Complex Systems, Complex Networks, and their Reliability

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Risk Institute Seminar

Research Interests

- Objects:
- Phenomena:
- Methods and Tools:

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  - Complex Systems
    - Structural systems (tall buildings, bridges)
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    - Technological networks (road networks, water distribution networks)
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  - Bayesian inference
  - Network theory

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System Reliability Problem

Reliability Problem: To estimate the probability of failure p_F

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Notation:

- $x \in \mathbb{R}^d$ represents the uncertain excitation of a system
  - x is a random vector with joint PDF $\pi(x)$
- $F \subset \mathbb{R}^d$ is a failure domain (unacceptable system performance)

$$F = \{x : g(x) \geq b^\star\}$$

- $g(x)$ is a performance function (loss function)
- $b^\star$ is a critical threshold for performance
- $I_F(x) = 1$ if $x \in F$ and $I_F(x) = 0$ if $x \notin F$

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Idea: To use advanced simulation methods e.g. Subset Simulation

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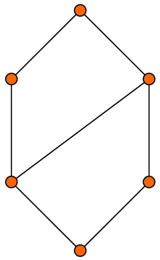
  - $V = \{v_1, \ldots, v_n\}$ set of all nodes
  - $E = \{e_1, \ldots, e_m\}$ set of all links



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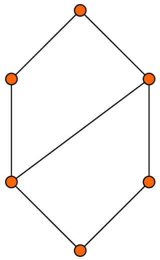

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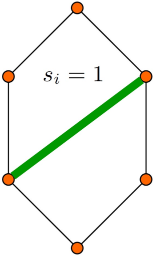

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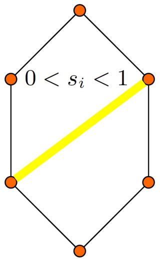

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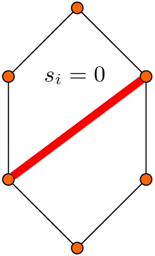

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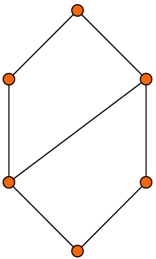

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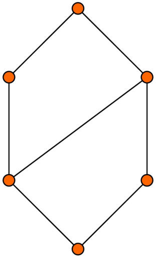

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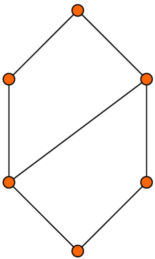

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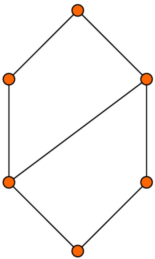

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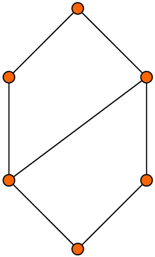

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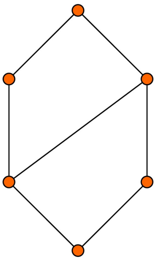

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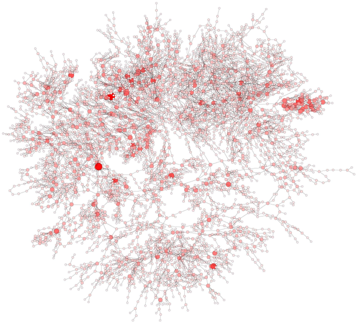

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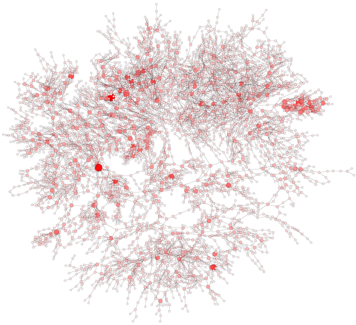

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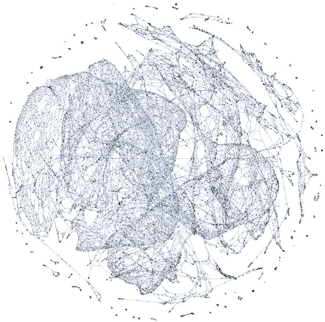

California Road Network, $m = 5,533,214$

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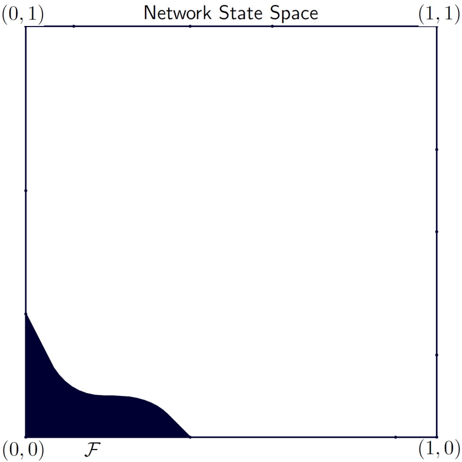
Idea: To use efficient methods developed by the "systems research community"

