Geometric insight into the challenges of solving high-dimensional reliability problems

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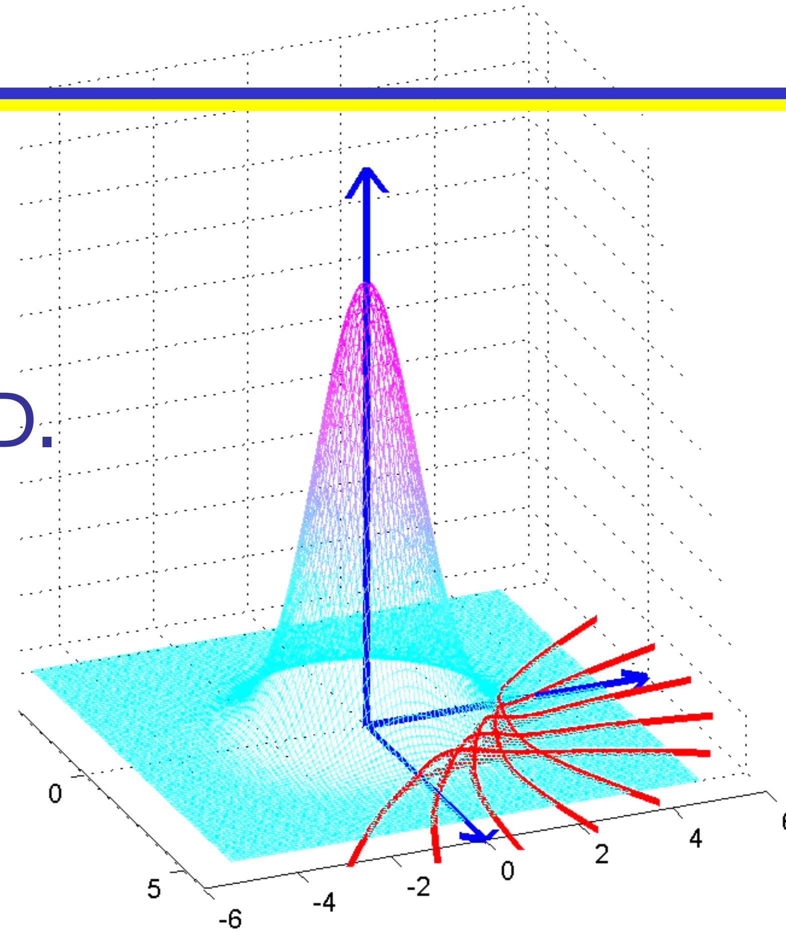


5th Computational Stochastic Mechanics Conference, Rodos, 21-23 June 2006

OUTLINE

- Geometric properties of high-dimensional Gaussian space
- Importance Sampling in high dimensions
- Design Points and nonlinear reliability problems
  - Duffing oscillator
  - Parabolic failure domain
- MCMC techniques in high dimensions
  - Original Metropolis Algorithm
  - Modified Metropolis Algorithm
  - MCMC of S^3

Geometric properties of high-dimensional Gaussian space

$$\theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N$$

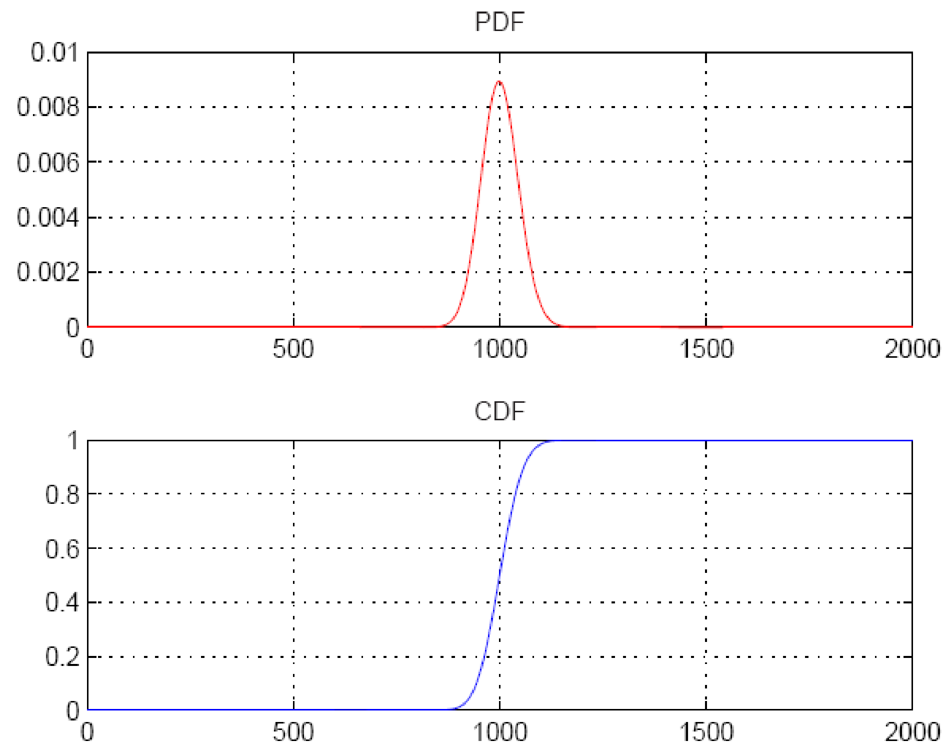
$$\theta_i \sim \mathcal{N}(0,1), \;\; i = 1, \ldots, N.$$

$$R^2 = \sum_{i=1}^{N} \theta_i^2 \sim \chi_N^2$$

$$R \overset{\text{app}}{\sim} \mathcal{N}\left(\sqrt{N}, 1/2\right), \;\; N \to \infty$$

<u>Important Ring</u>

$$\sqrt{N} - r < R < \sqrt{N} + r$$



PDF & CDF of chi-square distribution with N=1000

$$N = 10^3, \; r = 3.46 \;\; \Rightarrow \;\; P(\sqrt{N} - r < R < \sqrt{N} + r) > 1 - 10^{-6}$$

