Complex Systems, Complex Networks, and their Reliability

Konstantin Zuev

Institute for Risk and Uncertainty
University of Liverpool

http://www.liv.ac.uk/risk-and-uncertainty/staff/k-zuev/

14 Nov, 2013
Risk Institute Seminar
Research Interests

- Objects:
- Phenomena:
- Methods and Tools:
Research Interests

- **Objects:**
  - Complex Systems
    - Structural systems (tall buildings, bridges)
  - Complex Networks
    - Technological networks (road networks, water distribution networks)
    - Financial networks (stock market networks, banking networks)

- **Phenomena:**

- **Methods and Tools:**
Research Interests

- **Objects:**
  - Complex Systems
    - Structural systems (tall buildings, bridges)
  - Complex Networks
    - Technological networks (road networks, water distribution networks)
    - Financial networks (stock market networks, banking networks)

- **Phenomena:**
  - Reliability and Resilience
  - Self-organization
  - Emerging behaviour

- **Methods and Tools:**
Research Interests

**Objects:**
- Complex Systems
  - Structural systems (tall buildings, bridges)
- Complex Networks
  - Technological networks (road networks, water distribution networks)
  - Financial networks (stock market networks, banking networks)

**Phenomena:**
- Reliability and Resilience
- Self-organization
- Emerging behaviour

**Methods and Tools:**
- Simulation methods
- Bayesian inference
- Network theory
Research Interests

Objects:

- Complex Systems
  - Structural systems (tall buildings, bridges)
- Complex Networks
  - Technological networks (road networks, water distribution networks)
  - Financial networks (stock market networks, banking networks)

Phenomena:

- Reliability and Resilience
- Self-organization
- Emerging behaviour

Methods and Tools:

- Simulation methods
- Bayesian inference
- Network theory
System Reliability Problem

Reliability Problem: To estimate the probability of failure $p_F$

$$p_F = \mathbb{P}(x \in F) = \int_{\mathbb{R}^d} \pi(x) I_F(x) dx$$

Notation:
- $x \in \mathbb{R}^d$ represents the uncertain excitation of a system
- $x$ is a random vector with joint PDF $\pi(x)$
- $F \subset \mathbb{R}^d$ is a failure domain (unacceptable system performance)
- $g(x)$ is a performance function (loss function)
- $b^\star$ is a critical threshold for performance
- $I_F(x) = 1$ if $x \in F$ and $I_F(x) = 0$ if $x / \in F$
System Reliability Problem

Reliability Problem: To estimate the probability of failure $p_F$

$$p_F = \mathbb{P}(x \in F) = \int_{\mathbb{R}^d} \pi(x) I_F(x) \, dx$$

Notation:
- $x \in \mathbb{R}^d$ represents the uncertain excitation of a system
  - $x$ is a random vector with joint PDF $\pi(x)$
- $F \subset \mathbb{R}^d$ is a failure domain (unacceptable system performance)
  $$F = \{x : g(x) \geq b^*\}$$
- $g(x)$ is a performance function (loss function)
- $b^*$ is a critical threshold for performance
- $I_F(x) = 1$ if $x \in F$ and $I_F(x) = 0$ if $x \notin F$
Why is the System Reliability Problem Challenging?

\[ p_F = \int_{\mathbb{R}^d} \pi(x) I_F(x) dx, \quad F = \{x : g(x) \geq b^*\} \]
Why is the System Reliability Problem Challenging?

\[ p_F = \int_{\mathbb{R}^d} \pi(x) I_F(x) \, dx, \quad F = \{x : g(x) \geq b^*\} \]

Typically in Applications:

- We can compute \( I_F(x) \) for any \( x \), but this \textbf{computation is expensive}

[Professor Konstantin Zuev (UoL)]

Complex Systems, Complex Nets, and their Reliability

Risk Institute Seminar
Why is the System Reliability Problem Challenging?