e.g. Subset Simulation

Schematic Illustration



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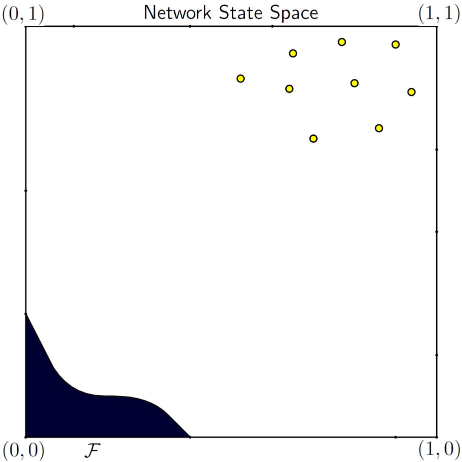
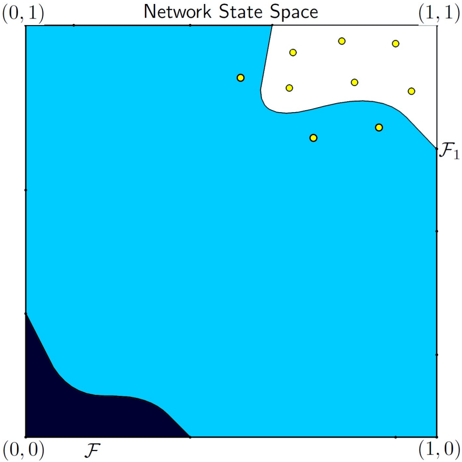


Monte Carlo Samples

Schematic Illustration



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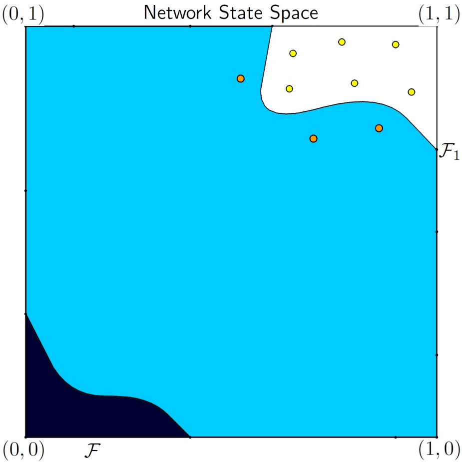

"seeds"

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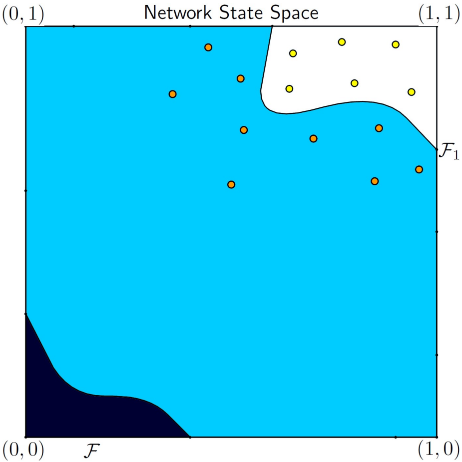

MCMC samples

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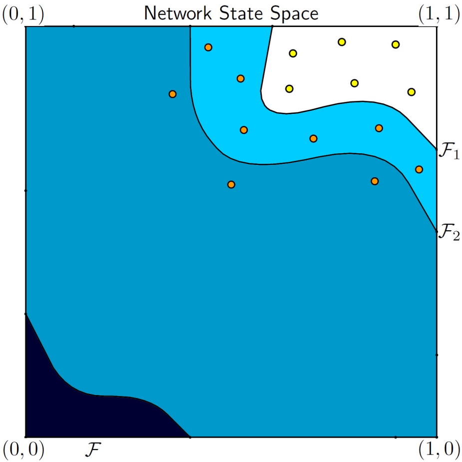

$$\mathbb{P}(\mathcal{F}_2|\mathcal{F}_1) \approx \frac{1}{N}\sum_{i=1}^{N} I_{\mathcal{F}_2}(x_1^{(i)}) = p$$

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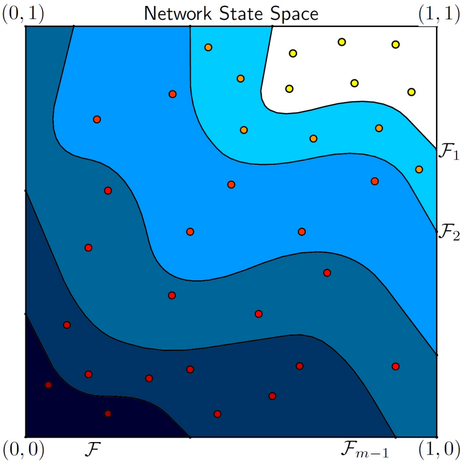

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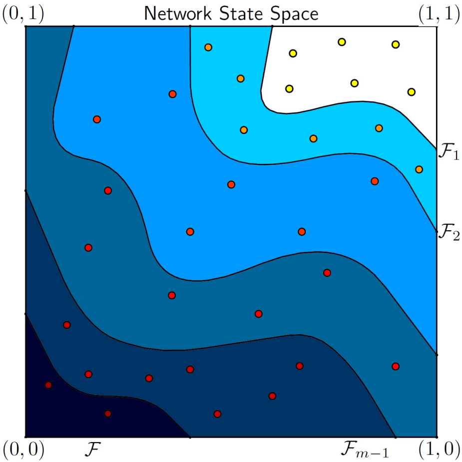

SS estimator : $\widehat{p_F} = p^{m-1}\frac{N_F}{N}$

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