Geometric properties of high-dimensional Gaussian space

$$e = (1, 0, \ldots, 0)$$

$$\alpha = \widehat{\theta e}$$

$$f_\alpha(\alpha) = \frac{\sin^{N-2}\alpha}{\int_0^\pi \sin^{N-2}\alpha d\alpha}$$

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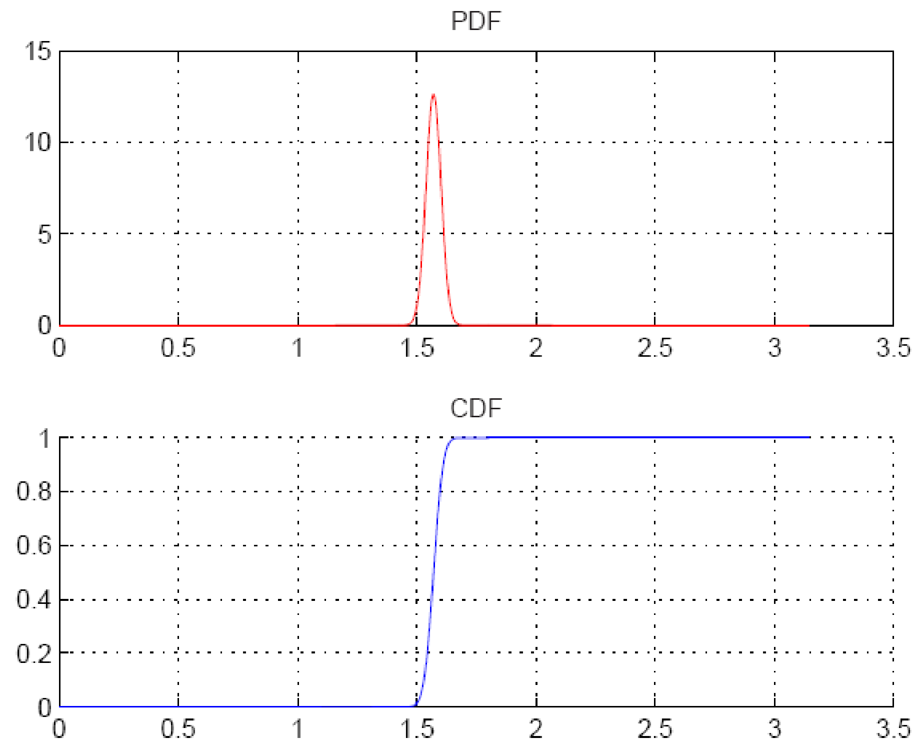


PDF & CDF of angle alpha, N=1000

If e is fixed direction and $\theta \sim \mathcal{N}(0,1) \Rightarrow \alpha \approx \pi/2$

Importance Sampling in high dimensions

Probability of failure: $p_F = \int_{\mathbb{R}^N} I_F(\theta) q(\theta) d\theta$

where

$F \subseteq \mathbb{R}^N$ is failure event

I_F is its indicator function

assumed Gaussian

θ is the random parameters with joint PDF q

$$p_F = \int_{\mathbb{R}^N} I_F(\theta) q(\theta) d\theta = \int_{\mathbb{R}^N} \frac{I_F(\theta) q(\theta)}{q_{is}(\theta)} q_{is}(\theta) d\theta = E_{q_{is}}\left[\frac{I_F q}{q_{is}}\right]$$

$$p_F^{is} = \frac{1}{n} \sum_{k=1}^{n} \frac{I_F(\theta^{(k)}) q(\theta^{(k)})}{q_{is}(\theta^{(k)})}, \qquad \theta^{(k)} \sim q_{is}(\cdot)$$

any PDF (ISD)

Importance Sampling in high dimensions

What does it mean: "good" ISD ?

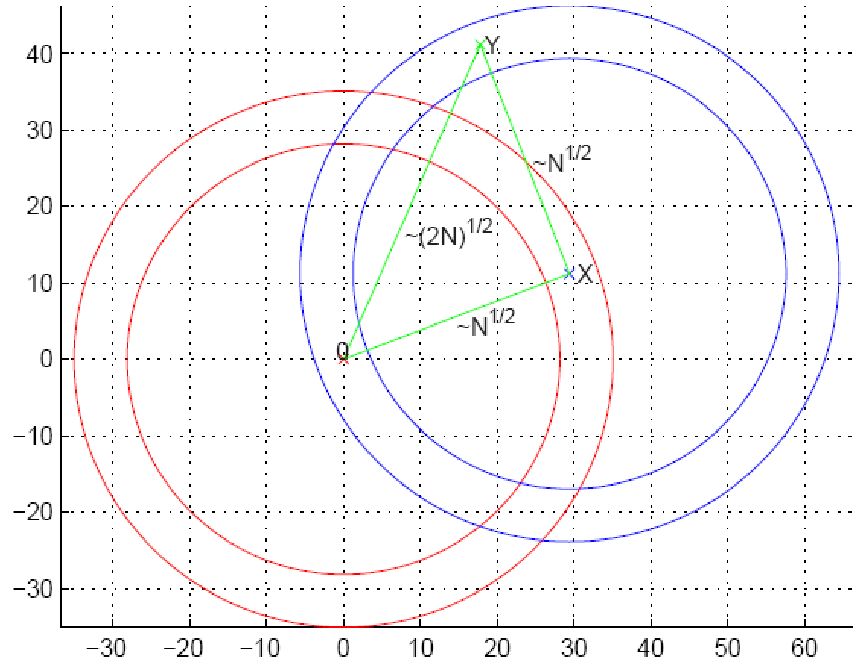
Example: Given $x \in F \quad \rightsquigarrow \quad q_{is}(\theta) = \frac{1}{(\sqrt{2\pi})^N} \exp\left(-\frac{\|\theta - x\|^2}{2}\right)$

