

# Two Applications of Twisted Floer Homology

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Given an irreducible closed three-manifold  $Y$ , we show that its twisted Heegaard Floer homology determines whether  $Y$  is a torus bundle over the circle. Another result we will prove is, if  $K$  is a genus-1 null-homologous knot in an  $L$ -space, and the zero surgery on  $K$  is fibered, then  $K$  itself is fibered. These two results are the missing cases of earlier results due to the second author.

## 1 Introduction

Heegaard Floer homology was introduced by Peter Ozsváth and Zoltán Szabó in [14]. It associates to every closed oriented three-manifold  $Y$ , a package of abelian groups, as invariants. This theory also provides an invariant, called knot Floer homology, for every null-homologous knot in a three-manifold [17, 22].

Heegaard Floer homology turns out to be very powerful. For example, it determines the Thurston norm of a three-manifold and the genus of a knot [19]. Ghiggini [6] and Ni [12] showed that knot Floer homology detects fibered knots. An analog of this result for closed manifolds was proved by Ni [13], namely, Heegaard Floer homology

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detects whether a closed three-manifold fibers over the circle when the genus of the fiber is greater than 1. The precise statement is as follows.

**Theorem** (Ni [13]). Suppose  $Y$  is a closed irreducible three-manifold and  $F \subset Y$  is a closed connected surface of genus  $g > 1$ . Let  $HF^+(Y, [F], g-1)$  denote the group

$$\bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y), \langle c_1(\mathfrak{s}), [F] \rangle = 2g-2} HF^+(Y, \mathfrak{s}).$$

If  $HF^+(Y, [F], g-1) \cong \mathbb{Z}$ , then  $Y$  fibers over the circle with  $F$  as a fiber.

The above theorem does not hold when  $g = 1$ . In this case, the group  $HF^+(Y, [F], 0)$  is always an infinitely generated group. Ozsváth and Szabó suggested that one may use Heegaard Floer homology with twisted coefficients in some Novikov ring. Some calculations for torus bundles have been done in an earlier paper [1] along this line.

As in [15], there is a twisted Heegaard Floer homology  $\underline{HF}^+(Y; \Lambda_\omega)$ , where  $\Lambda_\omega$  is the universal Novikov ring  $\Lambda$  equipped with a  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module structure, which will be defined in Section 2.1. In [1], this twisted Heegaard Floer homology is calculated for torus bundles.

**Theorem 1.1** [1]. Suppose  $\pi: Y \rightarrow S^1$  is a fiber bundle with torus fiber  $F$  and  $\omega \in H^2(Y; \mathbb{Z})$  is a cohomology class such that  $\omega([F]) \neq 0$ . Then, we have an isomorphism of  $\Lambda$ -modules

$$\underline{HF}^+(Y; \Lambda_\omega) \cong \Lambda.$$

The above theorem complements a result of Ozsváth and Szabó [16], which states that a surface bundle over circle with fiber of genus  $> 1$  has monic Heegaard Floer homology.

In the current paper, we prove the converse to the above theorem. Our main result is as follows.

**Theorem 1.2.** Suppose  $Y$  is a closed irreducible three-manifold and  $F \subset Y$  is an embedded torus. If there is a cohomology class  $\omega \in H^2(Y; \mathbb{Z})$  such that  $\omega([F]) \neq 0$  and  $\underline{HF}^+(Y; \Lambda_\omega) \cong \Lambda$ , then  $Y$  fibers over the circle with  $F$  as a fiber.

**Remark 1.3.** The proof of Theorem 1.2 is based on Ghiggini's argument in [6]. The only new ingredient we introduce here is twisted coefficients. In the setting of Monopole Floer homology, a corresponding version of this theorem was proved in [10, Theorem 42.7.1], following Ghiggini's argument.

Besides Theorem 1.2, we give an application of Theorem 1.1.

**Theorem 1.4.** Suppose  $Y$  is an  $L$ -space and  $K \subset Y$  is a genus-1 null-homologous knot. If the zero surgery on  $K$  fibers over  $S^1$ , then  $K$  itself is a fibered knot.

The case where  $K$  has genus  $g > 1$  has been proved in [12, Corollary 1.4].

This paper is organized as follows. In Section 2, we collect some preliminary results on Heegaard Floer homology with an emphasis on twisted coefficients. In Section 3, we prove a key proposition which relates the Euler characteristic of  $\omega$ -twisted Heegaard Floer homology with Turaev torsion. With the help of this proposition, we can prove a homological version (Proposition 3.5) of the main theorem. In Section 4, we give a proof of Theorem 1.2 following Ghiggini's argument. In Section 5, we prove Theorem 1.4 by using Theorem 1.1 and the exact sequence for  $\omega$ -twisted Floer homology from [1].

## 2 Preliminaries on Heegaard Floer Homology

We review some of the constructions in Heegaard Floer homology, which will be used throughout this paper. The details can be found in [1, 15, 17–19].

### 2.1 Heegaard Floer homology with twisted coefficients

Let  $Y$  be a closed oriented three-manifold and  $\mathfrak{t}$  be a  $\text{Spin}^c$  structure over  $Y$ . Ozsváth and Szabó [15] defined a universally twisted chain complex  $\underline{CF}^\circ(Y, \mathfrak{t})$  with coefficients in the group ring  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ . Its homology  $\underline{HF}^\circ(Y, \mathfrak{t})$  is an invariant of the pair  $(Y, \mathfrak{t})$ . Furthermore, given any module  $M$  over  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ , they defined Floer homology with coefficients twisted by  $M$ :

$$\underline{HF}^\circ(Y, \mathfrak{t}; M) = H_*(\underline{CF}^\circ(Y, \mathfrak{t}) \otimes_{\mathbb{Z}[H^1(Y; \mathbb{Z})]} M).$$

This construction recovers the ordinary Heegaard Floer homology if we take  $M$  to be the trivial  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module  $\mathbb{Z}$ .

A special twisted Floer homology is used to investigate torus bundles over the circle in [1]. Consider the universal Novikov ring [11, Section 11.1]

$$\Lambda = \left\{ \sum_{r \in \mathbb{R}} a_r t^r \mid a_r \in \mathbb{R}, \#\{a_r \mid a_r \neq 0, r \leq c\} < \infty \text{ for any } c \in \mathbb{R} \right\}.$$

Here  $\Lambda$  itself is a field. Given a cohomology class  $\omega \in H^2(Y; \mathbb{R})$ , there is a group homomorphism

$$\begin{aligned} H^1(Y; \mathbb{Z}) &\rightarrow \mathbb{R} \\ h &\mapsto \langle h \cup \omega, [Y] \rangle, \end{aligned}$$

which then induces a ring homomorphism

$$\begin{aligned} \mathbb{Z}[H^1(Y; \mathbb{Z})] &\rightarrow \Lambda \\ \sum a_h \cdot h &\mapsto \sum a_h \cdot t^{\langle h \cup \omega, [Y] \rangle}. \end{aligned}$$

In this way, we can equip  $\Lambda$  with an induced  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module structure. We denote this module by  $\Lambda_\omega$ .