\[ p_F = \int_{\mathbb{R}^d} \pi(x) I_F(x) dx, \quad F = \{ x : g(x) \geq b^* \} \]

Typically in Applications:

- We can compute \( I_F(x) \) for any \( x \), but this computation is expensive
- The probability of failure \( p_F \) is very small, \( p_F \sim 10^{-2} - 10^{-9} \)
Why is the System Reliability Problem Challenging?

\[ p_F = \int_{\mathbb{R}^d} \pi(x) I_F(x) dx, \quad F = \{ x : g(x) \geq b^* \} \]

Typically in Applications:

- We can compute \( I_F(x) \) for any \( x \), but this computation is expensive
- The probability of failure \( p_F \) is very small, \( p_F \sim 10^{-2} \, \text{to} \, 10^{-9} \)
- The dimension \( d \) is very large, \( d \sim 10^3 \)
Why is the System Reliability Problem Challenging?

\[ p_F = \int_{\mathbb{R}^d} \pi(x) I_F(x) \, dx, \quad F = \{ x : g(x) \geq b^* \} \]

Typically in Applications:
- We can compute \( I_F(x) \) for any \( x \), but this computation is expensive.
- The probability of failure \( p_F \) is very small, \( p_F \sim 10^{-2} - 10^{-9} \).
- The dimension \( d \) is very large, \( d \sim 10^3 \).

Consequences:
- Numerical integration is computationally infeasible.
- Monte Carlo method is too expensive.
Why is the System Reliability Problem Challenging?

\[
p_F = \int_{\mathbb{R}^d} \pi(x) I_F(x) dx, \quad F = \{x : g(x) \geq b^*\}
\]

Typically in Applications:
- We can compute \( I_F(x) \) for any \( x \), but this computation is expensive.
- The probability of failure \( p_F \) is very small, \( p_F \sim 10^{-2} - 10^{-9} \).
- The dimension \( d \) is very large, \( d \sim 10^3 \).

Consequences:
- Numerical integration is computationally infeasible.
- Monte Carlo method is too expensive.

Idea: To use advanced simulation methods e.g. Subset Simulation.
Network Reliability Problem

Network topology is represented by a graph $G = (V, E)$,

- $V = \{v_1, \ldots, v_n\}$ set of all nodes
- $E = \{e_1, \ldots, e_m\}$ set of all links

A network state is $s = (s_1, \ldots, s_m)$, where $0 \leq s_i \leq 1$

- $s_i = 1$ if link $e_i$ is fully operational
- $0 < s_i < 1$ if link $e_i$ is partially operational
- $s_i = 0$ if link $e_i$ is completely failed

The network state space is $S = \{(s_1, \ldots, s_m) : 0 \leq s_i \leq 1\} = [0, 1]^m$

Let $\pi(s)$ be a probability distribution on $S$, $s \sim \pi(s)$

Let $\mu : S \rightarrow \mathbb{R}$ be a performance function (utility function)

The failure domain is $F = \{s : \mu(s) < \mu^\star\} \subset S$

Network Reliability Problem: To estimate the probability of failure $p_F$

$$p_F = P(s \in F) = \int_S \pi(s) I_F(s) ds$$
Network Reliability Problem

- **Network topology** is represented by a graph \( G = (V, E) \)
  - \( V = \{v_1, \ldots, v_n\} \) set of all nodes
  - \( E = \{e_1, \ldots, e_m\} \) set of all links
Network Reliability Problem

- **Network topology** is represented by a graph $G = (V, E)$
  - $V = \{v_1, \ldots, v_n\}$ set of all nodes
  - $E = \{e_1, \ldots, e_m\}$ set of all links
- A **network state** is $s = (s_1, \ldots, s_m)$, where $0 \leq s_i \leq 1$
Network Reliability Problem

- Network topology is represented by a graph $G = (V, E)$
  - $V = \{v_1, \ldots, v_n\}$ set of all nodes
  - $E = \{e_1, \ldots, e_m\}$ set of all links
- A network state is $s = (s_1, \ldots, s_m)$, where $0 \leq s_i \leq 1$
  - $s_i = 1$ if link $e_i$ is fully operational