If $y \sim q_{is}(\cdot)$

$\Downarrow$ (in high dimensions)

$$\|x\| \approx \sqrt{N} \qquad \|y\| \approx \sqrt{2N}$$

$$\|y - x\| \approx \sqrt{N}$$



$$p_F^{is} = \frac{1}{n} \sum_{k=1}^{n} \frac{I_F(y^{(k)}) q(y^{(k)})}{q_{is}(y^{(k)})}$$

$$\frac{q(y^{(k)})}{q_{is}(y^{(k)})} \approx e^{-N/2} \ll 1$$

$\Rightarrow$ UNDERESTIMATION

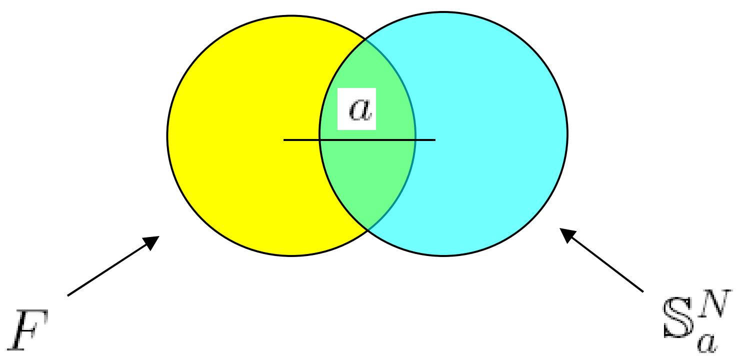
Importance Sampling in high dimensions

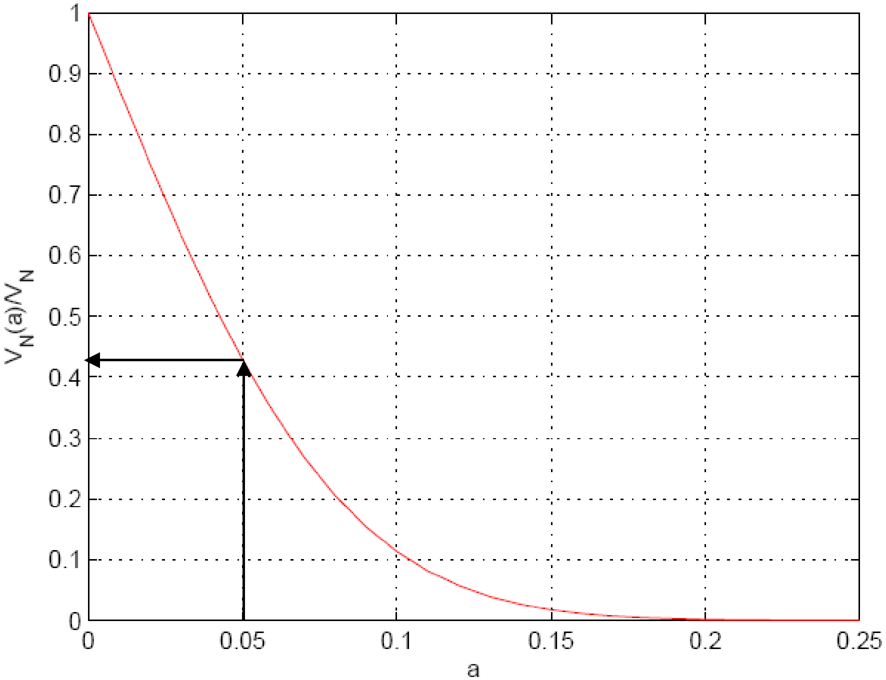
Example: $F = \mathbb{S}^N$

Question: How precisely should we find the center of $\mathbb{S}^N$ to cover the most part of it by another hypersphere of unit radius?

$$V_N = Vol(\mathbb{S}^N)$$

$$V_N(a) = Vol(\mathbb{S}^N \cap \mathbb{S}_a^N)$$





Dependence of ratio $V_N(a)/V_N$ on a, $N = 10^3$

Design points and nonlinear problems

Failure domain: $F = \{\theta : g(\theta) < 0\}$

Limit-state surface: $S_F = \{\theta : g(\theta) = 0\}$

Design point: $\theta^* = \arg\min_{\theta \in S_F} ||\theta||$

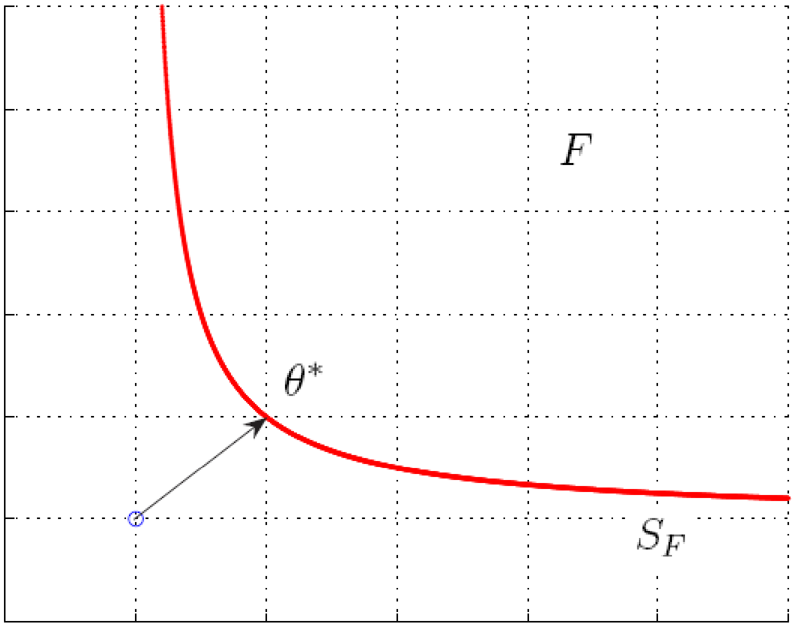
<u>Linear case</u>

$$g(\theta) = a^t\theta + b$$

$$\theta^* = -\frac{b}{||a||^2}a$$

$$\beta = ||\theta^*|| = \frac{b}{||a||}$$

$$p_F = 1 - \Phi(\beta)$$



Design points and nonlinear problems

Duffing oscillator

$$m\ddot{x}(t) + c\dot{x}(t) + k[x(t) + \gamma x(t)^3] = f(t)$$

Koo et al (2005)

$$\begin{cases} m = 1000 \text{ kg} \\ c = 200\pi \text{ Ns/m} \\ k = 1000(2\pi)^2 \text{ N/m} \\ \gamma = 1 \text{ m}^{-2} \end{cases}$$