This  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module  $\Lambda_\omega$  gives rise to a twisted Heegaard Floer homology  $\underline{HF}^+(Y; \Lambda_\omega)$ , which is called the  $\omega$ -twisted Heegaard Floer homology. More precisely, it is defined as follows (for more details, see [19] and [1]). Choose an admissible pointed Heegaard diagram  $(\Sigma, \alpha, \beta, z)$  for  $Y$ . Every Whitney disk  $\phi : \mathbb{D}^2 \rightarrow \text{Sym}^g(\Sigma)$  gives rise to a two-chain in  $Y$ . Let  $\eta$  be a closed two-cochain that represents  $\omega$ . The evaluation of  $\eta$  on  $\phi$  is denoted by  $\langle \eta, \phi \rangle$ . Take the  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module freely generated by all the pairs  $[\mathbf{x}, i]$  (where  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  and the integer  $i \geq 0$ ), its tensor product with the module  $\Lambda_\omega$  is the  $\omega$ -twisted chain complex  $\underline{CF}^+(Y; \Lambda_\omega)$ . We define the differential on the complex by the formula

$$\partial^+[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi)=1\}} \# \widehat{\mathcal{M}}(\phi)[\mathbf{y}, i - n_z(\phi)] \cdot t^{\langle \eta, \phi \rangle}.$$

The homology of this chain complex depends only on the cohomology class  $\omega$ . We call this homology the  $\omega$ -twisted Heegaard Floer homology  $\underline{HF}^+(Y; \Lambda_\omega)$ . This homology also has a  $\Lambda$ -module structure. Since  $\Lambda$  is a field, a  $\Lambda$ -module is actually a vector space over  $\Lambda$ . As is usual in Heegaard Floer homology, there is a map  $U$  on  $\underline{HF}^+(Y; \Lambda_\omega)$ , which makes  $\underline{HF}^+(Y; \Lambda_\omega)$  a  $\Lambda[U]$ -module.

Theorem 1.1 shows that this  $\omega$ -twisted Floer homology is very simple for torus bundles over  $S^1$  when  $\omega$  evaluates nontrivially on the fiber. The key ingredients in the proof are an exact sequence for  $\omega$ -twisted Heegaard Floer homology and the adjunction inequality.

**Theorem 2.1** [1]. Let  $K \subset Y$  be a framed knot in a three-manifold  $Y$  and  $\gamma \subset Y - K$  be a simple closed curve in the knot complement. For every rational number  $r$ ,  $\gamma$  is a curve in the surgery manifold  $Y_r(K)$ . Let  $\omega_r = \text{PD}(\gamma) \in H^2(Y_r(K); \mathbb{R})$  and  $\omega = \text{PD}(\gamma) \in H^2(Y; \mathbb{R})$ . Then,

we have the following exact sequence:

$$\begin{array}{ccc}
 \underline{HF}^+(Y; \Lambda_\omega) & \xrightarrow{\quad\quad\quad} & \underline{HF}^+(Y_0(K); \Lambda_{\omega_0}) \\
 & \nwarrow \quad \nearrow & \\
 & \underline{HF}^+(Y_1(K); \Lambda_{\omega_1}) &
 \end{array} \tag{1}$$

The maps in the above sequence are induced from cobordism.

The proof of the above theorem in [1] can be applied to general integer surgeries. As in [15, Theorem 9.19], suppose  $\mathfrak{s}$  is a  $\text{Spin}^c$  structure on  $Y$ ,  $\mathfrak{s}_k$  is any one of the  $p$   $\text{Spin}^c$  structures on  $Y_p(K)$ , which are  $\text{Spin}^c$ -cobordant to  $\mathfrak{s}$ , and  $Q: \text{Spin}^c(Y_0) \rightarrow \text{Spin}^c(Y_p)$  is the surjective map defined in the proof of [15, Theorem 9.19]. Let

$$\underline{HF}^+(Y_0(K), [\mathfrak{s}_k]; \Lambda_{\omega_0}) = \bigoplus_{\mathfrak{t} \in \text{Spin}^c(Y_0(K)), Q(\mathfrak{t}) = \mathfrak{s}_k} \underline{HF}^+(Y_0(K), \mathfrak{t}; \Lambda_{\omega_0}).$$

The proof of the following theorem is essentially the same as [15, Theorem 9.19].

**Theorem 2.2.** Notation is as in Theorem 2.1. For each positive integer  $p$ , the following sequence

$$\begin{array}{ccc}
 \underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega) & \xrightarrow{\quad\quad\quad} & \underline{HF}^+(Y_0(K), [\mathfrak{s}_k]; \Lambda_{\omega_0}) \\
 & \nwarrow \quad \nearrow & \\
 & \underline{HF}^+(Y_p(K), \mathfrak{s}_k; \Lambda_{\omega_p}) &
 \end{array}$$

is exact. Here, the map  $F_3^+$  is induced by a two-handle cobordism connecting  $Y_p(K)$  to  $Y$ .

Combining Theorem 2.1 and the fact that if  $Y$  contains a nonseparating sphere  $S$  and  $\omega([S]) \neq 0$  then  $\underline{HF}^+(Y; \Lambda_\omega) = 0$  (see [1, Proposition 2.2]), we can easily get the following corollary, which is used implicitly in [1] to prove Theorem 1.1.

**Corollary 2.3.** Suppose  $F$  is an embedded torus in a closed manifold  $Y$  and  $\omega \in H^2(Y; \mathbb{Z})$  is a cohomology class such that  $\omega([F]) \neq 0$ . Let  $Y'$  be the manifold obtained by cutting open  $Y$  along  $F$  and regluing by a self-homeomorphism of  $F$ . There is a cobordism  $W: Y \rightarrow Y'$  obtained by adding two-handles along knots in  $F$ . Take a closed curve  $\gamma \subset Y$ , missing the attaching regions of the two-handles such that  $\text{PD}(\gamma) = \omega \in H^2(Y; \mathbb{Z})$ . As in Theorem 2.1,  $\gamma$  also determines a cohomology class  $\omega'$  in  $Y'$  by setting  $\omega' = \text{PD}_{Y'}(\gamma)$ . Then, the map induced by cobordism is an isomorphism

$$\underline{HF}^+(Y'; \Lambda_{\omega'}) \cong \underline{HF}^+(Y; \Lambda_\omega).$$

## 2.2 Knot Floer homology

Suppose  $K \subset Y$  is a null-homologous knot in a rational homology 3-sphere and  $F$  is a fixed Seifert surface. There is a compatible doubly pointed Heegaard diagram  $(\Sigma, \alpha, \beta, w, z)$  for the knot  $K$  as in [17]. There is a map from intersection points between the two tori  $\mathbb{T}_\alpha, \mathbb{T}_\beta$  to relative  $\text{Spin}^c$  structures on  $Y - K$

$$s_{w,z}: \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow \underline{\text{Spin}}^c(Y, K) = \text{Spin}^c(Y_0(K)).$$

For each  $\text{Spin}^c$  structure  $\mathfrak{s}$  on  $Y$ , the knot Floer chain complex

$$C(\mathfrak{s}) = CFK^\infty(Y, K; \mathfrak{s})$$

is a free abelian group generated by  $[\mathbf{x}, i, j] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z} \times \mathbb{Z}$ , such that  $s_{w,z}(\mathbf{x})$  extends  $\mathfrak{s}$  and

$$\frac{\langle c_1(s_{w,z}(\mathbf{x})), [\widehat{F}] \rangle}{2} + (i - j) = 0,$$

and endowed with the differential

$$\partial[\mathbf{x}, i, j] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = 1\}} \# \widehat{\mathcal{M}}(\phi)[\mathbf{y}, i - n_w(\phi), j - n_z(\phi)].$$

This complex is given a filtration function  $\mathcal{F}[\mathbf{x}, i, j] = (i, j)$ . The forgetful map  $[\mathbf{x}, i, j] \mapsto [\mathbf{x}, i]$  induces an isomorphism between  $C(\mathfrak{s})$  and  $CF^\infty(Y, \mathfrak{s})$ , sending  $B_{\mathfrak{s}}^+ := C(\mathfrak{s})\{i \geq 0\}$  isomorphically to  $CF^+(Y, \mathfrak{s})$ . For each integer  $d$ , we define

$$\widehat{HFK}(Y, K, \mathfrak{s}; d) = H_*(C(\mathfrak{s})\{i = 0, j \leq d\} / C(\mathfrak{s})\{i = 0, j \leq d - 1\}, \partial).$$

Define  $A_{\mathfrak{s},k}^+ = C(\mathfrak{s})\{\max(i, j - k) \geq 0\}$ . There is a canonical chain map  $v_{\mathfrak{s},k}^+: A_{\mathfrak{s},k}^+ \rightarrow B_{\mathfrak{s}}^+$  which is the projection onto  $C(\mathfrak{s})\{i \geq 0\}$ .

