SCIENCE CHINA
Mathematics

Special Topic on Topology
• ARTICLES •

September 2017 Vol. 60 No. 9: 1591–1598
doi: 10.1007/s11425-016-9052-8

# Virtual Betti numbers and virtual symplecticity of 4-dimensional mapping tori, II

*Dedicated to Professor Boju Jiang on the Occasion of His 80th Birthday*

NI Yi

*Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA*

*Email: yini@caltech.edu*

Received December 23, 2016; accepted March 6, 2017; published online March 24, 2017

**Abstract** We show that if the fiber of a closed 4-dimensional mapping torus $X$ is reducible and not $S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$, then the virtual first Betti number of $X$ is infinite and $X$ is not virtually symplectic. This confirms two conjectures made by Li and Ni (2014) in an earlier paper.

**Keywords** virtual Betti number, virtual symplecticity, 4-dimensional mapping torus, reducible 3-manifold

**MSC(2010)** 57R17, 57M10

**Citation:** Ni Y. Virtual Betti numbers and virtual symplecticity of 4-dimensional mapping tori, II. Sci China Math, 2017, 60: 1591–1598, doi: 10.1007/s11425-016-9052-8

## 1 Introduction

In [7], Li and Ni computed the virtual first Betti numbers of 4-dimensional mapping tori with irreducible fibers, and also answered the question when such manifolds are virtually symplectic. But when the fibers are reducible, the corresponding questions are not answered. In this paper, we answer these questions when the fibers are reducible.

All manifolds we consider are oriented unless otherwise stated. If $E$ is an $F$-bundle over $S^1$ and $\varphi$ is the monodromy, then we write $E = F \rtimes_\varphi S^1$ or $E = F \rtimes S^1$.

The *virtual first Betti number* of a manifold $M$ is defined to be

$$vb_1(M) = \sup\{b_1(\widetilde{M}) \mid \widetilde{M} \text{ is a finite cover of } M\} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}.$$

Similarly, given a group $G$, let

$$vb_1(G) = \sup_{H<G,[G:H]<\infty} \operatorname{rank} H_1(H).$$

A 4-manifold $X$ is *virtually symplectic* if a finite cover of $X$ admits a symplectic structure.

Our main theorems confirm [7, Conjectures 5.1 and 5.2].

**Theorem 1.1.** *Suppose that $X = Y \rtimes S^1$ is a closed 4-manifold. If $Y$ is reducible, then $vb_1(X) = \infty$ unless $Y = S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$.*

 

**Theorem 1.2.** *Suppose that $X = Y \rtimes S^1$ is a closed symplectic 4-manifold. If $Y$ is reducible, then $Y = S^2 \times S^1$ and $X = S^2 \times T^2$ or $S^2 \widetilde{\times} T^2$, where $S^2 \widetilde{\times} T^2$ is the unique nontrivial oriented $S^2$-bundle over $T^2$.*

Theorem 1.2 was known when $X = Y \times S^1$ (see [8]).

**Remark 1.3.** The original statement of [7, Conjecture 5.2] is not correct, since it overlooks the case of $S^2 \widetilde{\times} T^2$.

**Corollary 1.4.** *Suppose that $X = Y \rtimes S^1$ is a closed 4-manifold. If $Y$ is reducible, then $X$ is virtually symplectic if and only if $Y = S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$.*

The strategy of the proof of Theorem 1.1 is as follows. We first consider the special case that $Y$ is the connected sum of several $S^2 \times S^1$'s. In this case $\pi_1(X)$ is a free-by-cyclic group. Using deep results in geometric group theory, we can show that $vb_1(X) = \infty$. In the general case, assume $Y$ is not $S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$. Then $Y$ has a finite cover $\widetilde{Y}$ with more than one $S^2 \times S^1$ summand. If there exists a degree one map from $\widetilde{Y} \rtimes S^1$ to $(\#^n S^2 \times S^1) \rtimes S^1$, then we can reduce this case to the case we have proved. In general, we do not know if such a degree one map exists. Instead, we will prove a weaker result (see Proposition 3.1), which will suffice for our propose. (It is possible that such a degree one map exists, but we do not need such a strong result here.)

Theorem 1.2 also follows from the above argument. If $Y$ is not $S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$, Proposition 3.1 implies that a finite cover of $X$ contains a homologically essential 2-sphere. Thus a standard argument in Seiberg-Witten theory shows that $X$ is not symplectic.

**Remark 1.5.** Yi Ni sketched most proofs to Caili Shen in 2014–2015. The idea of reducing the general case to the special case of $Y = \#^n S^2 \times S^1$ by using degree one maps was also got independently by Christoforos Neofytidis.

