Estimation of small failure probabilities in high dimensions by Adaptive Linked Importance Sampling

Lambros Katafygiotis
and
Konstantin Zuev
Outline

- Problem
- Proposed methodology
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  - Introduction of intermediate distribution
  - Bridging
  - ALIS procedure
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**Problem**

Input: \( x \in \mathbb{R}^N \) is a random vector with joint PDF \( \pi_0 \)

Output: \( f(x) \), where \( f : \mathbb{R}^N \rightarrow \mathbb{R}_+ \)

\[ F = \{ x \in \mathbb{R}^N \mid f(x) > b \} \] - failure domain, \( b \in \mathbb{R}_+ \) is prescribed critical threshold

Problem: To find the probability of failure \( p_F \)

\[ p_F = P(x \in F) = \int_F \pi_0(x) \, dx = \int_{\mathbb{R}^N} I_F(x) \pi_0(x) \, dx = E_{\pi_0}[I_F] \]

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stochastic simulation methods based on Monte Carlo
Proposed methodology

Reformulation of Problem

Consider two PDFs on $\mathbb{R}^N$:

$$\pi_0(x) = \frac{p_0(x)}{Z_0}, \quad \pi_1(x) = \frac{p_1(x)}{Z_1}$$

known pointwise

unknown normalizing constants

If

$$\begin{align*}
p_0(x) &= \pi_0(x) \\
p_1(x) &= \pi_0(x) I_F(x)
\end{align*}$$

$\Rightarrow Z_0 = 1$

$Z_1 = p_F$

$\Rightarrow p_F = \frac{Z_1}{Z_0}$

It can be shown, that

$$\frac{Z_1}{Z_0} = E_{\pi_0} \left[ \frac{p_1}{p_0} \right]$$

Standard Monte Carlo:

$$\frac{\hat{Z}_1}{Z_0} = \frac{1}{n} \sum_{k=1}^{n} \frac{p_1(x^{(k)})}{p_0(x^{(k)})}, \quad x^{(k)} \sim \pi_0$$

General framework:

Estimation of $Z_1/Z_0$

Estimation of $p_F$
Proposed methodology

Introduction of intermediate PDFs

Difficulty: Estimation of $Z_1/Z_0$ is hard if the distributions $\pi_0$ and $\pi_1$ are not close enough. General strategy: Introduce intermediate PDFs $\pi_0 = \pi_{\alpha_0}, \ldots, \pi_{\alpha_i}, \ldots, \pi_{\alpha_m} = \pi_1$ with $\pi_{\alpha_i}$ interpolating between $\pi_0$ and $\pi_1$.

If $\pi_{\alpha_i} = \frac{p_{\alpha_i}}{Z_{\alpha_i}}$, where $Z_{\alpha_i} = \int p_{\alpha_i}(x)dx$

$$\Rightarrow \frac{Z_1}{Z_0} = \frac{Z_{\alpha_1}}{Z_0} \frac{Z_{\alpha_2}}{Z_{\alpha_1}} \cdots \frac{Z_{\alpha_{m-1}}}{Z_{\alpha_{m-2}}} \frac{Z_1}{Z_{\alpha_{m-1}}}$$

$$\hat{Z}_1/Z_0 = \prod_{j=0}^{m-1} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{p_{\alpha_{j+1}}(x_j^{(k)})}{p_{\alpha_j}(x_j^{(k)})} \right), \quad x_j^{(k)} \sim \pi_{\alpha_j}$$

Generalized Importance Sampling
Proposed methodology

Introduction of intermediate PDFs

Subset Simulation (Au S.K & Beck J.L., 2001)

Let $\mathbb{R}^N = F_{\alpha_0} \supset F_{\alpha_1} \supset \ldots \supset F_{\alpha_m} = F$ be a filtration

$$p_F = \prod_{j=0}^{m-1} P(F_{\alpha_{j+1}}|F_{\alpha_j})$$

$$\hat{P}(F_{\alpha_{j+1}}|F_{\alpha_j}) = \frac{1}{n} \sum_{k=1}^{n} I_{F_{\alpha_{j+1}}}(x^{(k)}), \; x^{(k)} \sim \pi_0(\cdot|F_{\alpha_j})$$

Subset Simulation  
\[ \| \]
Generalized Importance Sampling with conditional intermediate distributions

$$\pi_{\alpha_j}(x) = \pi_0(x|F_{\alpha_j})$$
Proposed methodology

Introduction of intermediate PDFs

Two smooth families of intermediate PDFs designed for reliability problems

Limit state function: \( \Phi(x) = b - f(x) \), so that \( x \in F \iff \Phi(x) < 0 \)

\[
p^I_\alpha(x) = \pi_0(x) \min\{e^{-\alpha \Phi(x)}, 1\}
\]

\[
p^{II}_\alpha(x) = \frac{\pi_0(x)}{1 + e^{\alpha \Phi(x)}}
\]