The following theorem is a version of [17, Theorem 4.4], see also [21, Theorem 2.3].

**Theorem 2.4.** Let  $K \subset Y$  be a null-homologous knot in a rational homology sphere. There is an integer  $N$  with the property that for all  $m \geq N$  and all  $t \in \mathbb{Z}/m\mathbb{Z}$ ,  $CF^+(Y_m(K), \mathfrak{s}_t)$  is represented by the chain complex  $A_{\mathfrak{s},k}^+ = C(\mathfrak{s})\{\max(i, j - k) \geq 0\}$ , where  $k \equiv t \pmod{m}$  and  $|k| \leq \frac{m}{2}$ , in the sense that there are isomorphisms

$$\Psi_{m,k}^+: CF^+(Y_m(K), \mathfrak{s}_t) \rightarrow A_{\mathfrak{s},k}^+.$$

Moreover, let  $\mathfrak{s}_k$  and  $\mathfrak{y}_k$  denote the  $\text{Spin}^c$  structures over the cobordism

$$W'_m(K): Y_m(K) \rightarrow Y$$

with

$$\langle c_1(\mathfrak{x}_k), [\widehat{F}] \rangle + m = 2k \text{ and } \langle c_1(\mathfrak{y}_k), [\widehat{F}] \rangle - m = 2k,$$

respectively. Then,  $v_k^+$  corresponds to the map induced by the cobordism  $W'_m(K)$  endowed with the  $\text{Spin}^c$  structure  $\mathfrak{x}_k$ .

Let  $h_k^+$  denote the map induced by the cobordism  $W'_m(K)$  endowed with the  $\text{Spin}^c$  structure  $\mathfrak{y}_k$ .

### 2.3 Ozsváth–Szabó contact invariant in twisted Floer homology

Ozsváth and Szabó [18] defined an invariant  $c(\xi) \in \widehat{HF}(-Y)$  for every contact structure  $\xi$  on a closed three-manifold  $Y$ . It is defined up to sign and lies in the summand  $\widehat{HF}(-Y, \mathfrak{t}_\xi)$  corresponding to the canonical  $\text{Spin}^c$  structure  $\mathfrak{t}_\xi$  associated to  $\xi$ .

In [19], Ozsváth and Szabó point out that the contact invariant can be defined for twisted Heegaard Floer homology. In fact, for any module  $M$  over  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$  one can get an element

$$c(\xi; M) \in \widehat{HF}(-Y, \mathfrak{t}_\xi; M) / \mathbb{Z}[H^1(Y, \mathbb{Z})]^\times.$$

This is an element  $c(\xi; M) \in \widehat{HF}(-Y, \mathfrak{t}_\xi; M)$ , which is well defined up to an overall multiplication by a unit in the group ring  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ . Let  $c^+(\xi; M)$  denote the image of  $c(\xi; M)$  under the natural map  $\widehat{HF}(-Y, \mathfrak{t}_\xi; M) \rightarrow \underline{HF}^+(-Y, \mathfrak{t}_\xi; M)$ .

In particular, for  $\omega$ -twisted Heegaard Floer homology  $\underline{HF}^+(Y; \Lambda_\omega)$ , one gets a contact invariant  $c^+(\xi; \Lambda_\omega) \in \underline{HF}^+(-Y, \mathfrak{t}_\xi; \Lambda_\omega)$ . It is well defined up to multiplication by a term  $\pm t^n$  for some  $n \in \mathbb{R}$ . There is also a nonvanishing theorem for weakly fillable contact structures in this  $\omega$ -twisted version. The following theorem is essentially [19, Theorem 4.2], although the setting here is slightly different from [19].

**Theorem 2.5.** Let  $(W, \Omega)$  be a weak filling of a contact manifold  $(Y, \xi)$ , where  $\Omega \in H^2(W; \mathbb{R})$  is the symplectic two-form. Then, the contact invariant  $c^+(\xi; \Lambda_{\Omega|_Y})$  is nontrivial.

**Remark 2.6.** In [19, Theorem 4.2], Ozsváth and Szabó proved that if  $(W, \Omega)$  is a symplectic filling of a contact structure  $(Y, \xi)$ , then the contact invariant  $c^+(\xi; [\Omega|_Y])$  is nontrivial. That proof works for our  $\omega$ -twisted version as well, given the following description of the  $\omega$ -twisted contact invariant.

Take an open book decomposition  $(Y, K)$  compatible with the contact structure  $\xi$ . After positive stabilization, we can assume that the open book has connected binding and genus  $g > 1$ . Suppose  $W: Y \rightarrow Y_0(K)$  is the corresponding Giroux two-handle cobordism [7] and  $\beta \in H^2(W; \mathbb{R})$  is any cohomology class on  $W$ , which extends  $\omega$ . Then, we have

$$\underline{HF}^+(-Y_0(K), g-1; \Lambda_{\beta|_{-Y_0(K)}}) \cong \Lambda.$$

Moreover, the contact invariant  $c^+(\xi; \Lambda_\omega)$  is equal to the image of  $1 \in \Lambda$  under the map induced by cobordism

$$F_{W;[\beta]}^+: \underline{HF}^+(-Y_0(K), g-1; \Lambda_{\beta|_{-Y_0(K)}}) \rightarrow \underline{HF}^+(-Y, \mathfrak{t}_\xi; \Lambda_\omega).$$

In the untwisted case, such description for the contact invariant is proved in [18, Proposition 3.1]. In that proof, one constructed a Heegaard diagram for  $Y_0(K)$ , which is admissible with respect to all the  $\text{Spin}^c$  structures  $\mathfrak{t}$  such that

$$\langle c_1(\mathfrak{t}), [\widehat{F}] \rangle = 2g - 2.$$

In this diagram, there are only two intersection points representing  $\text{Spin}^c$  structures satisfying the above restriction. Notice that we can also use this Heegaard diagram to compute the  $\omega$ -twisted Heegaard Floer homology. So this argument can be used to prove the above statements.

The following corollary is a small modification of [6, Lemma 2.3].

**Corollary 2.7.** Suppose  $Y$  is a closed, connected, oriented three-manifold with  $b_1(Y) = 1$  and  $F$  is a closed surface in  $Y$ . Suppose  $\xi$  is a weakly fillable contact structure on  $Y$ , such that  $\xi|_F$  is  $C^0$ -close to the oriented tangent plane field of  $F$ , and  $\omega \in H^2(Y; \mathbb{R})$  is a cohomology class such that  $\omega([F]) > 0$ . Then, the  $\omega$ -twisted contact invariant  $c^+(\xi; \Lambda_\omega)$  is nonzero.

