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Characterizing slopes for torus knots, II

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ABSTRACT

A slope $\frac{p}{q}$ is called a characterizing slope for a given knot $K_0 \subset S^3$ if whenever the $\frac{p}{q}$-surgery on a knot $K \subset S^3$ is homeomorphic to the $\frac{p}{q}$-surgery on K_0 via an orientation preserving homeomorphism, then $K = K_0$. In a previous paper, we showed that, outside a certain finite set of slopes, only the negative integers could possibly be non-characterizing slopes for the torus knot $T_{5,2}$. More explicitly besides all negative integer slopes there are 247 slopes which were unknown to be characterizing for $T_{5,2}$, including 89 nontrivial L-space slopes. Applying recent work of Baldwin–Hu–Sivek, we improve our result by showing that a nontrivial slope $\frac{p}{q}$ is a characterizing slope for $T_{5,2}$ if $\frac{p}{q} > -1$ and $\frac{p}{q} \notin \{0, 1, \pm\frac{1}{2}, \pm\frac{1}{3}\}$. In particular every nontrivial L-space slope of $T_{5,2}$ is characterizing for $T_{5,2}$. More explicitly this work yields 121 new characterizing slopes for $T_{5,2}$. Another interesting consequence of this work is that if a nontrivial $\frac{p}{q}$-surgery on a non-torus knot in S^3 yields a manifold of finite fundamental group, then $|p| > 9$.

Keywords: Dehn surgery; characterizing slopes; $T_{5,2}$; L-spaces.

Mathematics Subject Classification 2020: 57K18, 57K30

1. Introduction

For a knot $K \subset S^3$ and a slope $\frac{p}{q} \in \mathbb{Q} \cup \{1/0\}$, let $S^3_{p/q}(K)$ be the manifold obtained by the $\frac{p}{q}$-surgery on K. A slope $\frac{p}{q}$ is said to be *characterizing* for a given knot $K_0 \subset S^3$ if whenever $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$ for a knot $K \subset S^3$, then $K = K_0$. Here, "$\cong$" stands for an orientation preserving homeomorphism. Obviously the trivial slope $\frac{1}{0}$ is never characterizing for any knot.

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A long standing conjecture due to Gordon states that all nontrivial slopes are characterizing for the unknot. This conjecture was originally proved using Monopole Floer homology [1], and there were also proofs via Heegaard Floer homology [2, 3]. Based on work of Ghiggini [4], Ozsváth and Szabó [5] proved the same result for the trefoil knot and the figure-8 knot. To date, the unknot, the trefoil knot and the figure-8 knot are the only knots for which it is known that all but finitely many slopes are characterizing.

The next simplest knot is the torus knot $T_{5,2}$. It is reasonable to expect that all nontrivial slopes are characterizing for $T_{5,2}$. It is proven in [6, Theorem 1.4] that a nontrivial slope $\frac{p}{q}$ is characterizing for $T_{5,2}$ if it belongs to the set

$$\left\{\frac{p}{q} > 1, |p| \geq 33\right\} \cup \left\{\frac{p}{q} < -6, |p| \geq 33, |q| \geq 2\right\} \cup \left\{\frac{p}{q}, |q| \geq 9\right\}$$
$$\cup \left\{\frac{p}{q}, |q| \geq 3, 2 \leq |p| \leq 2|q| - 3\right\}$$
$$\cup \left\{9, 10, 11, \frac{19}{2}, \frac{21}{2}, \frac{28}{3}, \frac{29}{3}, \frac{31}{3}, \frac{32}{3}\right\}.$$

In addition by [7, Theorem 1.6] the slopes 8 and 12 are characterizing for $T_{5,2}$. It also follows from known results that the slopes 17/2 and 23/2 are characterizing for $T_{5,2}$. Indeed let p/q represent 17/2 or 23/2 and assume that $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$, then since by [8] $S^3_{p/q}(T_{5,2})$ is a Seifert manifold with finite non-cyclic fundamental group, K cannot be a hyperbolic knot by [7, Theorem 1.3] or a satellite knot by [9, Corollary 1.4; 10, Theorem 7 and Table 1]. Hence K is a torus knot in which case one can easily see that $K = T_{5,2}$ by applying [8, Proposition 3.1].

To give an explicit counting of the slopes which were unknown to be characterizing for $T_{5,2}$ (before the current work), we identify each slope $p/q \in \mathbb{Q} \cup \{1/0\}$ (where we may assume $q \geq 0$, and p, q are relative prime) with the point (p, q) in the upper half xy-plane. Then the set of slopes which were unknown to be characterizing for $T_{5,2}$ is the set of green colored points in the upper half xy-plane as illustrated in Fig. 1. All these points are located in the region enclosed by the purple colored curve and the part of the x-axis from $-\infty$ to 32 (the rest of the slopes in this region are red colored points). To be exact, this set of slopes is the union of the following nine mutually disjoint subsets:

(1) All negative integer slopes;
(2) $\{p \,|\, p \in \{0, \ldots, 7\} \cup \{13, \ldots, 32\}\}$ (28 of them);
(3) $\{p/2 \,|\, p$ odd and $p \in \{-31, \ldots, 15\} \cup \{25, \ldots, 31\}\}$ (28 of them);
(4) $\{p/3 \,|\, \gcd(p, 3) = 1$, $p \neq -2$ or 2, and $p \in \{-32, \ldots, -1\} \cup \{1, \ldots, 26\}\}$ (38 of them);
(5) $\{p/4 \,|\, p$ odd and $p \in \{-31, \ldots, -7\} \cup \{7, \ldots, 31\}\}$ (26 of them);
(6) $\{p/5 \,|\, \gcd(p, 5) = 1$ and $p \in \{-32, \ldots, -8\} \cup \{8, \ldots, 32\}\}$ (40 of them);
(7) $\{p/6 \,|\, \gcd(p, 6) = 1$ and $p \in \{-35, \ldots, -11\} \cup \{11, \ldots, 31\}\}$ (17 of them);

(8) $\{p/7 \,|\, \gcd(p, 7) = 1 \text{ and } p \in \{-41, \ldots, -12\} \cup \{12, \ldots, 32\}\}$ (44 of them);
(9) $\{p/8 \,|\, p \text{ odd and } p \in \{-47, \ldots, -15\} \cup \{15, \ldots, 31\}\}$ (26 of them).

So besides all negative integer slopes, there are 247 nontrivial slopes which were unknown to be characterizing for $T_{5,2}$.

Recall that a rational homology 3-sphere Y is an *L-space* if the rank of its Heegaard Floer homology $\widehat{HF}(Y)$ is equal to the order of $H_1(Y;\mathbb{Z})$. We call a slope of a knot in S^3 an L-space slope of the knot if the corresponding surgery on the knot with the slope yields an L-space. For the knot $T_{5,2}$, its L-space slopes are exactly those slopes which are greater than or equal to $3 = 2g(T_{5,2}) - 1$. From Fig. 1, we see that for the knot $T_{5,2}$, there are 89 L-space slopes which were unknown to be characterizing before this paper.

In this paper, we further narrow down the range of possible non-characterizing slopes for $T_{5,2}$. Our main result is the following theorem.

Theorem 1.1. *A nontrivial slope* $\frac{p}{q}$ *is characterizing for* $T_{5,2}$ *if* $\frac{p}{q} > -1$ *and* $\frac{p}{q} \notin \{0, 1, \pm\frac{1}{2}, \pm\frac{1}{3}\}$.

From this theorem, we get 121 new characterizing slopes:

(2) $\{p \,|\, p \in \{2, \ldots, 7\} \cup \{13, \ldots, 32\}\}$ (26 of them);
(3) $\{p/2 \,|\, p \text{ odd and } p \in \{3, \ldots, 15\} \cup \{25, \ldots, 31\}\}$ (11 of them);
(4) $\{p/3 \,|\, \gcd(p, 3) = 1 \text{ and } p \in \{4, \ldots, 26\}\}$ (16 of them);
(5) $\{p/4 \,|\, p \text{ odd and } p \in \{7, \ldots, 31\}\}$ (13 of them);
(6) $\{p/5 \,|\, \gcd(p, 5) = 1 \text{ and } p \in \{8, \ldots, 32\}\}$ (20 of them);
(7) $\{p/6 \,|\, \gcd(p, 6) = 1 \text{ and } p \in \{11, \ldots, 31\}\}$ (8 of them);
(8) $\{p/7 \,|\, \gcd(p, 7) = 1 \text{ and } p \in \{12, \ldots, 32\}\}$ (18 of them);
(9) $\{p/8 \,|\, p \text{ odd and } p \in \{15, \ldots, 31\}\}$ (9 of them).

It follows that all the slopes represented by green colored dots in Fig. 1 which lie on the right-hand side of the line $x = y$ are actually characterizing. It also follows that all nontrivial L-space slopes of $T_{5,2}$ are characterizing.

Corollary 1.2. *If a nontrivial p/q-surgery, with* $|p| \leq 9$, *on a nontrivial knot K in S^3 produces a manifold of finite fundamental group, then K is one of the torus knots $T_{3,\pm 2}$ and $T_{5,\pm 2}$.*

Proof. When $S^3_{p/q}(K)$ has cyclic fundamental group, i.e., when it is a lens space, K cannot be a hyperbolic knot [11, Theorem 1.6] or a satellite knot [12, Corollary 1] (note that $|p|$ is the order of the lens space). Hence K is a torus knot $T_{s,t}$. Now it can be deduced from [8, Propositions 3.1 and 3.4] that p/q-surgery on a nontrivial torus knot $T_{s,t}$ gives a lens space if and only if $p = qst \pm 1$. So $9 \geq |p| \geq |qst| - 1$ from which one can easily see that $T_{s,t}$ is one of $T_{3,\pm 2}$ and $T_{5,\pm 2}$.

