithm with Delayed Rejection

Features of the Algorithm

- $S_2^j = S_2^j(\cdot|x_0^j, \xi_1^j)$
- Samples generated by MMHDR are less correlated then samples generated by MMH.

# Modified MH algorithm with Delayed Rejection

Features of the Algorithm

- $S_2^j = S_2^j(\cdot|x_0^j, \xi_1^j)$
- Samples generated by MMHDR are less correlated then samples generated by MMH.
- MMHDR needs more computational effort than MMH for generating the same number of samples.

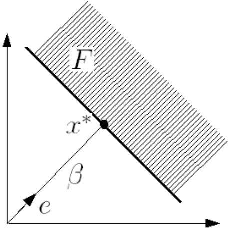
# Modified MH algorithm with Delayed Rejection

Features of the Algorithm

- $S_2^j = S_2^j(\cdot|x_0^j, \xi_1^j)$
- Samples generated by MMHDR are less correlated then samples generated by MMH.
- MMHDR needs more computational effort than MMH for generating the same number of samples.
- With fixed computational effort:
  - MMH: more Markov chains with more correlated states
  - MMHDR: fewer Markov chains with less correlated states

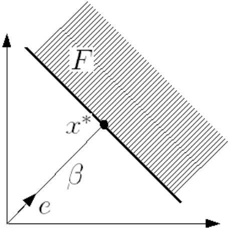
# Example

## Linear problem



# Example

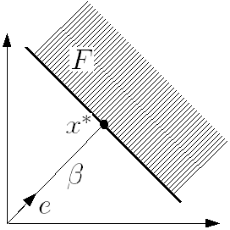
## Linear problem




- Geometry
  - $N = 1000$
  - $p_F = 10^{-5}$, $\beta = 4.265$
  - $x^* = e\beta$, $e \sim U(\mathbb{S}^{N-1})$

# Example

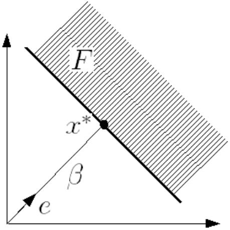
## Linear problem




- Geometry
  - $N = 1000$
  - $p_F = 10^{-5}$, $\beta = 4.265$
  - $x^* = e\beta$, $e \sim U(\mathbb{S}^{N-1})$
- Proposal PDFs
  - MMH: $S^j(\cdot|x_0^j) = \mathcal{N}(x_0^j, 1)$
  - MMHDR: $S^j_{1,2}(\cdot|x_0^j) = \mathcal{N}(x_0^j, 1)$

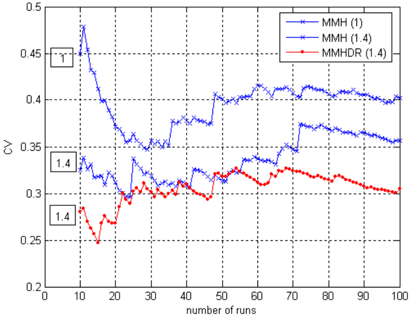
# Example

## Linear problem




- Geometry
  - $N = 1000$
  - $p_F = 10^{-5}$, $\beta = 4.265$
  - $x^* = e\beta$, $e \sim U(\mathbb{S}^{N-1})$
- Proposal PDFs
  - MMH: $S^j(\cdot|x_0^j) = \mathcal{N}(x_0^j, 1)$
  - MMHDR: $S^j_{1,2}(\cdot|x_0^j) = \mathcal{N}(x_0^j, 1)$

Subset Simulation



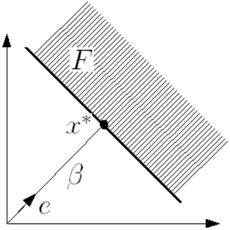

MMH(1): SS + MMH, $n = 10^3$

MMHDR(1.4): SS + MMHDR, $n = 10^3$

MMH(1.4): SS + MMH, $n = 1450$

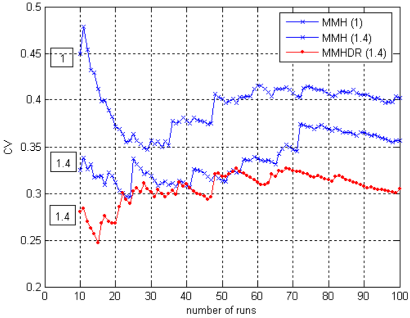
# Example

## Linear problem




- Geometry
  - $N = 1000$
  - $p_F = 10^{-5}$, $\beta = 4.265$
  - $x^* = e\beta$, $e \sim U(\mathbb{S}^{N-1})$
- Proposal PDFs
  - MMH: $S^j(\cdot|x_0^j) = \mathcal{N}(x_0^j, 1)$
  - MMHDR: $S^j_{1,2}(\cdot|x_0^j) = \mathcal{N}(x_0^j, 1)$

Subset Simulation




MMH(1): SS + MMH, $n = 10^3$

MMHDR(1.4): SS + MMHDR, $n = 10^3$

MMH(1.4): SS + MMH, $n = 1450$

Reduction in CV is 11%

# Summary

# Summary

- We propose a novel modification of the Metropolis-Hastings algorithm, called Modified Metropolis-Hastings algorithm with Delayed Rejection

# Summary

- We propose a novel modification of the Metropolis-Hastings algorithm, called Modified Metropolis-Hastings algorithm with Delayed Rejection

- MMHDR = MMH (Au & Beck 2001) + MHDR (Tierney & Mira 1999)

# Summary

- We propose a novel modification of the Metropolis-Hastings algorithm, called Modified Metropolis-Hastings algorithm with Delayed Rejection

- MMHDR = MMH (Au & Beck 2001) + MHDR (Tierney & Mira 1999)

- MMHDR is designed for sampling from high dimensional conditional PDFs

# Summary

- We propose a novel modification of the Metropolis-Hastings algorithm, called Modified Metropolis-Hastings algorithm with Delayed Rejection
- MMHDR = MMH (Au & Beck 2001) + MHDR (Tierney & Mira 1999)
- MMHDR is designed for sampling from high dimensional conditional PDFs
- The efficiency of the MMHDR is demonstrated with a numerical example

# Thank You for attention!