**Proof.** Let  $(W, \Omega)$  be a weak filling of the contact structure  $(Y, \xi)$  and  $[\Omega] \in H^2(W)$  be the cohomology class represented by the closed two-form  $\Omega$ . By the definition of weak fillability,  $\Omega$  is positive on each plane in  $\xi$ , so  $\langle [\Omega]|_Y, [F] \rangle > 0$ . Since  $b_1(Y) = 1$  and  $\omega([F]) > 0$ , we may assume  $[\Omega]|_Y = k\omega \in H^2(Y; \mathbb{R})$  for some positive real number  $k$ . By Theorem 2.5,  $c^+(\xi; \Lambda_\Omega) \neq 0$ . Notice that the map  $t \mapsto t^k$  induces an isomorphism between chain complexes

$$\underline{CF}^+(-Y; \Lambda_\omega) \rightarrow \underline{CF}^+(-Y; \Lambda_\Omega),$$

it thus induces an isomorphism between homology groups

$$\underline{HF}^+(-Y; \Lambda_\omega) \rightarrow \underline{HF}^+(-Y; \Lambda_\Omega).$$

Under this isomorphism,  $c^+(\xi; \Lambda_\omega)$  is taken to  $c^+(\xi; \Lambda_\Omega)$ . This forces the contact invariant  $c^+(\xi; \Lambda_\omega)$  to be nonzero.

Given a contact manifold  $(Y, \xi)$  and a Legendrian knot  $K \subset Y$ , we can do contact  $(+1)$ -surgery to produce a new contact manifold  $(Y_1(K), \xi')$ . In [9, 18], it is shown that the untwisted contact invariant behaves well with respect to contact  $(+1)$ -surgery. Similarly, the  $\omega$ -twisted contact invariants are related as follows. (See [9, Theorem 2.3] for the untwisted version.)

**Proposition 2.8.** Suppose  $(Y', \xi')$  is obtained from  $(Y, \xi)$  by contact  $(+1)$ -surgery along a Legendrian knot  $K \subset Y$ . Suppose  $\gamma \subset Y - K$  is a closed curve, denote by  $\omega = PD(\gamma) \in H^2(Y; \mathbb{R})$ . View  $\gamma$  as a curve in the surgery manifold  $Y'$  and denote its Poincaré dual by  $\omega' \in H^2(Y'; \mathbb{R})$ . Let  $-W$  be the cobordism obtained from  $Y \times I$  by adding a two-handle along  $K$  with  $+1$  framing and with orientation reversed. Then,

$$F_{-W; PD(\gamma \times I)}^+(c^+(Y, \xi; \Lambda_\omega)) = c^+(Y', \xi'; \Lambda_{\omega'}).$$

### 3 Euler Characteristic of $\omega$ -Twisted Floer Homology

The goal of this section is to prove a homological version of Theorem 1.2. In order to do so, we first study the Euler characteristic of twisted Heegaard Floer homology.

In [15], Ozsváth and Szabó prove that the Euler characteristic of  $HF^+(Y, \mathfrak{s})$  is equal to the Turaev torsion when  $\mathfrak{s}$  is a nontorsion  $\text{Spin}^c$  structure. More precisely, they prove the following theorem.

**Theorem** [15, Theorem 5.11, Theorem 5.2]. If  $\mathfrak{s}$  is a nontorsion  $\text{Spin}^c$  structure over an oriented three-manifold  $Y$  with  $b_1(Y) \geq 1$ , then  $HF^+(Y, \mathfrak{s})$  is finitely generated and

$$\chi(HF^+(Y, \mathfrak{s})) = \pm T(Y, \mathfrak{s}).$$

When  $b_1(Y) = 1$ , the Turaev torsion in the above equation is defined with respect to the chamber containing  $c_1(\mathfrak{s})$ .

**Remark 3.1.** The proof of the above theorem can be modified to show that when  $\mathfrak{s}$  is a nontorsion  $\text{Spin}^c$  structure, the  $\omega$ -twisted Heegaard Floer homology  $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$  is a

finitely generated vector space over  $\Lambda$ , and

$$\chi(\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)) = \pm T(Y, \mathfrak{s}).$$

For a torsion  $\text{Spin}^c$  structure  $\mathfrak{s}$ , however,  $HF^+(Y, \mathfrak{s})$  is infinitely generated. Ozsváth and Szabó [15] introduce truncated Euler characteristic for torsion  $\text{Spin}^c$  structures and prove when  $b_1(Y) = 1$  or 2, it is related to the Turaev torsion.

When we use  $\omega$ -twisted Heegaard Floer homology, the situation is much simpler. In fact, we have the following proposition.

**Proposition 3.2.** Suppose  $\mathfrak{s}$  is a torsion  $\text{Spin}^c$  structure over an oriented three-manifold  $Y$  with  $b_1(Y) \geq 1$ . Then, for any nonzero cohomology class  $\omega \in H^2(Y; \mathbb{R})$ , the twisted Floer homology  $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$  is a finitely generated vector space over  $\Lambda$  and

$$\chi(\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)) = \pm T(Y, \mathfrak{s}).$$

Here, the Euler characteristic is taken by viewing  $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$  as a vector space over  $\Lambda$ .

Notice that when  $b_1(Y) = 1$  and  $\mathfrak{s}$  is a torsion  $\text{Spin}^c$  structure,  $T(Y, \mathfrak{s})$  does not depend on the choice of a chamber, see [24].

The following lemma is an analog of [8, Corollary 8.5].

**Lemma 3.3.** Let  $\mathfrak{s}$  be a torsion  $\text{Spin}^c$  structure on a three-manifold  $Y$  and  $\omega \in H^2(Y; \mathbb{R})$  be a nonzero cohomology class. Then,

$$\underline{HF}^\infty(Y, \mathfrak{s}; \Lambda_\omega) \cong 0.$$

**Proof.** We prove the lemma by induction on  $b_1(Y)$ .

(1) When  $b_1(Y) = 1$ , let  $\mathcal{R} = \mathbb{Q}[H^1(Y; \mathbb{Z})]$ , which is a principal ideal domain. Since  $\mathfrak{s}$  is torsion, the rational universally twisted Heegaard Floer chain complex  $\underline{CF}^\infty(Y, \mathfrak{s}; \mathcal{R})$  admits a relative  $\mathbb{Z}$ -grading

$$\underline{CF}^\infty(Y, \mathfrak{s}; \mathcal{R}) = \bigoplus_{i \in \mathbb{Z}} \underline{CF}_i^\infty(Y, \mathfrak{s}; \mathcal{R}).$$

By definition,

$$\underline{CF}^\infty(Y, \mathfrak{s}; \Lambda_\omega) = \underline{CF}^\infty(Y, \mathfrak{s}; \mathcal{R}) \otimes_{\mathcal{R}} \Lambda_\omega,$$

so the Universal Coefficients theorem [23] implies the following exact sequence

$$0 \rightarrow \underline{HF}^\infty(Y, \mathfrak{s}; \mathcal{R}) \otimes_{\mathcal{R}} \Lambda_\omega \rightarrow \underline{HF}^\infty(Y, \mathfrak{s}; \Lambda_\omega) \rightarrow \text{Tor}_1^{\mathcal{R}}(\underline{HF}^\infty(Y, \mathfrak{s}; \mathcal{R}), \Lambda_\omega) \rightarrow 0.$$

It follows from [15, Theorem 10.12] that the universally twisted Heegaard Floer homology

$$\underline{HF}^\infty(Y, \mathfrak{s}; \mathcal{R}) \cong \mathbb{Q}[U, U^{-1}],$$

and the latter group is endowed with a trivial  $\mathcal{R}$ -module structure. Since  $\omega \neq 0$ ,  $\Lambda_\omega$  is a divisible  $\mathcal{R}$ -module. Thus,

$$\underline{HF}^\infty(Y, \mathfrak{s}; \mathcal{R}) \otimes_{\mathcal{R}} \Lambda_\omega = 0, \quad \text{Tor}_1^{\mathcal{R}}(\underline{HF}^\infty(Y, \mathfrak{s}; \mathcal{R}), \Lambda_\omega) \cong 0.$$

Hence,  $\underline{HF}^\infty(Y, \mathfrak{s}; \Lambda_\omega) \cong 0$ .