## 2 Virtual Betti numbers of free-by-cyclic groups

Let $n > 1$ be an integer, let $F = F_n$ be the free group on $n$ generators $x_1, x_2, \ldots, x_n$, and let $\Phi : F \to F$ be an isomorphism. A *free-by-cyclic group*

$$F \rtimes_\Phi \mathbb{Z}$$

has the presentation

$$\langle x_1, \ldots, x_n, t \,|\, \Phi(x_i) = t x_i t^{-1}, i = 1, \ldots, n \rangle.$$

The isomorphism $\Phi$ is *atoroidal* if no iterate of $\Phi$ stabilizes a nontrival conjugacy class in $F$. Otherwise, we say $\Phi$ is *toroidal*. It is known that $\Phi$ is atoroidal if and only if $F \rtimes_\Phi \mathbb{Z}$ is word hyperbolic (see [2, 4]).

**Proposition 2.1.** *Let $\Gamma = F \rtimes_\Phi \mathbb{Z}$ be a free-by-cyclic group. Then $vb_1(\Gamma) = \infty$. In fact, for any $N > 0$, there exists a finite index subgroup $\widetilde{F}$ of $F$ and an integer $m > 0$ such that $\Phi^m(\widetilde{F}) = \widetilde{F}$ and the rank of $H_1(\widetilde{F} \rtimes_{\Phi^m} \mathbb{Z})$ is at least $N$.*

The rest of this section is devoted to the proof of Proposition 2.1. The following lemma is obvious. We leave its proof to the reader.

**Lemma 2.2.** *Let $\Gamma = F \rtimes_\Phi \mathbb{Z}$ be a free-by-cyclic group.*

(1) *For any positive integer $m$, $F \rtimes_{\Phi^m} \mathbb{Z}$ is a finite index subgroup of $\Gamma$.*

(2) *Suppose that $\widetilde{F}$ is a finite index subgroup of $F$ satisfying $\Phi(\widetilde{F}) = \widetilde{F}$. Then $\widetilde{F} \rtimes_\Phi \mathbb{Z}$ is a finite index subgroup of $\Gamma$.*

**Lemma 2.3.** *Let $S$ be a compact surface with $\chi(S) < 0$, and let $c \looparrowright S$ be a $\pi_1$-injective immersed circle. Then for any integer $N > 0$, there exists a finite covering map $p_N : S_N \to S$, such that*

$$p_N^{-1}(c) = c_1 \cup c_2 \cup \cdots \cup c_k$$

*is the union of embedded circles, and the homology classes $[c_1], \ldots, [c_k]$ span a subgroup of $H_1(S_N)$ with rank $\geqslant N$.*

*Proof.* Since $\pi_1(S)$ is locally extended residually finite (LERF) (see [10]), $c$ lifts to an embedded circle in a finite cover of $S$. So we may assume $c$ is embedded. Taking a finite cover of $S$ if necessary, we may also assume $\chi(S) < -1$.

If $[c] \neq 0$ in $H_1(S)$, there exists a properly embedded curve $c' \subset S$ such that $c'$ intersects $c$ transversely at exactly one point. Since $\chi(S) < -1$, there exists a properly embedded nonseparating curve $a \subset S$ such that

$$a \cap (c \cup c') = \emptyset.$$

Let $p_N : S_N \to S$ be the $N$-fold cyclic cover dual to $a$. Then

$$p_N^{-1}(c) = c_1 \cup \cdots \cup c_N, \quad p_N^{-1}(c') = c'_1 \cup \cdots \cup c'_N,$$

such that $c_i \cap c'_j = \emptyset$ when $i \neq j$, and $c_i$ intersects $c'_i$ transversely at exactly one point for each $i$. It follows that $[c_1], \dots, [c_N]$ are linearly independent in $H_1(S_N)$, and we are done.

If $[c] = 0$ in $H_1(S)$, then $c$ separates $S$ into two subsurfaces $T_1$ and $T_2$, such that $\partial T_1 = c$. Let $a_1 \subset T_1$ be a nonseparating simple closed curve. If $\partial T_2 \neq c$, choose the double cover $\widetilde{S}$ of $S$ dual to $a_1$, then any lift of $c$ in $\widetilde{S}$ is homologically nonzero. If $\partial T_2 = c$, let $a_2 \subset T_2$ be a nonseparating simple closed curve. Choose the double cover $\widetilde{S}$ of $S$ dual to $a_1 \cup a_2$, then any lift of $c$ in $\widetilde{S}$ is homologically nonzero. Thus we can reduce to the case in the last paragraph. □

**Lemma 2.4.** *If $\Phi$ is toroidal, then $vb_1(\Gamma) = \infty$.*

*Proof.* Identify $F$ as the fundamental group of a compact surface $S$ with $\chi(S) < 0$, then there exists a map $\varphi : S \to S$ inducing $\Phi$. Since $\Phi$ is toroidal, a power of $\Phi$ stabilizes a nontrival conjugacy class in $F$, which is represented by an immersed curve $c \subset S$. Then for any integer $N > 0$, there exists a finite covering map $p_N : S_N \to S$ as in Lemma 2.3.