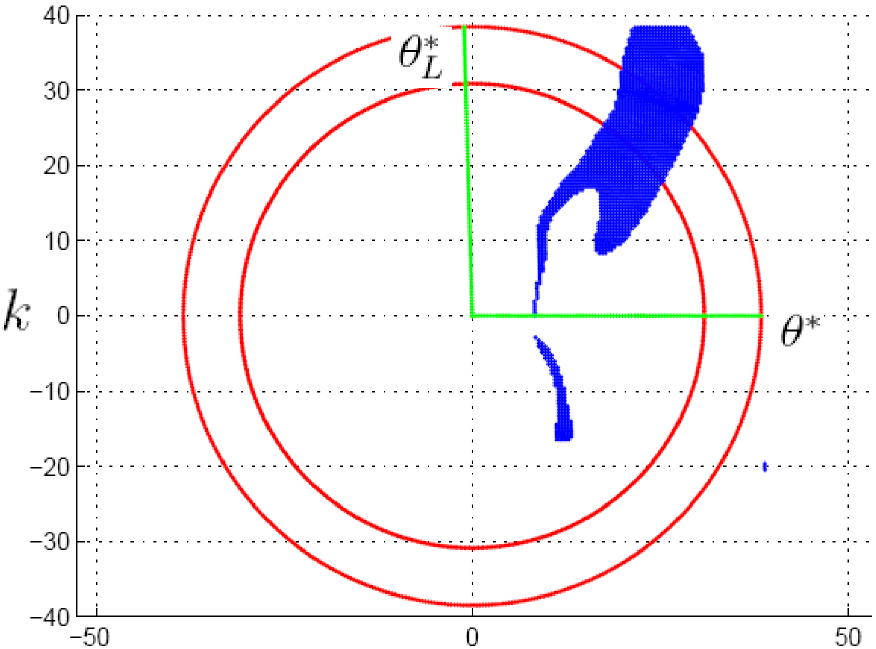
white noise

$$S_0 = 10^6 \text{ N}^2\text{s/rad}$$
$$f = (f_1, \ldots, f_N) = \sigma\theta$$
$$\sigma = \sqrt{2\pi S_0/\Delta t}$$
$$\Delta t = 0.01$$

$$\boxed{F_\tau = \{\theta : x(\tau) > x_0\}}$$

$$\tau = 12 \text{ s}$$
$$x_0 = 5\sigma_0$$
$$\sigma_0^2 = \pi S_0/ck$$

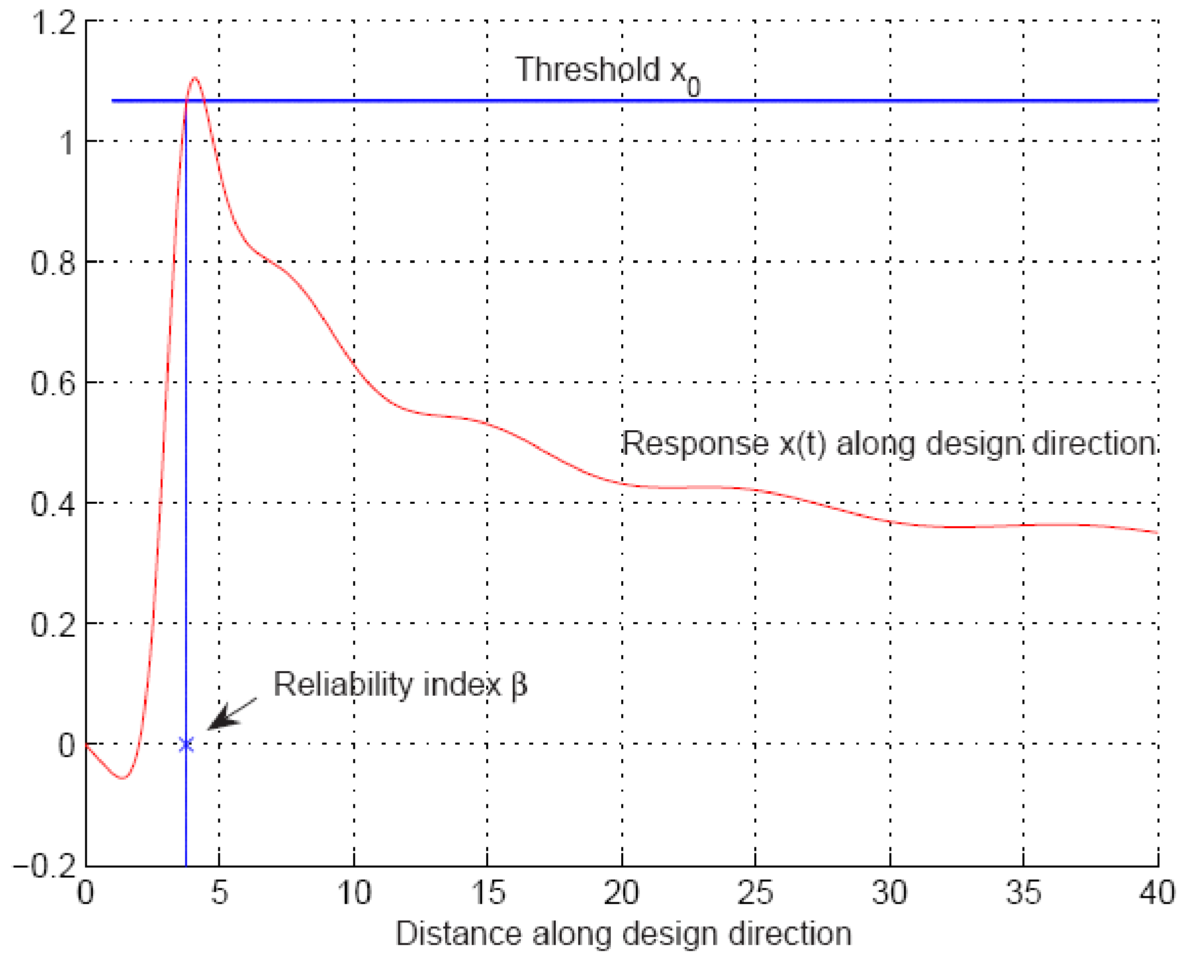
$$F_\tau \cap \pi(\theta^*, \theta_L^*)$$