(2) When  $b_1(Y) > 1$ , we can choose a knot  $K \subset Y$  such that its image in  $H_1(Y; \mathbb{Z})/\text{Tors}$  is primitive and the image of  $\text{PD}(\omega)$  under the natural projection

$$\pi: H_1(Y; \mathbb{R}) \rightarrow H_1(Y; \mathbb{R})/\langle [K] \rangle$$

is nonzero. Fix a longitude  $l$  of  $K$ , let  $\mu$  denote a meridian of  $K$ . By Poincaré duality, there exists a homology class  $\Phi \in H_2(Y)$  such that  $\Phi \cdot [K] = 1$ . So  $\mu$  bounds a surface in  $Y - K$ .

By the exact sequence for the pair  $(Y, Y - K)$ ,

$$\cdots \rightarrow H_2(Y; \mathbb{Z}) \rightarrow H_2(Y, Y - K)(\cong \mathbb{Z}) \xrightarrow{\partial_*} H_1(Y - K; \mathbb{Z}) \rightarrow H_1(Y; \mathbb{Z}) \rightarrow 0.$$

Since  $\partial_*$  maps the generator of  $H_2(Y, Y - K)$  to  $[\mu]$ , we have  $H_1(Y - K; \mathbb{Z}) \cong H_1(Y; \mathbb{Z})$ . Let  $Y' = Y_l(K)$  be the three-manifold obtained from  $Y$  by Dehn surgery along the knot  $K$  with frame  $l$ , and let  $\mathfrak{s}'$  be the induced torsion  $\text{Spin}^c$  structure on  $Y'$ . We have  $b_1(Y') = b_1(Y) - 1$ , and  $Y$  is obtained by zero surgery on a null-homologous knot  $K' \subset Y'$ .

As in [15, Propositions 10.5, 10.11], one can find a Heegaard diagram  $(\Sigma, \alpha' = \{\alpha_2, \dots, \alpha_g\}, \beta = \{\beta_1, \dots, \beta_g\})$  representing the knot complement  $Y - \nu^\circ(K)$ . Choosing  $\alpha_1$  and  $\gamma$  to represent  $\mu$  and  $l$ , respectively. Then the diagrams  $(\Sigma, \{\alpha_1\} \cup \alpha', \beta)$  and  $(\Sigma, \{\gamma\} \cup \alpha', \beta)$  represent  $Y$  and  $Y'$ , respectively. Wind  $\alpha_1$  along  $\gamma$  and pick a base point in the winding region as in [15, Section 5]. Let  $\underline{R}^\infty$  and  $\underline{L}^\infty$  denote the subspace of  $\underline{CF}^\infty(Y, \mathfrak{s}; \Lambda_\omega)$  generated by the  $\gamma$ -induced intersection points to the right and to the left of the curve  $\gamma$ , respectively. By [15, Lemma 5.12], if we wind  $\alpha_1$  sufficiently many times, then  $\underline{L}^\infty$  is a subcomplex of  $\underline{CF}^\infty(Y, \mathfrak{s}; \Lambda_\omega)$ .

Let  $\omega' \in H^2(Y'; \mathbb{R})$  satisfy that  $\text{PD}(\omega') = \pi(\text{PD}(\omega))$ . Then,  $\omega' \neq 0$ , so  $\underline{HF}^\infty(Y', \mathfrak{s}'; \Lambda_{\omega'}) = 0$  by the induction hypothesis.

Now the proof of [15, Lemma 10.6] gives us that  $H_*(\underline{R}^\infty) \cong \underline{HF}^\infty(Y', \mathfrak{s}'; \Lambda_{\omega'}) = 0$ . The short exact sequence

$$0 \rightarrow \underline{L}^\infty \rightarrow \underline{CF}^\infty(Y, \mathfrak{s}; \Lambda_\omega) \rightarrow \underline{R}^\infty \rightarrow 0$$

gives rise to a long exact sequence

$$\cdots \rightarrow \underline{HF}^\infty(Y, \mathfrak{s}; \Lambda_\omega) \rightarrow H_*(\underline{R}^\infty) \xrightarrow{\underline{\delta}} H_*(\underline{L}^\infty) \rightarrow \cdots, \quad (2)$$

where the homomorphism  $\underline{\delta}$  is induced by the natural map  $\underline{\delta}: \underline{R}^\infty \rightarrow \underline{L}^\infty$  given by taking the  $\underline{L}^\infty$ -component of the boundary of each element in  $\underline{R}^\infty$ .

By [15, Lemma 5.12], for each pair of intersection points  $\mathbf{x}^+ \in \underline{R}^\infty$  and  $\mathbf{y}^- \in \underline{L}^\infty$  inducing  $\mathfrak{s}$ , there are at most two homotopy classes modulo the action of  $\text{PD}(\gamma)^\perp$ ,

$$[\phi^{\text{in}}], [\phi^{\text{out}}] \in \pi_2(\mathbf{x}^+, \mathbf{y}^-)/\text{PD}(\gamma)^\perp$$

with Maslov index one and with only nonnegative multiplicities. Moreover,  $\text{PD}(\gamma)^\perp$  is generated by periodic domains whose boundary does not involve  $\alpha_1$ . Let  $\underline{\delta}_1$  be the map defined using those holomorphic disks, which are congruent to  $[\phi^{\text{in}}]$  modulo the action of  $\text{PD}(\gamma)^\perp$ . These holomorphic disks are precisely the holomorphic disks whose boundary uses  $\alpha_1 \cap \gamma$  exactly once.

On the chain level, one can construct the twisted chain complex so that  $\underline{\delta}_1$  has the form

$$\underline{\delta}_1[\mathbf{x}^+, i] = t^a[\mathbf{x}^-, i - n_z(\phi_{\mathbf{x}^+, \mathbf{x}^-})] + \text{lower order},$$

for some fixed constant  $a$ . As in [15, Lemma 10.7], this shows that  $\underline{\delta}_1$  induces an isomorphism on homology, thus  $H_*(\underline{L}^\infty) \cong H_*(\underline{R}^\infty) \cong 0$ . Finally, by the exact sequence (2) we have  $\underline{HF}^\infty(Y, \mathfrak{s}; \Lambda_\omega) \cong 0$ . This finishes the induction.

Recall that for a torsion  $\text{Spin}^c$  structure  $\mathfrak{s}$ , there is an absolute  $\mathbb{Q}$ -grading on  $\underline{HF}^+(Y, \mathfrak{s})$  which lifts the relative  $\mathbb{Z}$ -grading defined in [15], see [20]. This is also the case for our  $\omega$ -twisted Floer homology  $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$ . Suppose the absolute grading is supported in  $\mathbb{Z} + d$  for some constant  $d \in \mathbb{Q}$ . The Euler characteristic of  $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$  is defined to be

$$\chi(\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)) = \sum_{n \in \mathbb{Z}} (-1)^n \text{rank } \underline{HF}_{d+n}^+(Y, \mathfrak{s}; \Lambda_\omega).$$

Notice that our  $d$  is unique up to adding an integer, so the Euler characteristic is defined up to sign.

This absolute  $\mathbb{Q}$ -grading and Lemma 3.3 lead to the following corollary.

**Corollary 3.4.** Let  $\mathfrak{s}$  be a torsion  $\text{Spin}^c$  structure on a three-manifold  $Y$  and  $\omega \in H^2(Y; \mathbb{R})$  be a nonzero cohomology class. Then, for all sufficiently large  $N \in \mathbb{Z}$ ,

$$\underline{HF}_{d+N}^+(Y, \mathfrak{s}; \Lambda_\omega) = 0.$$

**Proof.** The elements in  $\underline{HF}^-(Y, \mathfrak{s}; \Lambda_\omega)$  have absolute  $\mathbb{Q}$ -grading bounded from above. So by the exact sequence relating  $\underline{HF}^-$ ,  $\underline{HF}^\infty$ , and  $\underline{HF}^+$ , we see that for all sufficiently large  $N \in \mathbb{Z}$ ,

$$\underline{HF}_{d+N}^+(Y, \mathfrak{s}; \Lambda_\omega) \cong \underline{HF}_{d+N}^\infty(Y, \mathfrak{s}; \Lambda_\omega),$$

which is zero by Lemma 3.3.