Note that $\pi_1(S_N)$ is a finite index subgroup of $F$. Since $F$ has only finitely many subgroups of a fixed index, a power of $\Phi$ sends $\pi_1(S_N)$ to itself. Moreover, a power of $\Phi$ sends the conjugacy class represented by each $c_i$ to itself. Let $\Phi^m$ be a common power which fixes $\pi_1(S_N)$ and the conjugacy class represented by each $c_i$. Using Lemma 2.2, $\Gamma_N = \pi_1(S_N) \rtimes_{\Phi^m} \mathbb{Z}$ is a finite index subgroup of $\Gamma$. Since $[c_1], \dots, [c_k]$ span a subgroup of $H_1(S_N)$ with rank $\geqslant N$ and $\varphi_*([c_i]) = [c_i]$, $[c_1], \dots, [c_k]$ also span a subgroup of $H_1(\Gamma_N)$ with rank $\geqslant N$, thus $b_1(\Gamma_N) \geqslant N$. Since $N$ can be arbitrarily large, our conclusion holds. □

*Proof of Proposition* 2.1. When $\Phi$ is toroidal, it follows from Lemma 2.4 that $vb_1(\Gamma) = \infty$.

When $\Phi$ is atoroidal, combining the main results in [1, 6], we get that $\Gamma$ is virtually the fundamental group of a compact special cube complex. It follows from [13, Theorem 14.10] that $\Gamma$ is virtually large, hence $vb_1(\Gamma) = \infty$.

Suppose that $\Gamma_N$ is a finite index subgroup of $\Gamma$ such that $b_1(\Gamma_N) \geqslant N$. Let $\widetilde{F} = \Gamma \cap F$. Then $\widetilde{F}$ is a finite index subgroup of $F$. There exists an integer $m > 0$ such that $\Phi^m(\widetilde{F}) = \widetilde{F}$ and $\widetilde{F} \rtimes_{\Phi^m} \mathbb{Z}$ is a finite index subgroup of $\Gamma_N$. Hence $b_1(\widetilde{F} \rtimes_{\Phi^m} \mathbb{Z}) \geqslant N$. □

## 3 Projecting the maps of reducible 3-manifolds

Suppose that

$$Y = W_1 \# \cdots \# W_k \# (\#_{j=1}^l S^2 \times S^1)$$

is a closed, connected, oriented, reducible 3-manifold, where each $W_i$ is irreducible. We construct $Y$ as follows. Removing $k + 2l$ open 3-balls from $S^3$, we get a compact manifold $U$ with boundary components

$$R_1, \dots, R_k, \quad S'_1, S''_1, S'_2, S''_2, \dots, S'_l, S''_l.$$

Let $W_i^*$ be the manifold obtained by removing an open 3-ball from $W_i$, $i = 1, \dots, k$. Let $W^* = \bigsqcup_{i=1}^k W_i^*$. Let $S_j$ be a copy of $S^2$, $j = 1, \dots, l$. We glue each $W_i^*$ to $U$ along $R_i$, and glue each $S_j \times [0, 1]$ to $U$ along $S'_j$ and $S''_j$. The resulting closed manifold is $Y$. We choose a basepoint $y$ in the interior of $U$.

Let

$$Z = \#_{j=1}^{l} S^2 \times S^1.$$

We define a map $p : Y \to Z$ as follows. Regard $Z$ as the manifold obtained from $Y$ by replacing each $W_i^*$ with a 3-ball $B_i$. The map $p$ restricts to the identity map on $Y \setminus \mathrm{int}(W^*)$, and sends each $W_i^*$ to $B_i$. It is easy to see $p$ is uniquely defined up to homotopy. We call this map $p$ a *pinch*.

For any continuous map $h$ of pointed spaces, let $h_{\pi q}$ be the induced map on $\pi_q$, and $h_{*q}$ be the induced map on $H_q$.