For both families:

\[
p_0(x) \propto \pi_0(x)
\]

\[
\lim_{\alpha \to +\infty} p_\alpha(x) \propto \pi_0(x|F)
\]
Proposed methodology

Problem: If \( \{\pi_\alpha\} \) are NOT nested, we cannot hope to estimate \( Z_{\alpha_{j+1}} / Z_{\alpha_j} \) by sampling just from \( \pi_{\alpha_j} \)

Example: Uniform PDFs

\[
Z_{\alpha_j} = \text{Vol}(U_j), \quad Z_{\alpha_{j+1}} = \text{Vol}(U_{j+1}) \\
Z_* = \text{Vol}(U_j \cap U_{j+1})
\]

If \( x_j^{(k)} \sim \pi_{\alpha_j} \) \( \Rightarrow \)
\[
\frac{1}{n} \sum_{k=1}^{n} \frac{p_{\alpha_{j+1}}(x_j^{(k)})}{p_{\alpha_j}(x_j^{(k)})} = \frac{1}{n} \sum_{k=1}^{n} \frac{I_{U_{j+1}}(x_j^{(k)})}{I_{U_j}(x_j^{(k)})} \quad \Rightarrow \quad \frac{Z_*}{Z_{\alpha_j}} \]

If \( x_{j+1}^{(k)} \sim \pi_{\alpha_{j+1}} \) \( \Rightarrow \)
\[
\frac{1}{n} \sum_{k=1}^{n} \frac{p_{\alpha_{j+1}}(x_{j+1}^{(k)})}{p_{\alpha_{j+1}}(x_{j+1}^{(k)})} = \frac{1}{n} \sum_{k=1}^{n} \frac{I_{U_{j+1}}(x_{j+1}^{(k)})}{I_{U_{j+1}}(x_{j+1}^{(k)})} \quad \Rightarrow \quad \frac{Z_*}{Z_{\alpha_{j+1}}}
\]
Proposed methodology

Bridging: \[ \frac{Z_{\alpha_{j+1}}}{Z_{\alpha_j}} = \frac{\frac{Z_*}{Z_{\alpha_j}}}{\frac{Z_*}{Z_{\alpha_{j+1}}}} , \]

where \( Z_* \) is normalizing constant for ”bridge distribution” \( \pi_*(x) = \frac{p_*(x)}{Z_*} \)

“Geometric” bridge
(Bennett C.H., 1976)
\[ p_{geo}^*(x) = \sqrt{p_{\alpha_j}(x) p_{\alpha_{j+1}}(x)} \]

\[ \frac{\overline{Z_{\alpha_{j+1}}}}{Z_{\alpha_j}} = \frac{\frac{1}{n} \sum_{k=1}^{n} \frac{p_*(x_j^{(k)})}{p_{\alpha_j}(x_j^{(k)})}}{\frac{1}{n} \sum_{k=1}^{n} \frac{p_*(x_{j+1}^{(k)})}{p_{\alpha_{j+1}}(x_{j+1}^{(k)})}} \]

\( x_j^{(k)} \sim \pi_{\alpha_j} \)
\( x_{j+1}^{(k)} \sim \pi_{\alpha_{j+1}} \)
Proposed methodology

1dim Example: \( \pi_\alpha = \mathcal{N}(\alpha, 1) \) is Gaussian with mean \( \alpha \) and variance \( \sigma = 1 \)

\[
\frac{Z_\alpha}{Z_0} = 1
\]

\[
\frac{\hat{Z}_\alpha}{Z_0} = \frac{1}{n} \sum_{k=1}^{n} \frac{\pi_\alpha(x_0^{(k)})}{\pi_0(x_0^{(k)})}, x_0^{(k)} \sim \pi_0
\]

\[
\frac{\hat{Z}_\alpha}{Z_0} = \frac{1}{n} \sum_{k=1}^{n} \frac{p_*(x_0^{(k)})}{\pi_0(x_0^{(k)})}, x_0^{(k)} \sim \pi_0
\]

\[
\frac{\hat{Z}_\alpha}{Z_0} = \frac{1}{n} \sum_{k=1}^{n} \frac{p_*(x_\alpha^{(k)})}{\pi_\alpha(x_\alpha^{(k)})}, x_\alpha^{(k)} \sim \pi_\alpha
\]
**Proposed methodology**