**Proof of Proposition 3.2.** (Compare the proof of [15, Theorem 5.2].)

From Corollary 3.4,  $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$  is a finitely generated vector space over  $\Lambda$ . So we can talk about its Euler characteristic. As in [15, Section 5.3], we can construct a Heegaard diagram  $(\Sigma, \alpha, \beta, z)$  for  $Y$  such that there is a periodic domain  $P_1$  containing  $\alpha_1$  with multiplicity one in its boundary. Extend  $\{P_1\}$  to a basis  $\{P_1, P_2, \dots, P_b\}$  for periodic domains such that  $P_2, \dots, P_b$  does not contain  $\alpha_1$  in their boundaries. This can be achieved by adding proper multiples of  $P_1$  to each  $P_i$ . Choose a set of dual simple closed curves  $\{a_i^*\}$  for  $\{\alpha_i\}$ , namely,  $a_i^*$  meets  $\alpha_i$  transversely in a single point and misses all other  $\alpha_j$ . Wind  $\alpha_1$  along  $a_1^*$   $n$  times and put the base point  $z$  in this winding region, to the right of  $a_1^*$  and in the  $\frac{n}{2}$ th subregion of the winding region (see [15, Figure 6]). Wind  $\alpha_2, \dots, \alpha_g$  along  $a_2^*, \dots, a_g^*$  sufficiently many times, such that any nontrivial linear combination of  $P_2, \dots, P_n$  has both large positive and negative local multiplicities as in [14, Lemma 5.4].

When  $n$  is sufficiently large, by [15, Lemma 5.4], if an intersection point represents the fixed torsion  $\text{Spin}^c$  structure  $\mathfrak{s}$ , then its  $\alpha_1$ -component must lie in the winding region, corresponding to  $\alpha_1$ . These intersection points are partitioned into two subsets according to whether they lie to the left or right side of  $a_1^*$ . Let  $L^+$  ( $R^+$ ) denote the subgroup of  $\underline{CF}^+(Y, \mathfrak{s}; \Lambda_\omega)$  generated by the points which lie to the left (right) of  $a_1^*$ .

As proved in [15, Lemma 5.5 and Lemma 5.6],  $L^+$  is a subcomplex of  $\underline{CF}^+(Y, \mathfrak{s}; \Lambda_\omega)$ , and  $R^+$  is a quotient complex. We have a short exact sequence

$$0 \longrightarrow L^+ \longrightarrow \underline{CF}^+(Y, \mathfrak{s}; \Lambda_\omega) \longrightarrow R^+ \longrightarrow 0,$$

which gives rise to a long exact sequence

$$\cdots \rightarrow H_{d+i}(L^+) \rightarrow \underline{HF}_{d+i}^+(Y, s; \Lambda_\omega) \longrightarrow H_{d+i}(R^+) \xrightarrow{\delta} H_{d+i-1}(L^+) \rightarrow \cdots.$$

By Corollary 3.4, for sufficiently large  $i$ ,  $\underline{HF}_{d+i}^+(Y, s; \Lambda_\omega) = 0$ . It follows that for all sufficiently large  $N$ ,

$$\chi(\underline{HF}_{\leq d+N}^+(Y, s; \Lambda_\omega)) = \chi(H_{\leq d+N}(L^+)) + \chi(H_{\leq d+N+1}(R^+)). \quad (3)$$

On the other hand, define  $f_1: R^+ \rightarrow L^+$  to be

$$f_1([x_i^+, i]) = [x_i^-, i - n_z(\phi)]t^{(\eta, \phi)}.$$

Here,  $\phi$  is the disk connecting  $x_i^+$  to  $x_i^-$ , which is supported in the winding region corresponding to  $\alpha_1$ , and  $\eta$  is the cochain (representing  $\omega$ ) used to define  $CF^\circ(Y; \Lambda_\omega)$ . Then, we have another short exact sequence

$$0 \longrightarrow \ker f_1 \longrightarrow R^+ \xrightarrow{f_1} L^+ \longrightarrow 0,$$

which induces a long exact sequence

$$\cdots \rightarrow H_{d+i}(\ker f_1) \longrightarrow H_{d+i}(R^+) \xrightarrow{f_1} H_{d+i-1}(L^+) \longrightarrow H_{d+i-1}(\ker f_1) \rightarrow \cdots.$$

The argument in [15, Section 5.2] shows that  $\ker f_1$  is a finite-dimensional graded vector space over  $\Lambda$  and has Euler characteristic  $\chi(\ker f_1) = \pm T(Y, s)$ . So for all sufficiently large  $N$ ,

$$\chi(\ker f_1) = \chi(H_{\leq d+N}(L^+)) + \chi(H_{\leq d+N+1}(R^+)). \quad (4)$$

Combining equations (3), (4), and Corollary 3.4, we obtain the desired result.

Having proved Proposition 3.2, we can use the same argument as in [13, Section 3] to prove the following homological version of Theorem 1.2.

**Proposition 3.5.** Suppose  $Y$  is a closed three-manifold and  $F \subset Y$  is an embedded torus. Let  $M$  be the three-manifold obtained by cutting  $Y$  open along  $F$ . The two boundary components of  $M$  are denoted by  $F_-, F_+$ . If there is a cohomology class  $\omega \in H^2(Y; \mathbb{Z})$  such that  $\omega([F]) \neq 0$  and  $\underline{HF}^+(Y; \Lambda_\omega) = \Lambda$ , then  $M$  is a homology product, namely, the two maps

$$i_{\pm*}: H_*(F_{\pm}) \rightarrow H_*(M)$$

are isomorphisms.

#### 4 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. We will essentially follow Ghiggini's argument in [6], with little modifications when necessary.

**Remark 4.1.** Before we get into the proof, we make a remark on the smoothness of foliations. In [5], the foliations constructed are smooth, except possibly along torus components of the given taut surface. In the proof of [6, Theorem 3.8], one also modifies a foliation further by replacing a compact leaf  $F$  with an  $F \times I$ , which is foliated by  $F \times t$ 's. The new foliation may not be smooth if  $F$  is a torus. However, by [3, Proposition 2.9.4], the new foliation can be approximated in  $C^0$ -topology by smooth weakly semi-fillable (hence, weakly fillable by [2, 4]) contact structures. Hence, one can now run the standard argument as in [10, Section 41] and [19] to get the nontriviality of the corresponding Heegaard Floer homology.

**Lemma 4.2.** Conditions are as in Theorem 1.2. Cut  $Y$  open along  $F$ , the resulting manifold  $M$  has two boundary components  $F_+, F_-$ . Suppose  $c \subset F_+$  is an essential simple closed curve, then there exists an embedded annulus  $A \subset M$ , such that  $c$  is one component of  $\partial A$ , and the other component of  $\partial A$  lies on  $F_-$ .

**Proof.** First notice that  $M$  is a homological product by Proposition 3.5. We can glue the two boundary components of  $M$  by a homeomorphism  $\psi$  to get a new manifold  $Y_\psi$  with  $b_1(Y_\psi) = 1$ . This homeomorphism  $\psi$  can be realized as a product of Dehn twists, thus  $Y_\psi$  can be obtained from  $Y$  by Dehn surgery on a link  $\mathcal{L}$ , which is contained in a neighborhood of  $F$ . Let  $\gamma \subset Y - \mathcal{L}$  be a closed curve which is Poincaré dual to  $\omega$  in  $Y$ . Then,  $\gamma$  also lies in  $Y_\psi$ . We still denote its Poincaré dual in  $Y_\psi$  by  $\omega$ , and we have  $\omega([F]) \neq 0$  in the new manifold  $Y_\psi$ . By Corollary 2.3,

$$\underline{HF}^+(Y_\psi; \Lambda_\omega) \cong \underline{HF}^+(Y; \Lambda_\omega) \cong \Lambda.$$

$Y_\psi$  satisfies all the conditions in Theorem 1.2, so we can work with  $Y_\psi$  instead of  $Y$ .