**Proposition 3.1.** *Let $Y, Z$ and $p$ be as above. Let $f : Y \to Y$ be a diffeomorphism fixing the basepoint. Then there exists a diffeomorphism $g : Z \to Z$ such that*

$$(p \circ f)_{\pi 1} = (g \circ p)_{\pi 1}, \tag{3.1}$$

$$(p \circ f)_{\pi 2} = (g \circ p)_{\pi 2}. \tag{3.2}$$

**Corollary 3.2.** *There exists a surjective map $\varphi : \pi_1(Y \rtimes_f S^1) \to \pi_1(Z \rtimes_g S^1)$.*

*Proof.* We have

$$\pi_1(Y \rtimes_f S^1) = \langle \pi_1(Y), t \mid f_{\pi 1}(\eta) = t\eta t^{-1} \text{ for any } \eta \in \pi_1(Y)\rangle,$$
$$\pi_1(Z \rtimes_g S^1) = \langle \pi_1(Z), s \mid g_{\pi 1}(\zeta) = s\zeta s^{-1} \text{ for any } \zeta \in \pi_1(Z)\rangle.$$

Using (3.1), we can extend $p_{\pi 1}$ to a surjective map

$$\varphi : \pi_1(Y \rtimes_f S^1) \to \pi_1(Z \rtimes_g S^1)$$

by letting $\varphi(t) = s$. □

**Remark 3.3.** Proposition 3.1 holds if we can find $g$ such that

$$p \circ f \simeq g \circ p, \tag{3.3}$$

where "$\simeq$" denotes homotopy of maps. In [3], it is showed that (3.3) implies that there exists a degree one map between $Y \times_f S^1$ and $Z \rtimes_g S^1$.

In order to prove Proposition 3.1, we need to understand the generators of the mapping class group of $Y$. Let us briefly recall the generators from [9, Section 3].

The mapping class group of $Y$ is generated by the following four types of homeomorphisms:

1. *Homeomorphisms preserving summands.* These are the homeomorphisms that restrict to the identity on $U$.

2. *Interchanges of homeomorphic summands.* Suppose that $W_{i_1}$ and $W_{i_2}$ are homeomorphic via an orientation preserving homeomorphism. Then we can construct a homeomorphism of $Y$ fixing all other summands, leaving $U$ invariant, and interchanging $W_{i_1}^*$ and $W_{i_2}^*$. Similarly, we can interchange any $S_{j_1} \times [0,1]$ and $S_{j_2} \times [0,1]$, leaving $U$ invariant.

3. *Spins of $S^2 \times S^1$ summands.* For each $j \in \{1, \dots, l\}$, there is a homeomorphism of $Y$ fixing all other summands, leaving $U$ invariant, interchanging $S_j'$ and $S_j''$, and restricting to an orientation preserving homeomorphism of $S_j \times [0,1]$ that interchanges its two boundary components.

4. *Slide homeomorphisms.* For $i \in \{1, \dots, k\}$, let $Y_i$ be obtained from $Y$ by replacing $W_i^*$ with a 3-ball $E$. Let $\alpha$ be an arc in $Y_i$ which meets $E$ only in its endpoints. There exists an isotopy $J_t$ of $Y_i$, such that $J_t$ moves $E$ around $\alpha$, $J_0 = \mathrm{id}_{Y_i}$ and $J_1 \mid E = \mathrm{id}_E$. We can extend $J_1 \mid (Y_i \backslash E)$ to a homeomorphism $h$ of $Y$ such that $h$ restricts to the identity on $W_i^*$. We call this $h$ the *slide homeomorphism* that slides $W_i^*$ around $\alpha$. Let $T$ be the frontier of a neighborhood of $W_i^* \cup \alpha$ in $Y$, then $h$ is isotopic to a certain Dehn twist about $T$.

Similarly, we can slide either end of $S_j \times [0,1]$ around an arc in $Y \setminus (S_j \times (0,1))$.

**Remark 3.4.** Strictly speaking, to define a slide homeomorphism, we need to fix a frame over $\alpha$. Thus two slide homeomorphisms around $\alpha$ may differ by a rotation about $R_i$ or $S_j$. It turns out that each $\alpha$ corresponds to two isotopy classes of homeomorphisms. This ambiguity will not affect our later proofs.

**Remark 3.5.** If $\alpha_1$ and $\alpha_2$ are two proper arcs in $Y \setminus \mathrm{int}(W_i^*)$, and $\alpha$ is an arc representing the product of $\alpha_1$ and $\alpha_2$ in $\pi_1(Y, W_i^*)$, then the slide around $\alpha$ is isotopic to a composition of slides around $\alpha_1$ and $\alpha_2$. There is a similar product relation for sliding ends of $S_j \times [0,1]$.

In Lemmas 3.6 and 3.7 below, we prove Proposition 3.1 for two special types of slide homeomorphisms.