Design points and nonlinear problems

$$\boxed{f_{\theta^*} = \sigma(s\theta^*)}$$

excitation along design point direction



Design points and nonlinear problems

In the first passage problem:

$$\left.\begin{aligned} F &= \cup F_i \\ F_i &= \{\theta : x(t_i) > x_0\} \\ \theta_i^* &\text{ is design point for } F_i \\ i &= 0, \ldots, N = \frac{\tau}{\Delta t} \end{aligned}\right\} \quad \rightsquigarrow \quad q_{is}(\theta) = \sum_{i=1}^{N} w_i \mathcal{N}_{(0,1)}(\theta - \theta_i^*)$$

Observation: when failure domain is strongly nonlinear
Importance Sampling with this natural ISD
can be feeble

Design points and nonlinear problems

Nonlinear failure domain of parabolic shape

$$F = \left\{ x \in \mathbb{R}^N : x_1 > a \sum_{i=2}^{N} x_i^2 - b \right\} \qquad \begin{aligned} a &= 0.025 \\ b &= 20.27 \end{aligned} \qquad N = 1000$$

Monte Carlo

$$p_F = 0.00142$$
$$\delta = 0.24$$

Importance Sampling

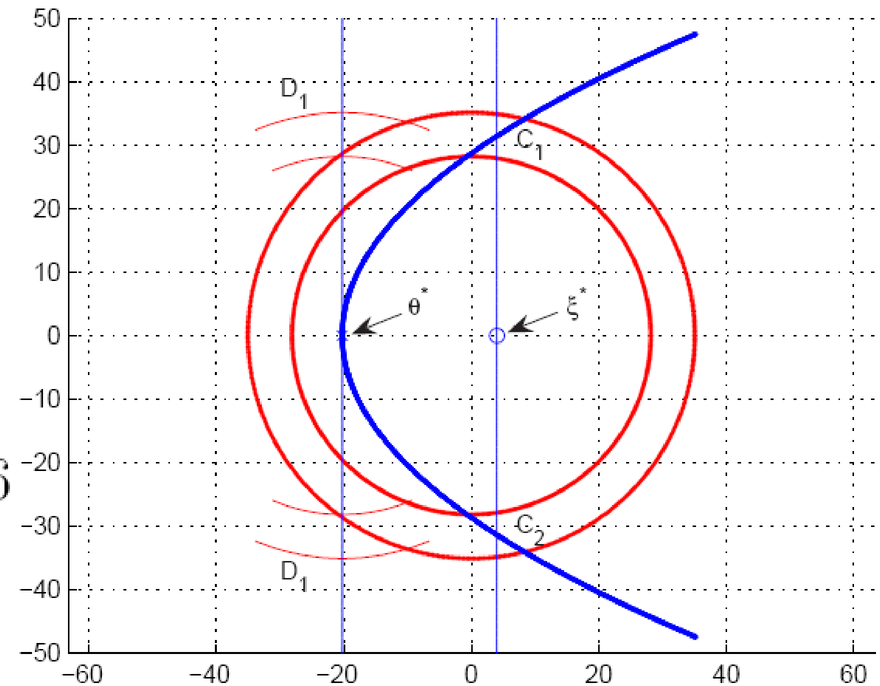
$$\mathcal{N}(\theta^*, 1)$$

$$p_F^{is} = 0$$

Importance Sampling

$$\mathcal{N}\left(\xi^*, \frac{N - (\xi^*)^2}{N}\right)$$

$$p_F^{is} = 0.00146$$
$$\delta = 0.32$$



MCMC techniques in high dimensions

Original Metropolis Algorithm (OMA)

$\theta^{(k)} \rightsquigarrow \theta^{(k+1)}$

N-dimensional proposal PDF

1. *Generate candidate state* $\tilde{\theta}$

Simulate ξ according to $p(\cdot|\theta)$

Compute the ratio $r = q(\xi)/q(\theta^{(k)})$

$$\text{Set } \tilde{\theta} = \begin{cases} \xi & \text{with } \min\{1, r\} \\ \theta^{(k)} & \text{with } 1 - \min\{1, r\} \end{cases}$$

2. *Accept/Reject* $\tilde{\theta}$

$$\theta^{(k+1)} = \begin{cases} \tilde{\theta} & \text{if } \tilde{\theta} \in F_i \\ \theta^{(k)} & \text{otherwise} \end{cases}$$

Au and Beck (2001)

SS with MMA does not work in high dimensions

Geometric reason

$r = q(\xi)/q(\theta^{(k)}) \ll 1$

MCMC techniques in high dimensions

Modified Metropolis Algorithm (MMA) Au & Beck (2001)

$\boxed{1.\ \textit{Generate candidate state } \tilde{\theta}}$

1-dimensional
proposal PDF

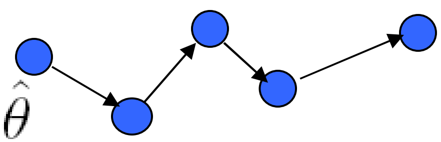
For each component simulate $\xi_j \sim p(\cdot|\theta_j)$