From now on we assume  $b_1(Y) = 1$ . We also assume that  $\omega([F]) > 0$ , otherwise we can change the orientation of  $F$ .

If the conclusion of the lemma does not hold, suppose  $c = c_+ \subset F_+$  is an essential simple closed curve such that there does not exist an annulus  $A$  as in the statement of the lemma. Since  $M$  is a homology product, we can find a simple closed curve  $c_- \subset F_-$  homologous to  $c_+$  in  $M$ . We fix an arc  $\delta \subset M$  connecting  $F_-$  to  $F_+$ . Let  $S_m(+c)$  be the

set of properly embedded surfaces  $S \subset M$  such that  $\partial S = (-c_-) \cup c_+$  and the algebraic intersection number of  $S$  with  $\delta$  is  $m$ .  $S_m(+c) \neq \emptyset$  since  $M$  is a homology product.

For any surface  $S \in S_m$ , its norm  $x(S) > 0$ . Otherwise one component of  $S$  must be an annulus  $A$  connecting  $c_-$  to  $c_+$ , which contradicts our assumption.

By [12, Lemma 6.4], when  $m$  is sufficiently large, there is a connected surface  $S_1 \in S_m(+c)$  which gives a taut decomposition of  $M$ . If we reverse the orientation of  $c$ , when  $n$  is sufficiently large, as before there is  $S_2 \in S_n(-c)$  which gives a taut decomposition of  $M$ . As in [5], using these two decompositions, one can then construct two taut foliations  $\mathcal{G}_1, \mathcal{G}_2$  of  $M$ , such that  $F_-, F_+$  are leaves of them. These two foliations are glued to get two taut foliations  $\mathcal{F}_1, \mathcal{F}_2$  of  $Y$ , such that  $F$  is a leaf of them. Suppose  $S$  is a surface in  $S_0(+c)$ , then  $-S \in S_0(-c)$ . We have

$$\begin{aligned} e(\mathcal{F}_1, S) &= e(\mathcal{F}_1, S_1 - mF) = \chi(S_1) - m \cdot 0 < 0, \\ e(\mathcal{F}_2, -S) &= e(\mathcal{F}_2, S_2 - nF) = \chi(S_2) - n \cdot 0 < 0, \end{aligned}$$

where  $e(\mathcal{F}, S)$  is defined in [6, Definition 3.7]. Thus, we conclude that

$$e(\mathcal{F}_1, S) \neq e(\mathcal{F}_2, S). \quad (5)$$

Choose a diffeomorphism  $\phi: F_+ \rightarrow F_-$  such that  $\phi(c_+) = c_-$ . Let  $Y_\phi$  be the three-manifold obtained from  $M$  by gluing  $F_+$  to  $F_-$  by  $\phi$ . Decompose  $\phi$  as a product of positive Dehn twists along nonseparating curves  $\{c_1, \dots, c_k\}$  on  $F$ , then  $Y_\phi$  is obtained from  $Y$  by doing  $(-1)$ -surgeries along these curves. Let  $-W$  be the cobordism obtained by adding two-handles to  $Y \times I$  along these curves with  $-1$  framing. As in the beginning of this proof,  $\omega$  also denotes an element in  $H^2(Y_\phi; \mathbb{Z})$ .

Since  $\omega([F]) \neq 0$ , by Corollary 2.3, the map

$$\underline{F}_{-W; \text{PD}(Y \times I)}^+ : \underline{HF}^+(Y; \Lambda_\omega) \rightarrow \underline{HF}^+(Y_\phi; \Lambda_\omega)$$

induced by the cobordism  $W$  is an isomorphism.

By Remark 4.1, one can approximate the foliations  $\mathcal{F}_1, \mathcal{F}_2$  on  $-Y$  by smooth weakly fillable contact structures  $\xi_1, \xi_2$  on  $-Y$ . We can realize the above curves  $\{c_1, \dots, c_k\}$  to be Legendrian knots in both  $\xi_1$  and  $\xi_2$ . Let  $\xi'_i$  be the contact structure on  $-Y_\phi$  obtained from  $(-Y, \xi_i)$  by doing  $(+1)$ -contact surgeries along these Legendrian knots. By Proposition 2.8

$$\underline{F}_{-W; \text{PD}(Y \times I)}^+(c^+(-Y, \xi_i; \Lambda_\omega)) = c^+(-Y_\phi, \xi'_i; \Lambda_\omega)$$

for  $i = 1, 2$ . The hypothesis  $b_1(Y) = 1$ ,  $\omega([F]) > 0$ , the fact that  $\xi_i$  is weakly fillable and Corollary 2.7 force  $c^+(-Y, \xi_i; \Lambda_\omega) \neq 0$  for  $i = 1, 2$ . Since the map  $\underline{F}_{-W; \text{PD}(Y \times I)}^+$  is an

isomorphism as stated above,

$$c^+(-Y_\phi, \xi'_i; \Lambda_\omega) = \underline{F}_{W; \text{PD}(\gamma \times I)}^+(c^+(-Y, \xi_i; \Lambda_\omega)) \neq 0$$

for  $i = 1, 2$ .

Let  $\bar{S}$  be the closed surface in  $Y_\phi$  obtained by gluing the two boundary components of  $S$  together. As in [6, Lemma 3.10], (5) implies that

$$\langle c_1(\xi'_1), [\bar{S}] \rangle \neq \langle c_1(\xi'_2), [\bar{S}] \rangle,$$

so the  $\text{Spin}^c$  structures  $\mathfrak{s}_{\xi'_1}$  and  $\mathfrak{s}_{\xi'_2}$  are different. Therefore, the two elements

$$c^+(-Y_\phi, \xi'_1; \Lambda_\omega) \in \underline{HF}^+(-Y_\phi, \mathfrak{s}_{\xi'_1}; \Lambda_\omega)$$

and

$$c^+(-Y_\phi, \xi'_2; \Lambda_\omega) \in \underline{HF}^+(-Y_\phi, \mathfrak{s}_{\xi'_2}; \Lambda_\omega)$$

are linearly independent. Hence,  $c^+(-Y, \xi_1; \Lambda_\omega)$  and  $c^+(-Y, \xi_2; \Lambda_\omega)$  are also linearly independent. We therefore get a contradiction to the assumption that  $\underline{HF}^+(Y; \Lambda_\omega) = \Lambda$ . ■

**Proof of Theorem 1.2.** Cut  $Y$  open along  $F$ , we get a compact manifold  $M$ . By Lemma 4.2, we can find two embedded annuli  $A_1, A_2 \subset M$ , each has one boundary component on  $F_-$  and the other boundary component on  $F_+$ , and  $\partial A_1 \cap \partial A_2$  consists of two points lying in  $F_-$  and  $F_+$ , respectively. A further isotopy will ensure that  $A_1 \cap A_2$  consists of exactly one arc, so a regular neighborhood of  $A_1 \cup A_2$  in  $M$  is homeomorphic to  $(T^2 - D^2) \times I$ . Now the irreducibility of  $Y$  implies that  $M = T^2 \times I$ . This finishes the proof.

## 5 Proof of Theorem 1.4

We turn our attention to the proof of Theorem 1.4. We will use the following lemmas, which are very similar in spirit.