We first set up the notation for Lemma 3.6. Let $\alpha \subset U \cup W_1^*$ be a proper arc such that $\partial\alpha \subset S_1'$ and $\alpha \cap W_1^*$ is a single arc. Let $T$ be the frontier of a neighborhood of $S_1' \cup \alpha$ in $U \cup W_1^*$, and let $\tau_\alpha$ be a Dehn twist about $T$, which is isotopic to the slide homeomorphism that slides $S_1 \times \{0\}$ around $\alpha$. Let $\beta \subset U \cup B_1$ be a proper arc which coincides with $\alpha$ in $U$, and $\beta \cap B_1$ is a trivial arc parallel to the boundary. Let $T'$ be the frontier of a neighborhood of $S_1' \cup \beta$ in $U \cup B_1$, and let $\tau_\beta$ be a Dehn twist about $T'$ corresponding to the slide homeomorphism.

**Lemma 3.6.** *With the above notation, we have*

$$(p \circ \tau_\alpha)_{\pi 1} = (\tau_\beta \circ p)_{\pi 1}, \tag{3.4}$$

$$(p \circ \tau_\alpha)_{\pi 2} = (\tau_\beta \circ p)_{\pi 2}. \tag{3.5}$$

*Proof.* The fundamental group $\pi_1(Y)$ is generated by two types of loops:

• a Type-$W$ loop is a loop $\gamma$ supported in $U \cup W_i^*$ for some $i$, and $\gamma$ intersects $R_i$ transversely exactly twice;

• a Type-$Z$ loop is a loop $\delta$ supported in $U \cup (S_j \times [0,1])$ for some $j$, and $\delta$ intersects $S_j'$ transversely exactly once.

In order to prove (3.4), we only need to evaluate its both sides on each of the above loops and check the evaluations are equal.

Let $\gamma$ be a Type-$W$ loop. We may homotope $\gamma$ so that it is disjoint from $T$ and $p(\gamma)$ is disjoint from $T'$. Then $p\tau_\alpha(\gamma) = p(\gamma)$, and $\tau_\beta p(\gamma) = p(\gamma)$.

Checking (3.4) for a Type-$Z$ loop $\delta$ is trivial when $j \neq 1$, since $p$ fixes $\delta$ and $\delta$ can be homotoped to be disjoint from both $T$ and $T'$.

When $j = 1$, write $\delta$ as the composition of two paths $\delta_1$ and $\delta_2$, where $\delta_1 \cap \delta_2 = \delta \cap S_1'$. Let $\widehat{\alpha}$ be a loop obtained from $\alpha$ by connecting the two ends of $\alpha$ by an arc in $S_1'$ containing $\delta_1 \cap \delta_2$. Define a loop $\widehat{\beta}$ similarly. Then $p(\delta) = \delta$, $\tau_\alpha(\delta) = \delta_1\widehat{\alpha}\delta_2$, and $\tau_\beta(\delta) = \delta_1\widehat{\beta}\delta_2$. Since $p(\widehat{\alpha})$ and $\widehat{\beta}$ differ only in $B_1$, $p(\delta_1\widehat{\alpha}\delta_2) = \delta_1 p(\widehat{\alpha})\delta_2$ is homotopic to $\delta_1\widehat{\beta}\delta_2$. So the evaluation of (3.4) on $\delta$ holds.

Now we proceed to prove (3.5). Let $\xi \subset U$ be a path connecting the base point $y$ to a point $r_i \in R_i$, such that $\xi$ is generic in the sense that it is disjoint from $\alpha \cup S_1'$, and $p(\xi)$ is disjoint from $\beta$. We then construct a map $\rho_i : (D^2, \partial D^2) \to (Y, y)$, such that $\rho_i$ collapses a product neighborhood $\partial D^2 \times [0, \epsilon]$ of $\partial D^2$ onto $\xi$, and sends $D^2 \setminus (\partial D^2 \times [0, \epsilon])$ homeomorphically to $R_i \setminus \{r_i\}$. Similarly, using $S_j''$ and a generic path in $U$ connecting $y$ to $S_j''$, we can construct a map $\sigma_j : (D^2, \partial D^2) \to (Y, y)$.

We claim that as a $\pi_1(Y)$-module $\pi_2(Y)$ is generated by the homotopy classes $[\rho_i]$, $i = 1, \ldots, k$, and $[\sigma_j]$, $j = 1, \ldots, l$. To see this, let $\overline{Y}$ and $\overline{W_i^*}$ be the universal covers of $Y$ and $W_i^*$, and we only need to check that $H_2(\overline{Y}) \cong \pi_2(Y)$ is generated by the homology classes of the lifts of $R_i$'s and $S_j''$'s. Note that $\overline{Y}$ is obtained by gluing copies of $\overline{W_i^*}, U$ and $S_j \times [0,1]$ together. In each of the copies, $H_2$ is generated by the homology classes of the lifts of $R_i$'s, $S_j'$'s and $S_j''$'s, and $S_j'$ and $S_j''$ are homologous through the homology $S_j \times [0,1]$. So the claim is proved.