Intermediate PDFs are given

<table>
<thead>
<tr>
<th>I. Initialization</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0^{(k)} \sim \pi_0, \text{ for } k = 1, \ldots, n )</td>
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<tr>
<th>II. Link State</th>
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<tbody>
<tr>
<td>( x^{(k)}<em>{1/2} \sim w_k = \frac{p</em>{1/2*1}(x^{(k)}<em>{1/2})}{p</em>{1/2}(x^{(k)}_{1/2})} )</td>
</tr>
<tr>
<td>( x_{1/2*1} = x^{(k)}_{1/2} \text{ w.p. } w_k/\sum w_k )</td>
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</tbody>
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<thead>
<tr>
<th>III. Sampling using MCMC</th>
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<tbody>
<tr>
<td>( x_1^{(1)} = x_{1/2*1}, \ x^{(k)}_1 \sim \pi_1, \text{ for } k = 2 \ldots n )</td>
</tr>
</tbody>
</table>

**ALIS procedure**

\( \pi_0 \)

\( \pi_{1/2} \)

\( \pi_1 = \pi_0(\cdot | F) \)

\( \pi_{0*1/2} \)

\( \pi_{1/2*1} \)

\[ \widehat{PF} = \frac{\frac{1}{n} \sum_{k=1}^{n} p_{0*1/2}(x_0^{(k)})}{p_0(x_0^{(k)})} \]  

\[ \frac{\frac{1}{n} \sum_{k=1}^{n} p_{1/2*1}(x_{1/2}^{(k)})}{p_{1/2}(x_{1/2}^{(k)})} \]

\[ \frac{\frac{1}{n} \sum_{k=1}^{n} p_{1/2*1}(x_1^{(k)})}{p_1(x_1^{(k)})} \]

Neal R.M., (2005): Estimate is exactly unbiased

\[ \frac{\widehat{Z_{1/2}}}{Z_0} \]

\[ \frac{\widehat{Z_1}}{Z_{1/2}} \]
Proposed methodology

**Problem:** Usually Intermediate PDFs are not given a priori

**General strategy:** Given $\pi_{\alpha_j}$ and $x_j^{(k)} \sim \pi_{\alpha_j}, k = 1, \ldots, N$

construct $\pi_{\alpha_{j+1}}$ adaptively

$$p_F = \frac{Z_1}{Z_0} = \frac{Z_{\alpha_1}}{Z_0} \frac{Z_{\alpha_2}}{Z_{\alpha_1}} \cdots \frac{Z_{\alpha_{j+1}}}{Z_{\alpha_j}} \cdots \frac{Z_{\alpha_{m-1}}}{Z_{\alpha_{m-2}}} \frac{Z_1}{Z_{\alpha_{m-1}}}$$

very small

$$\| T \approx 0.1$$

$$\frac{Z_{\alpha_{j+1}}}{Z_{\alpha_j}} = E_{\pi_{\alpha_j}} \left[ \frac{p_{\alpha_{j+1}}}{p_{\alpha_j}} \right] \approx \frac{1}{N} \sum_{k=1}^{n} \frac{p_{\alpha_{j+1}}(x_j^{(k)})}{p_{\alpha_j}(x_j^{(k)})} = T$$

equation on $\alpha_{j+1}$
Examples

One-dimensional ray

\[ x \sim \mathcal{N}(0, 1) \]

\[ F = \{ x \in \mathbb{R} \mid x > b \} \quad p_F = 1 - \Phi(b) \]

We want to estimate

\[ p_F = 10^{-k}, \quad k = 2, \ldots, 10 \]

\[ b_k = \Phi^{-1}(1 - 10^{-k}) \]

100 runs are used

10^4 samples for each \( \pi_\alpha \) are used
Examples

$x \in \mathbb{R}^N \sim \mathcal{N}(I, 0), \; N = 100$

$\mathcal{C}_{a,\varphi} = \{ x \in \mathbb{R}^N \mid \hat{x}, a < \varphi \}$

$F = \mathcal{C}_{a,\varphi} \cap S^{N-1}_{\sqrt{N}}$

We want to estimate $p_F = 10^{-k}, \; k = 2, \ldots, 7$

100 runs are used

$10^4$ samples for each $\pi_\alpha$ are used
Examples

\[ x \in \mathbb{R}^N \sim \mathcal{N}(I, 0), \; N = 1000 \]
\[ C_{a,\varphi} = \{ x \in \mathbb{R}^N | \langle x, a \rangle < \varphi \} \]

We want to estimate
\[ p_F = 10^{-k}, \; k = 2, \ldots, 7 \]
100 runs are used
10^4 samples for each \( \pi_\alpha \) are used
Conclusions and remarks

- A novel simulation approach, called Adaptive Linked Importance Sampling (ALIS), was introduced.
- It has been shown that this methodology generalizes the Subset Simulation algorithm (Au S.K. & Beck J.L., 2001) : instead of using conditional distributions ALIS allows for a much richer choice of artificial intermediate distributions (AID’s). The case of non-nested distributions can be overcome by the concept of bridging.
- The accuracy and efficiency of the ALIS is demonstrated with numerical examples. In particular, we show that the choice of AID’s prescribed by Subset Simulation is far from optimal.
Thank You for attention!