The failure domain is $F = \{s : \mu(s) < \mu^\star\} \subset S$
Network Reliability Problem

- **Network topology** is represented by a graph \( G = (V, E) \)
  - \( V = \{v_1, \ldots, v_n\} \) set of all nodes
  - \( E = \{e_1, \ldots, e_m\} \) set of all links
- A **network state** is \( s = (s_1, \ldots, s_m) \), where \( 0 \leq s_i \leq 1 \)
  - \( s_i = 1 \) if link \( e_i \) is **fully operational**
  - \( 0 < s_i < 1 \) if link \( e_i \) is **partially operational**
Network Reliability Problem

- **Network topology** is represented by a graph \( G = (V, E) \)
  - \( V = \{v_1, \ldots, v_n\} \) set of all nodes
  - \( E = \{e_1, \ldots, e_m\} \) set of all links

- A network state is \( s = (s_1, \ldots, s_m) \), where \( 0 \leq s_i \leq 1 \)
  - \( s_i = 1 \) if link \( e_i \) is fully operational
  - \( 0 < s_i < 1 \) if link \( e_i \) is partially operational
  - \( s_i = 0 \) if link \( e_i \) is completely failed
Network Reliability Problem

- **Network topology** is represented by a graph $G = (V, E)$
  - $V = \{v_1, \ldots, v_n\}$ set of all nodes
  - $E = \{e_1, \ldots, e_m\}$ set of all links

- A network state is $s = (s_1, \ldots, s_m)$, where $0 \leq s_i \leq 1$
  - $s_i = 1$ if link $e_i$ is fully operational
  - $0 < s_i < 1$ if link $e_i$ is partially operational
  - $s_i = 0$ if link $e_i$ is completely failed

- The network state space is $\mathcal{S} = \{(s_1, \ldots, s_m) : 0 \leq s_i \leq 1\}$
Network Reliability Problem

- **Network topology** is represented by a graph $G = (V, E)$
  - $V = \{v_1, \ldots, v_n\}$ set of all nodes
  - $E = \{e_1, \ldots, e_m\}$ set of all links

- A **network state** is $s = (s_1, \ldots, s_m)$, where $0 \leq s_i \leq 1$
  - $s_i = 1$ if link $e_i$ is fully operational
  - $0 < s_i < 1$ if link $e_i$ is partially operational
  - $s_i = 0$ if link $e_i$ is completely failed

- The **network state space** is $S = \{(s_1, \ldots, s_m) : 0 \leq s_i \leq 1\} = [0, 1]^m$
Network Reliability Problem

- **Network topology** is represented by a graph $G = (V, E)$
  - $V = \{v_1, \ldots, v_n\}$ set of all nodes
  - $E = \{e_1, \ldots, e_m\}$ set of all links
- A **network state** is $s = (s_1, \ldots, s_m)$, where $0 \leq s_i \leq 1$
  - $s_i = 1$ if link $e_i$ is fully operational
  - $0 < s_i < 1$ if link $e_i$ is partially operational
  - $s_i = 0$ if link $e_i$ is completely failed
- The **network state space** is $S = \{(s_1, \ldots, s_m) : 0 \leq s_i \leq 1\} = [0, 1]^m$
- Let $\pi(s)$ be a probability distribution on $S$, $s \sim \pi(s)$
Network Reliability Problem

- **Network topology** is represented by a graph $G = (V, E)$
  - $V = \{v_1, \ldots, v_n\}$ set of all nodes
  - $E = \{e_1, \ldots, e_m\}$ set of all links
- A network state is $s = (s_1, \ldots, s_m)$, where $0 \leq s_i \leq 1$
  - $s_i = 1$ if link $e_i$ is fully operational
  - $0 < s_i < 1$ if link $e_i$ is partially operational
  - $s_i = 0$ if link $e_i$ is completely failed
- The network state space is $S = \{(s_1, \ldots, s_m) : 0 \leq s_i \leq 1\} = [0, 1]^m$
- Let $\pi(s)$ be a probability distribution on $S$, $s \sim \pi(s)$
- Let $\mu : S \rightarrow \mathbb{R}$ be a performance function (utility function)
Network Reliability Problem