In light of the above claim and (3.4), to prove (3.5) we only need to check its evaluation on each $[\rho_i]$ and $[\sigma_j]$. Clearly, $p_{\pi 2}([\sigma_j]) = [\sigma_j]$. Since $T \subset U \cup W_1^*$ is always disjoint from $S_j''$, we have $(\tau_\alpha)_{\pi 2}([\sigma_j]) = [\sigma_j]$. Similarly, $(\tau_\beta)_{\pi 2}([\sigma_j]) = [\sigma_j]$.

For $[\rho_i]$, when $i \neq 1$, the result can be proved as in the last paragraph. When $i = 1$, note that the only difference between $p\tau_\alpha\rho_1(D^2)$ and $\tau_\beta p\rho_1(D^2)$ is in $B_1$, where each restriction is a pair of immersed annuli. Clearly, the two pairs of annuli are homotopic rel $\partial$ in $B_1$, so

$$(p\tau_\alpha)_{\pi 2}([\rho_1]) = (\tau_\beta p)_{\pi 2}([\rho_1]).$$

So (3.5) is proved. □

Now we set up the notation for Lemma 3.7. Assume $k \geqslant 2$. Let $\alpha \subset U \cup W_1^*$ be a proper arc such that $\partial\alpha \subset R_2$ and $\alpha \cap W_1^*$ is a single arc. Let $\beta \subset U \cup B_1$ be a proper arc which coincides with $\alpha$ in $U$, and $\beta \cap B_1$ is a trivial arc parallel to the boundary. Define $T$ and $T'$ similarly as before. Let $\tau_\alpha$ be the Dehn twist about $T$ which slides $W_2^*$ around $\alpha$, and $\tau_\beta$ be the Dehn twist about $T'$ which slides $B_2$ around $\beta$. It is worth pointing out that $\tau_\beta$ is isotopic to the identity since it just slides a ball in $Z$.

**Lemma 3.7.** *With the above notation, we also have* (3.4) *and* (3.5).

*Proof.* The proof is similar to the proof of Lemma 3.6, so we omit most details. The only place that we need to address is the case that $\gamma$ is a Type-$W$ loop in $U \cup W_2^*$ in the proof of (3.4). In this case, both $p\tau_\alpha(\gamma)$ and $\tau_\beta p(\gamma)$ are loops in $U \cup (B_1 \cup B_2)$ which is simply connected. So the evaluation of (3.4) holds in this case. □

*Proof of Proposition* 3.1. Notice that if the conclusion holds for two homeomorphisms $f_1$ and $f_2$ of $Y$, and the corresponding homeomorphisms of $Z$ are $g_1$ and $g_2$, then the conclusion also holds for $f_1 \circ f_2$, since

$$(p \circ f_1 \circ f_2)_{\pi q} = (g_1 \circ p \circ f_2)_{\pi q} = (g_1 \circ g_2 \circ p)_{\pi q}, \quad q = 1, 2.$$

So we only need to prove our result for each of the four types of homeomorphisms listed before.

If $f$ is one of the first three types, we claim that we can always find a homeomorphism $g$ of $Z$ satisfying (3.3). In fact, if $f$ restricts to the identity on $U$ or $f$ interchanges two homeomorphic summands $W_{i_1}$ and $W_{i_2}$, it is clear that $g = \mathrm{id}$ will suffice. If $f$ interchanges $S_{j_1} \times [0,1]$ and $S_{j_2} \times [0,1]$, $g$ can be chosen to be the corresponding interchange in $Z$. If $f$ is the spin of an $S^2 \times S^1$ summand, we can choose $g$ to be the corresponding spin in $Z$.

If $f$ is a slide homeomorphism that slides $S_1 \times \{0\}$ around a proper arc $\alpha \subset Y_1' = Y \setminus (S_1 \times (0,1))$, we can use Remark 3.5 to factorize $\alpha$ into simpler arcs. In $\pi_1(Y_1', S_1')$, $\alpha$ is the product of two types of arcs: arcs supported in $U \cup W_i^*$ for some $i$, such that the intersection of each arc with $W_i^*$ is a single arc; and arcs disjoint from $W^*$. If $f$ is the slide around an arc of the first type, we can use Lemma 3.6. If $f$ is the slide around an arc disjoint from $W^*$, since $p$ restricts to the identity outside $W^*$, we can choose $g$ to be the same slide in $Z$.