Compute the ratio $r_j = q_j(\xi_j)/q_j(\theta_j)$

$$\text{Set } \tilde{\theta}_j = \begin{cases} \xi_j & \text{with } \min\{1, r_j\} \\ \theta_j & \text{with } 1 - \min\{1, r_j\} \end{cases}$$

Example

$\boxed{\text{"Free"}}$ Markov chain $\hat{\theta}$



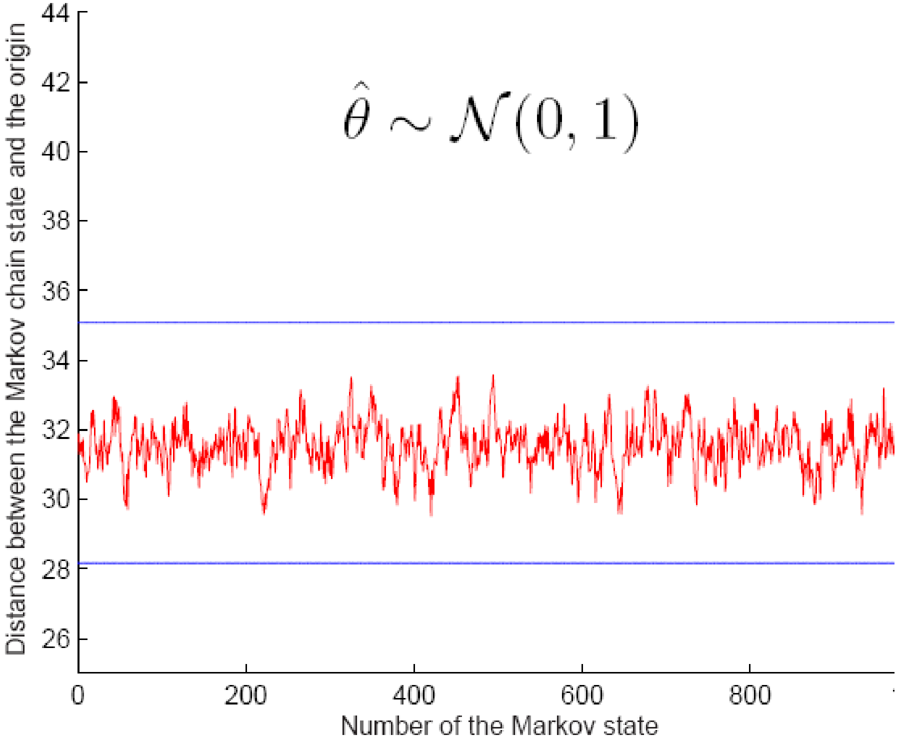
$p(\cdot|\theta_j) = \mathcal{N}(\theta_j, 1)$

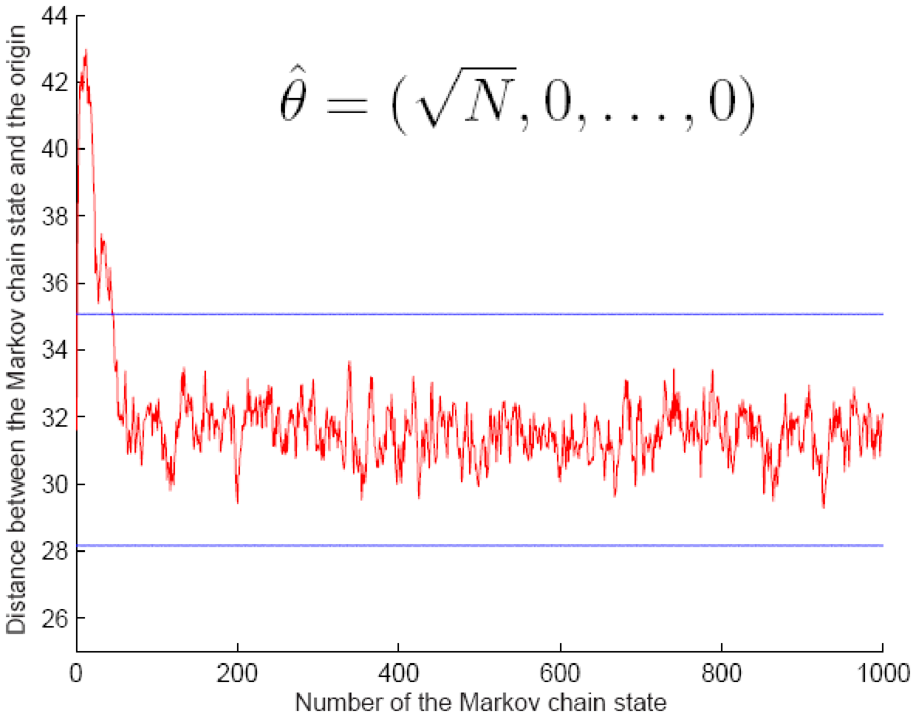
$\boxed{\theta^{(k+1)} = \tilde{\theta}}$

$\hat{\theta} \sim \mathcal{N}(0, 1)$

$\hat{\theta} = (\sqrt{N}, 0, \ldots, 0)$

MCMC techniques in high dimensions





<u>Geometric explanation:</u>

In the MMA there are special directions:

x_j-direction $= (0, \ldots, 0, 1, 0, \ldots, 0)$

$\uparrow \; j = 1, \ldots, N$

MCMC techniques in high dimensions

Two linear problems

Design points

$$\begin{cases} \theta_1^* = \beta \frac{x}{||x||} \\ \theta_2^* = (\beta, 0, \dots, 0) \end{cases}$$