**Lemma 5.1.** Let  $A, B$  be  $\mathbb{Z}$ -graded abelian groups such that  $A_{(i)}$  and  $B_{(i)}$  are finitely generated for each  $i \in \mathbb{Z}$ . Let  $v, h: A \rightarrow B$  be homogeneous homomorphisms of the same degree  $d$ . Suppose  $v$  has a right inverse  $\iota$ , i.e.  $v \circ \iota = \text{Id}_B$ . (Hence, the degree of  $\iota$  is  $-d$ .) Then, the map

$$v + th: A \otimes \Lambda \rightarrow B \otimes \Lambda$$

is surjective and  $\ker(v + th) \cong \ker(v) \otimes \Lambda$ .

**Proof.** Define a map

$$P = \sum_{j=0}^{\infty} \iota \circ (h\iota)^j (-t)^j.$$

It is well defined by the fact that the composition  $h\iota$  is of degree 0 and the assumption that  $B_{(i)}$  is finitely generated for each  $i \in \mathbb{Z}$ . Clearly,

$$(v + th) \circ P = \text{Id}_B.$$

So  $v + th$  is surjective. Define a map  $F: \ker(v) \otimes \Lambda \rightarrow \ker(v + th)$  by

$$F(a) = \sum_{i=0}^{\infty} (-t\iota \circ h)^i(a).$$

It has a two sided inverse  $G: \ker(v + th) \rightarrow \ker(v) \otimes \Lambda$  defined by

$$G(b) = (1 + t\iota \circ h)b.$$

$F$  and  $G$  are also well defined. They define an isomorphism between  $\ker(v + th)$  and  $\ker(v) \otimes \Lambda$ .

**Lemma 5.2.** Let  $A, B$  be  $\mathbb{Z}$ -graded abelian groups as in the above lemma such that their gradings are bounded from below. Suppose  $f_1: A \rightarrow B$  and  $f_2: A \rightarrow B$  are homomorphisms such that  $f_1$  is homogeneous, and for every homogeneous element  $a \in A$  each term in  $f_2(a)$  has grading strictly lower than  $f_1(a)$ . If  $f_1$  is surjective, then  $f_1 + f_2$  is also surjective. Furthermore,  $\ker(f_1 + f_2) \cong \ker(f_1)$ .

**Proof.** See [15, Proposition 5.8].

**Proof of Theorem 1.4.** If the zero surgery  $Y_0(K)$  is a torus bundle, take  $\omega = \text{PD}(-\mu) \in H^2(Y_0(K); \mathbb{Z})$ , where  $\mu$  is the meridian of  $K$  and PD is the Poincaré duality map. By Theorem 1.1,

$$\underline{HF}^+(Y_0(K); \Lambda_\omega) \cong \Lambda,$$

and it is supported in a single torsion  $\text{Spin}^c$  structure  $\mathfrak{s}_0$ . For every  $m > 0$ ,  $\mathfrak{s}_0$  induces unique  $\text{Spin}^c$  structures on  $Y$  and  $Y_m(K)$ , respectively. We denote these  $\text{Spin}^c$  structures

by  $\mathfrak{s}, \mathfrak{s}_m$ . According to Theorem 2.2, we have the following exact sequence:

$$\begin{array}{ccc} \underline{HF}^+(Y_0(K), \mathfrak{s}_0; \Lambda_\omega) & \longrightarrow & HF^+(Y_m(K), \mathfrak{s}_m) \otimes \Lambda \\ & \nwarrow & \nearrow F^+ \\ & HF^+(Y, \mathfrak{s}) \otimes \Lambda, \end{array} \quad (6)$$

where the map  $F^+$  is induced by the cobordism  $W'_m: Y_m(K) \rightarrow Y$ ,

$$F^+ = \sum_{\{t \in \text{Spin}^c(W'_m), t|_Y = \mathfrak{s}, t|_{Y_m} = \mathfrak{s}_m\}} \underline{F}_{W'_m(K), t; \text{PD}(-\mu \times I)}^+.$$

Note  $W'_m$  is a cobordism between two rational homology 3-spheres, as in [19, Section 3.1], the above maps are related to the untwisted case by the formula

$$F^+ = \pm t^c \cdot \sum_{\{t \in \text{Spin}^c(W'_m), t|_Y = \mathfrak{s}, t|_{Y_m} = \mathfrak{s}_m\}} F_{W'_m(K), t}^+ \cdot t^{\frac{\langle c_Y(t) \cup \text{PD}[-\mu \times I], [W'_m] \rangle}{2}}. \quad (7)$$

$F^+$  has two distinguished summands  $\underline{F}_{W'_m, \mathfrak{k}; \text{PD}(-\mu \times I)}^+$  and  $\underline{F}_{W'_m, \mathfrak{n}; \text{PD}(-\mu \times I)}^+$ ; a simple calculation shows they are homogeneous maps of the same degree, and their degree is strictly larger than the degree of any other summand of  $F^+$  [21, Lemma 4.4].

Taking  $m$  to be sufficiently large, by Theorem 2.4, the exact triangle (6) can be identified with

$$\begin{array}{ccc} \underline{HF}^+(Y_0(K), \mathfrak{s}_0; \Lambda_\omega) & \longrightarrow & H_*(C(\mathfrak{s})\{\max(i, j) \geq 0\}) \otimes \Lambda \\ & \nwarrow & \nearrow F^+ \\ & H_*(C(\mathfrak{s})\{i \geq 0\}) \otimes \Lambda. \end{array} \quad (8)$$

Under this identification, using equation (7),  $F^+$  can be written as

$$F^+ = \pm t^c \cdot (v_* + th_* + \text{lower degree summands}).$$

On the other hand, there is a short exact sequence (see [17, Corollary 4.5]):

$$0 \rightarrow C(\mathfrak{s})\{i < 0 \text{ and } j \geq 0\} \rightarrow C(\mathfrak{s})\{i \geq 0 \text{ or } j \geq 0\} \rightarrow C(\mathfrak{s})\{i \geq 0\} \rightarrow 0,$$

which induces an exact triangle:

$$\begin{array}{ccc} \widehat{HFK}(Y, K, \mathfrak{s}, 1) & \longrightarrow & H_*(C(\mathfrak{s})\{\max(i, j) \geq 0\}) \\ & \nwarrow & \nearrow v_* \\ & H_*(C(\mathfrak{s})\{i \geq 0\}). \end{array} \quad (9)$$

By assumption,  $Y$  is an  $L$ -space, so  $H_*(C(s)\{i \geq 0\}) \cong \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$ . It follows that  $v_*$  is surjective since it is  $U$ -equivariant and is an isomorphism at sufficiently large gradings. Moreover,  $v_*$  has a right inverse since its image is a free abelian group.

Using Lemmas 5.1 and 5.2,  $v_* + th_*$  and  $F^+$  are surjective, and

$$\ker(F^+) \cong \ker(v_* + th_*) \cong \ker(v_*) \otimes \Lambda.$$

From exact sequences (8) and (9),

$$\widehat{HF}^+(Y_0(K), s; \Lambda_\omega) \cong \ker(F^+), \quad \widehat{HFK}(Y, K, s, 1) \cong \ker(v_*).$$

Hence,  $\widehat{HFK}(Y, K, s, 1)$  has rank 1. The same argument shows that for any other  $\text{Spin}^c$  structure  $s'$  on  $Y$ ,  $\widehat{HFK}(Y, K, s', 1) \cong 0$ . So,

$$\widehat{HFK}(Y, K, [F], 1) = \bigoplus_{\tau \in \text{Spin}^c(Y)} \widehat{HFK}(Y, K, \tau, 1)$$

is of rank one; hence,  $K$  is a fibered knot by [12].

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