If $f$ is a slide homeomorphism that slides $W_i^*$ around a proper arc $\alpha \subset Y_i$, we can use Lemma 3.7 and Remark 3.5 as before to get our conclusion. □

## 4 Proofs of the main theorems

The proofs of Theorems 1.1 and 1.2 are based on the following lemma.

**Lemma 4.1** (See [7, Lemma 5.4])**.** *Suppose that $Y$ is a closed orientable reducible 3-manifold, and $Y$ is not $S^2 \times S^1$ or $\mathbb{R}P^3\#\mathbb{R}P^3$. Then $Y$ has a finite cover of the form $(S^2 \times S^1)\#(S^2 \times S^1)\#Y'$ for some 3-manifold $Y'$.*

*Proof of Theorem* 1.1. If $Y$ is not $S^2 \times S^1$ or $\mathbb{R}P^3\#\mathbb{R}P^3$, using Lemma 4.1, $Y$ has a finite cover $\widetilde{Y}$ of the form $W\#(\#_{j=1}^l S^2 \times S^1)$ for some 3-manifold $W$ containing no $S^2 \times S^1$ summand, and $l \geqslant 2$. By [7, Lemma 2.1], $X$ has a finite cover $\widetilde{X}$ of the form $\widetilde{Y} \rtimes S^1$. By Corollary 3.2, $\pi_1(\widetilde{X})$ surjects onto the fundamental group of a manifold of the form $(\#_{j=1}^l S^2 \times S^1) \rtimes S^1$. It follows from Proposition 2.1 that $vb_1(X) = vb_1(\widetilde{X}) = \infty$. □

The following lemma is elementary and we leave its proof to the reader.

**Lemma 4.2.** *Let $Y$ be a closed oriented 3-manifold, $f: Y \to Y$ be an orientation preserving homeomorphism, $X = Y \rtimes_f S^1$. Let*

$$b = \mathrm{rank}\ \ker(f_{*1} - \mathrm{id}) = \mathrm{rank}\ \ker(f_{*2} - \mathrm{id}).$$

*Then $b_1(X) = b + 1$, $b_2^+(X) = b_2^-(X) = b$.*

*Proof of Theorem* 1.2. Suppose $X = Y \rtimes_f S^1$ is symplectic, then $b_2^+(X) > 0$. By Lemma 4.2, $b_1(Y) > 0$. If $Y = S^2 \times S^1$, Lemma 4.2 implies that $f$ induces the identity on $H_*(Y)$. Up to isotopy, we may assume that $f$ preserves each $S^2 \times \{t\}$. By [7, Lemma 2.2], $X$ is an $S^2$-bundle over $T^2$. So $X = S^2 \times T^2$ or $S^2 \widetilde{\times} T^2$.

If $Y \neq S^2 \times S^1$, as in the proof of Theorem 1.1, replacing $X$ with its finite cover if necessary, we may assume

$$Y = W_1 \# \cdots \# W_k \# Z$$

as in Section 3, where $Z = \#_{j=1}^{l} S^2 \times S^1$ for some $l \geqslant 2$.

Let $p : Y \to Z$ be a pinch. By Proposition 3.1, there exists a homeomorphism $g$ of $Z$ satisfying (3.1) and (3.2). By Proposition 2.1, there exists a finite index subgroup $G$ of $\pi_1(Z)$ and an integer $m > 0$, such that $g_{\pi 1}^m$ preserves $G$ and

$$b_1(G \rtimes_{g_{\pi 1}^m} \mathbb{Z}) > 2. \tag{4.1}$$

Let $\widetilde{Z}$ be the cover of $Z$ corresponding to $G$, and let $\widetilde{Y}$ be the cover of $Z$ corresponding to $p_{\pi 1}^{-1}(G)$. The pinch $p$ lifts to a map $\widetilde{p} : \widetilde{Y} \to \widetilde{Z}$. Since $p_{\pi 1}$ kills each $\pi_1(W_i^*)$, $\pi_1(W_i^*) < p_{\pi 1}^{-1}(G)$, so $W_i^*$ lifts to disjoint copies of $W_i^*$ in $\widetilde{Y}$, and $\widetilde{p}$ collapses each copy of $W_i^*$ to a ball. It is clear that $\widetilde{p}$ restricts to the identity outside the copies of the $W_i^*$'s. Let $H_2^h(\cdot)$ be the image of the Hurewicz homomorphism $\pi_2(\cdot) \to H_2(\cdot)$. It follows that $\widetilde{p}_{*2}$ restricts to an isomorphism $H_2^h(\widetilde{Y}) \to H_2^h(\widetilde{Z})$.