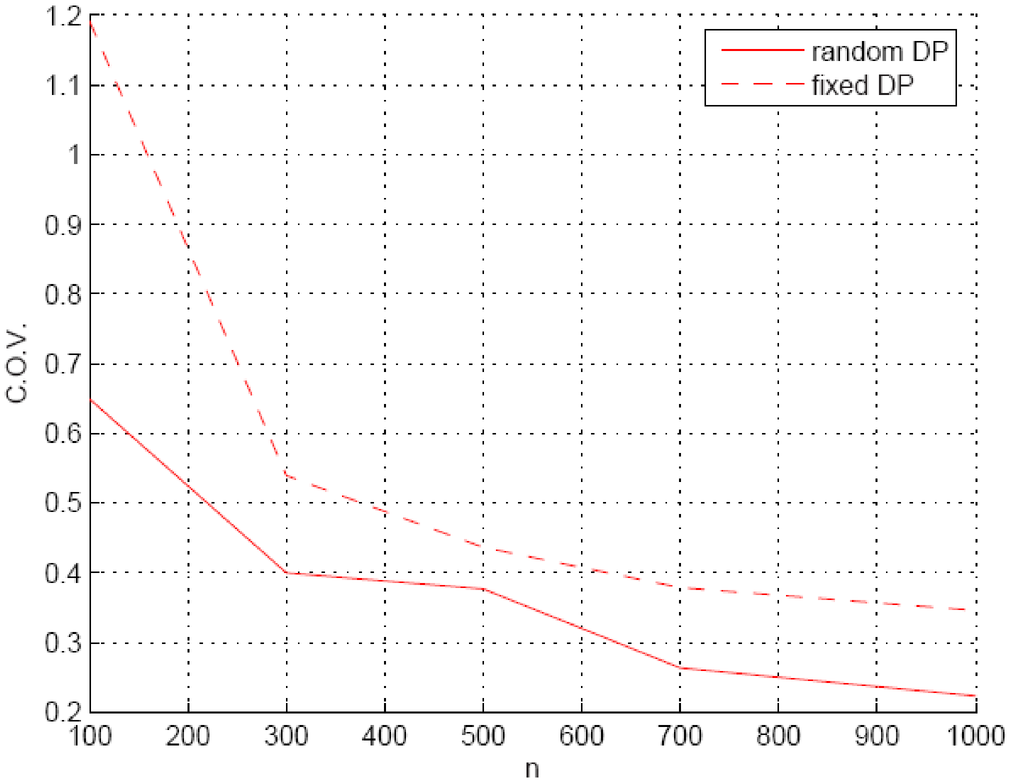
$$x \sim \mathcal{N}(0,1)$$

Here

$$\beta = 3$$

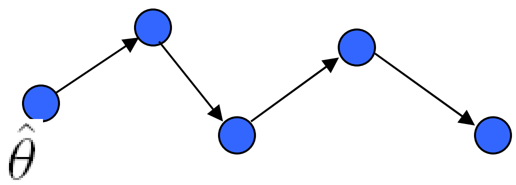
$$N = 10^3$$

50 runs of algorithm is used



MCMC techniques in high dimensions

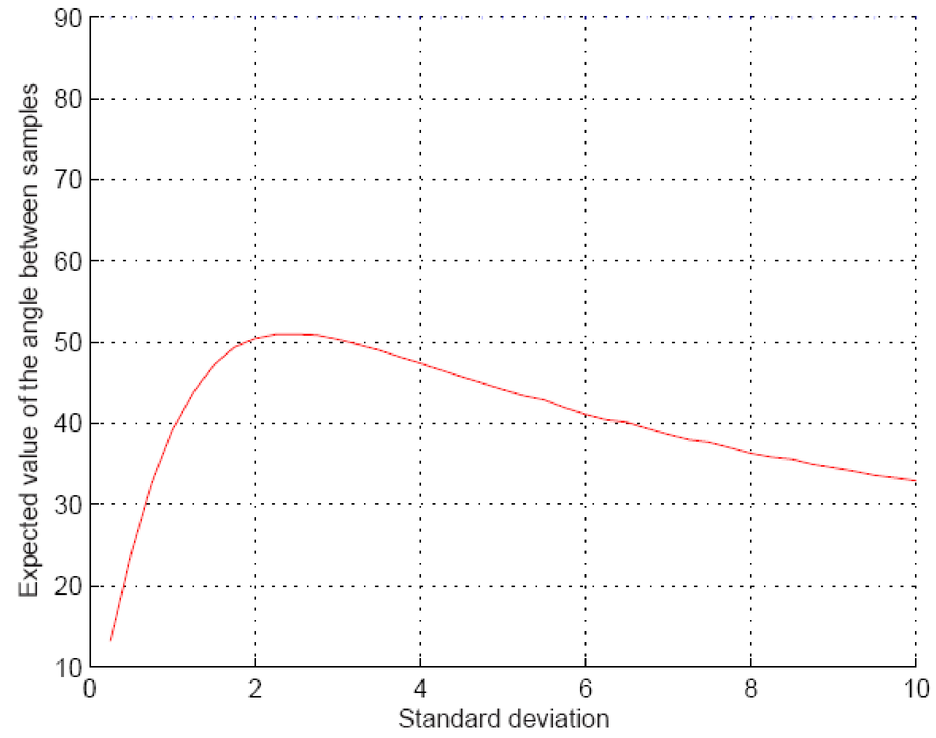
"Free" Markov chain



$$\hat{\theta} \sim \mathcal{N}(0,1)$$

$$\boxed{p(\cdot|\theta_j) = \mathcal{N}(\theta_j, \sigma^2)}$$

$$\theta^{(k+1)} = \tilde{\theta}$$



σ

α is an angle between samples

Monte Carlo $\longrightarrow$ $E[\alpha] = \pi/2$

MCMC techniques in high dimensions

MCMC of S^3 (Katafygiotis and Cheung, 2006)