- **Network topology** is represented by a graph $G = (V, E)$
  - $V = \{v_1, \ldots, v_n\}$ set of all nodes
  - $E = \{e_1, \ldots, e_m\}$ set of all links
- A network state is $s = (s_1, \ldots, s_m)$, where $0 \leq s_i \leq 1$
  - $s_i = 1$ if link $e_i$ is fully operational
  - $0 < s_i < 1$ if link $e_i$ is partially operational
  - $s_i = 0$ if link $e_i$ is completely failed
- The network state space is $S = \{(s_1, \ldots, s_m) : 0 \leq s_i \leq 1\} = [0, 1]^m$
- Let $\pi(s)$ be a probability distribution on $S$, $s \sim \pi(s)$
- Let $\mu : S \to \mathbb{R}$ be a performance function (utility function)
- The failure domain is $F = \{s : \mu(s) < \mu^*\} \subset S$
Network Reliability Problem

- Network topology is represented by a graph $G = (V, E)$
  - $V = \{v_1, \ldots, v_n\}$ set of all nodes
  - $E = \{e_1, \ldots, e_m\}$ set of all links
- A network state is $s = (s_1, \ldots, s_m)$, where $0 \leq s_i \leq 1$
  - $s_i = 1$ if link $e_i$ is fully operational
  - $0 < s_i < 1$ if link $e_i$ is partially operational
  - $s_i = 0$ if link $e_i$ is completely failed
- The network state space is $\mathcal{S} = \{(s_1, \ldots, s_m) : 0 \leq s_i \leq 1\} = [0, 1]^m$
- Let $\pi(s)$ be a probability distribution on $\mathcal{S}$, $s \sim \pi(s)$
- Let $\mu : \mathcal{S} \rightarrow \mathbb{R}$ be a performance function (utility function)
- The failure domain is $\mathcal{F} = \{s : \mu(s) < \mu^*\} \subset \mathcal{S}$

Network Reliability Problem: To estimate the probability of failure $p_\mathcal{F}$

$$p_\mathcal{F} = \mathbb{P}(s \in \mathcal{F}) = \int_{\mathcal{S}} \pi(s) I_\mathcal{F}(s) ds$$
Why is the Network Reliability Problem Challenging?

\[ p_F = \int_S \pi(s)I_F(s)ds, \quad F = \{ s : \mu(s) < \mu^* \} \subset [0, 1]^m \]
Why is the Network Reliability Problem Challenging?

\[ p_F = \int_S \pi(s)I_F(s)ds, \quad F = \{ s : \mu(s) < \mu^* \} \subset [0, 1]^m \]

Typically in Applications:

- The number of links \( m \) is very large
Why is the Network Reliability Problem Challenging?

\[ p_F = \int_S \pi(s) I_F(s) ds, \quad F = \{ s : \mu(s) < \mu^* \} \subset [0, 1]^m \]

Typically in Applications:

- The number of links \( m \) is very large

US Western States Power Grid, \( m = 6,594 \)
Why is the Network Reliability Problem Challenging?

\[ p_F = \int_S \pi(s) I_F(s) ds, \quad F = \{ s : \mu(s) < \mu^* \} \subset [0, 1]^m \]

Typically in Applications:

- The number of links \( m \) is very large

US Western States Power Grid, \( m = 6,594 \) 
California Road Network, \( m = 5,533,214 \)
Why is the Network Reliability Problem Challenging?

\[ p_F = \int_S \pi(s) I_F(s) ds, \quad F = \{ s : \mu(s) < \mu^* \} \subset [0, 1]^m \]

Typically in Applications:

- The number of links \( m \) is very large

Konstantin Zuev (UoL) Complex Systems, Complex Nets, and their Reliability Risk Institute Seminar 6 / 8
Why is the Network Reliability Problem Challenging?

\[ p_{\mathcal{F}} = \int_{\mathcal{S}} \pi(s) I_{\mathcal{F}}(s) ds, \quad \mathcal{F} = \{ s : \mu(s) < \mu^* \} \subset [0, 1]^m \]

Typically in Applications:

- The number of links \( m \) is very large
- The probability of failure \( p_{\mathcal{F}} \) is very small
Why is the Network Reliability Problem Challenging?

\[ p_F = \int_{S} \pi(s) I_F(s) ds, \quad F = \{s : \mu(s) < \mu^*\} \subset [0, 1]^m \]

Typically in Applications:
- The number of links \( m \) is very large
- The probability of failure \( p_F \) is very small
- The computational effort for evaluating \( \mu(s) \) is significant
Why is the Network Reliability Problem Challenging?

\[ p_\mathcal{F} = \int_{\mathcal{F}} \pi(s) I_\mathcal{F}(s) ds, \quad \mathcal{F} = \{ s : \mu(s) < \mu^* \} \subset [0, 1]^m \]

Typically in Applications:
- The number of links \( m \) is very large
- The probability of failure \( p_\mathcal{F} \) is very small
- The computational effort for evaluating \( \mu(s) \) is significant

Consequences:
- Combinatorial exhaustive search methods are not applicable
- Numerical integration is computationally infeasible
- Monte Carlo method is too expensive
Why is the Network Reliability Problem Challenging?

\[ p_{\mathcal{F}} = \int_{\mathcal{F}} \pi(s) I_{\mathcal{F}}(s) ds, \quad \mathcal{F} = \{ s : \mu(s) < \mu^* \} \subset [0, 1]^m \]

Typically in Applications:

- The number of links \( m \) is very large
- The probability of failure \( p_{\mathcal{F}} \) is very small
- The computational effort for evaluating \( \mu(s) \) is significant

Consequences:

- Combinatorial exhaustive search methods are not applicable
- Numerical integration is computationally infeasible
- Monte Carlo method is too expensive

Idea: To use efficient methods developed by the “systems research community”

e.g. Subset Simulation
Schematic Illustration

\[ \mathcal{F} = \{ s : \mu(s) < \mu^* \} \subset S \]
Schematic Illustration

Monte Carlo Samples
\[ P(F_1) \approx \frac{1}{N} \sum_{i=1}^{N} I_{F_1}(x_0^{(i)}) = p \]
Schematic Illustration

Network State Space

\( \mathcal{F} \)

“seeds”
Schematic Illustration

MCMC samples
\[ P(\mathcal{F}_2 | \mathcal{F}_1) \approx \frac{1}{N} \sum_{i=1}^{N} I_{\mathcal{F}_2}(x_1^{(i)}) = p \]
Schematic Illustration
SS estimator: \[ \hat{p}_F = p^{m-1} \frac{N_F}{N} \]
Brief Summary
Brief Summary

- A strong mathematical similarity between reliability problems for complex systems and complex networks is highlighted.
Brief Summary

- A strong **mathematical similarity** between **reliability problems** for complex **systems** and complex **networks** is highlighted.

- A framework for **quantitative assessment** of network reliability is described and a general **network reliability problem** within this framework is formulated.
Brief Summary

- A strong mathematical similarity between reliability problems for complex systems and complex networks is highlighted.

- A framework for quantitative assessment of network reliability is described and a general network reliability problem within this framework is formulated.

- Subset Simulation can be used for solving the network reliability problem.
Brief Summary

- A strong **mathematical similarity** between **reliability problems** for complex **systems** and complex **networks** is highlighted.

- A framework for **quantitative assessment** of network reliability is described and a general **network reliability problem** within this framework is formulated.

- **Subset Simulation** can be used for solving the network reliability problem.