Since $g_{\pi 1}^m$ preserves $G$, there exists a homeomorphism $\psi : \widetilde{Z} \to \widetilde{Z}$ which is a lift of $g^m$. By (3.1), $f_{\pi 1}^m$ preserves $p_{\pi 1}^{-1}(G)$, so $f^m$ also lifts to $\varphi : \widetilde{Y} \to \widetilde{Y}$. Since $\pi_2$ is unchanged after taking covers, (3.2) implies that

$$(\widetilde{p} \circ \varphi)_{\pi 2} = (\psi \circ \widetilde{p})_{\pi 2}. \tag{4.2}$$

It follows from (4.2) that

$$(\widetilde{p} \circ \varphi)_{*2} = (\psi \circ \widetilde{p})_{*2} \quad \text{on} \quad H_2^h(\widetilde{Y}). \tag{4.3}$$

By Lemma 4.2 and (4.1), rank $\ker(\psi_{*2} - \mathrm{id}) > 1$. Using (4.3) and the fact that $\widetilde{p}_{*2}$ restricts to an isomorphism on $H_2^h$,

$$\mathrm{rank}(\ker(\varphi_{*2} - \mathrm{id}) \cap H_2^h(\widetilde{Y})) > 1.$$

So $b_2^+(\widetilde{X}) > 1$, and there exists an embedded 2-sphere $S \subset \widetilde{Y}$ such that

$$[S] \in \ker(\varphi_{*2} - \mathrm{id}).$$

It follows that $[S] \neq 0$ in $H_2(\widetilde{Y} \rtimes_\varphi S^1)$, and the self-intersection number $[S]^2$ is zero. Using [5, Lemma 5.1] and [11], $\widetilde{Y} \rtimes_\varphi S^1$ is not symplectic. Thus $X$ is not symplectic. □

*Proof of Corollary* 1.4. If $Y$ is $S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$, then $Y$ is covered by $S^2 \times S^1$. Using [7, Lemmas 2.1 and 2.2], $X$ is covered by an $S^2$-bundle over $T^2$, which is symplectic by [12].

If $Y$ is not $S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$, then none of the covers of $Y$ is. It follows from Theorem 1.2 that none of the finite covers of $X$ is symplectic. □

**Acknowledgements** This work was supported by National Science Foundation of USA (Grant No. DMS-1252992) and an Alfred P. Sloan Research Fellowship. The author thanks a referee for clarifying some arguments in geometric group theory.

## References


1 Agol I. The virtual Haken conjecture, with an appendix by Ian Agol, Daniel Groves, and Jason Manning. Doc Math, 2013, 18: 1045–1087

2 Bestvina M, Feighn M. A combination theorem for negatively curved groups. J Differential Geom, 1992, 35: 85–101

3 Boileau M, Ni Y, Wang S. Pseudo-Anosov extensions and degree one maps between hyperbolic surface bundles. Math Z, 2007, 256: 913–923

4 Brinkmann P. Hyperbolic automorphisms of free groups. Geom Funct Anal, 2000, 10: 1071–1089

5 Fintushel R, Stern R. Immersed spheres in 4-manifolds and the immersed Thom conjecture. Turkish J Math, 1995, 19: 145–157

6 Hagen M, Wise D. Cubulating hyperbolic free-by-cyclic groups: The general case. Geom Funct Anal, 2015, 25: 134–179

7 Li T, Ni Y. Virtual Betti numbers and virtual symplecticity of 4-dimensional mapping tori. Math Z, 2014, 277: 195–208

8 McCarthy J. On the asphericity of a symplectic $M^3 \times S^1$. Proc Amer Math Soc, 2001, 129: 257–264

9 McCullough D. Mappings of reducible 3-manifolds. In: Geometric and Algebraic Topology. Banach Center Publications, vol. 18. Warszawa: Polish Scientific Publishers, 1986, 61–76

10 Scott P. Subgroups of surface groups are almost geometric. J Lond Math Soc (2), 1978, 17: 555–565

11 Taubes C. The Seiberg-Witten invariants and symplectic forms. Math Res Lett, 1994, 1: 809–822

12 Thurston W. Some simple examples of symplectic manifolds. Proc Amer Math Soc, 1976, 55: 467–468

13 Wise D. From Riches to Raags: 3-Manifolds, Right-Angled Artin Groups, and Cubical Geometry. CBMS Regional Conference Series in Mathematics, vol. 117. Providence: Amer Math Soc, 2012