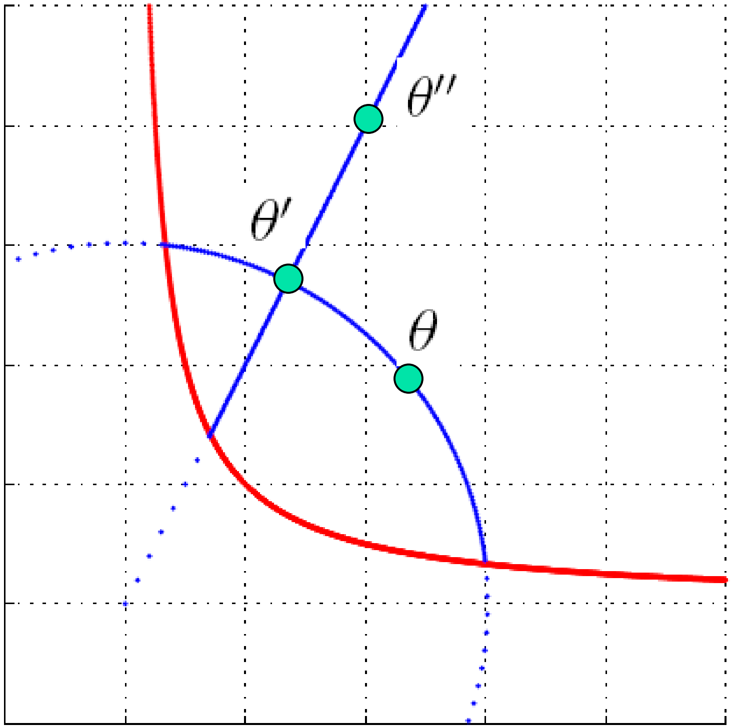
$$\theta = Ru \quad \rightsquigarrow \quad \theta' = Ru' \quad \rightsquigarrow \quad \theta'' = R'u'$$

$$u = \theta / ||\theta||$$

$$R = ||\theta||$$

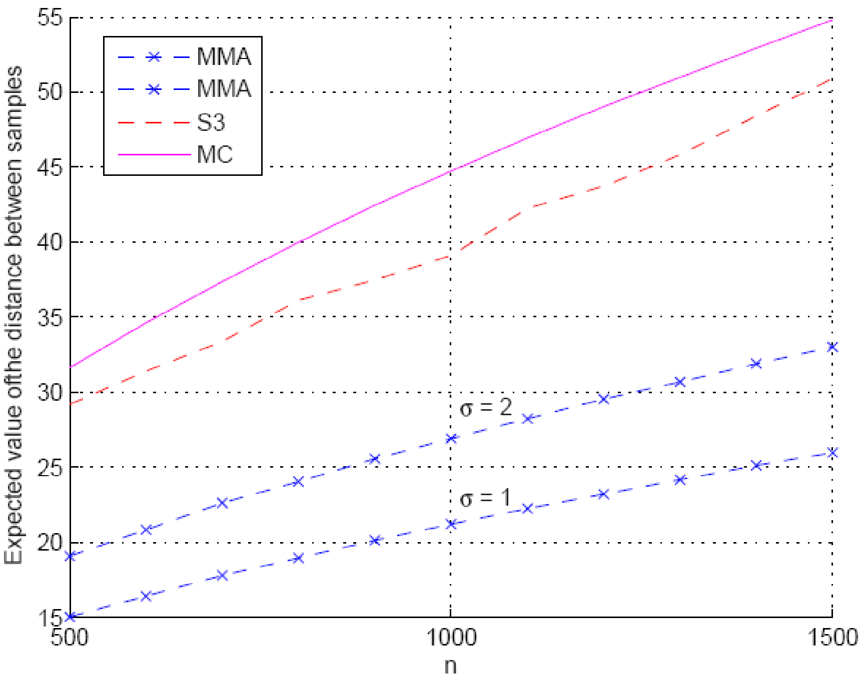
Geometric advantage:

Consistence with nature of Gaussian space there is no preferable directions

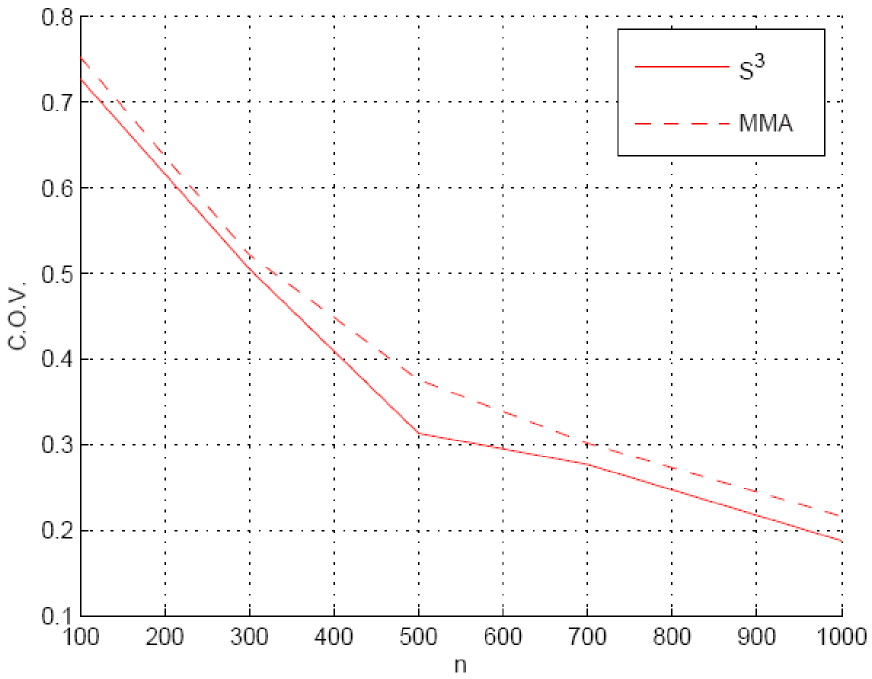


MCMC techniques in high dimensions

Comparison of MMA and MCMC of S^3



Distances between samples of ”free” Markov chains



C.o.v. of estimators for liner failure domain probabilities

Concluding Remarks

- Geometric understanding as to why Important Sampling does generally not work in high dimensions was given.
- Design point seems not to be relevant when dealing with strongly nonlinear problems.
- A study of the correlation of samples obtained using MMA as a function of different parameters leads to recommendations for fine-tuning the MMA.
- Finally, the MMA algorithm was compared with the MCMC algorithm proposed by Katafygiotis and Cheung (2006).

Thank You!