When $S^3_{p/q}(K)$ has finite but non-cyclic fundamental group, by changing K to its mirror image, we may assume that p/q is positive. It is shown in [13, Theorem 2]

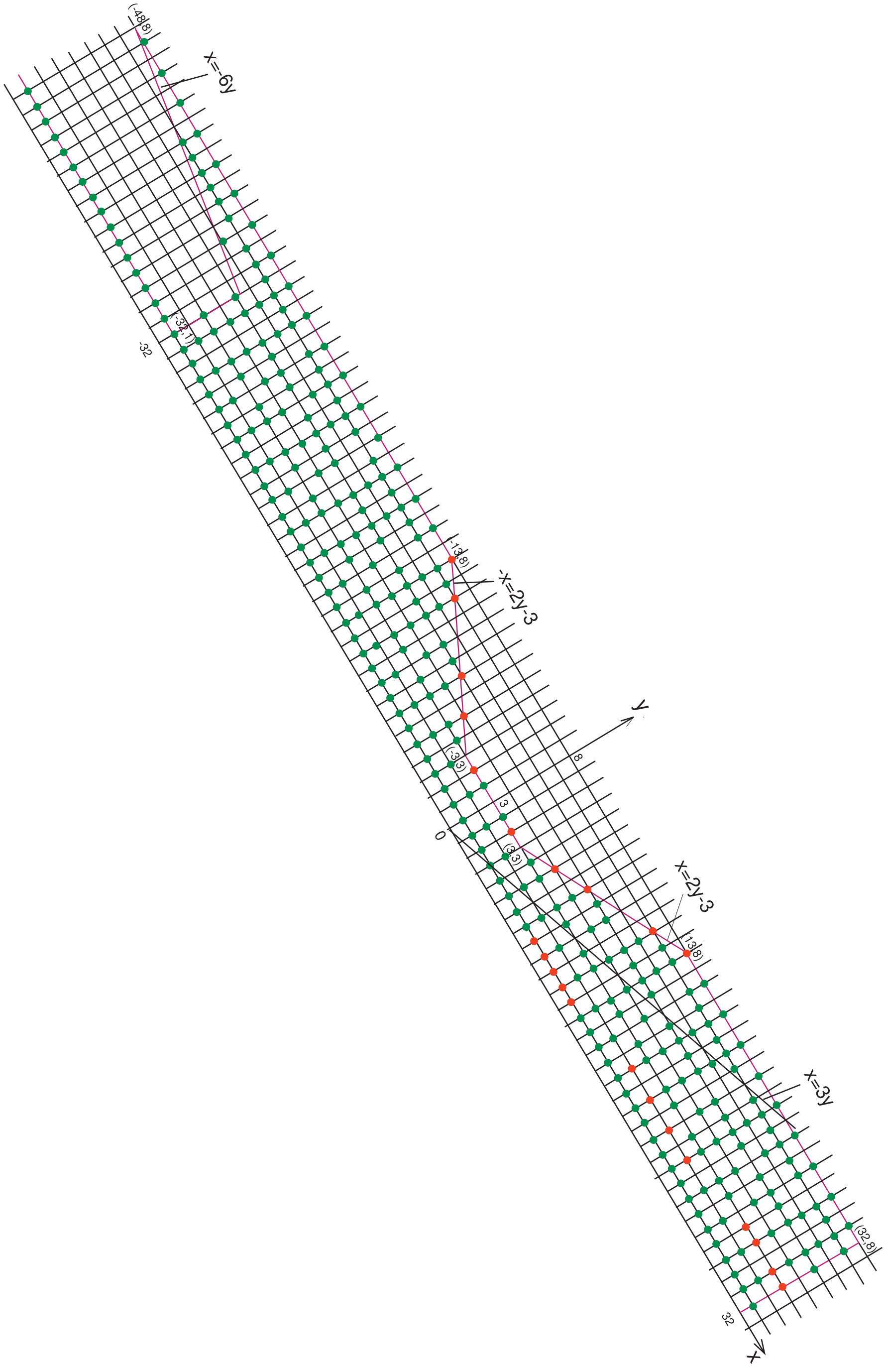

Fig. 1. (Color online) Green colored dots are slopes which were unknown to be characterizing for $T_{5,2}$.

and its proof that either K is $T_{3,2}$ or $T_{5,2}$ unless $p/q = 7$ or 8 and $S^3_{p/q}(K)$ is homeomorphic to $S^3_{p/q}(T_{5,2})$.

In case of $p/q = 7$, we may apply [14, Table 3 and Theorem 1.2] to see the homeomorphism from $S^3_7(K)$ to $S^3_7(T_{5,2})$ must be orientation preserving. Now we can apply Theorem 1.1 to conclude that $K = T_{5,2}$.

In case of $p/q = 8$, we may apply [7, Theorem 1.6] to conclude that the homeomorphism from $S^3_8(K)$ to $S^3_8(T_{5,2})$ must be orientation preserving and $K = T_{5,2}$. □

The bound nine in Corollary 1.2 appears to be the best one could get for hyperbolic knots in S^3 with the current techniques. The 10-surgery on $T_{4,3}$ gives a manifold with finite non-cyclic fundamental group, and the current techniques could not rule out the possibility that the same surgery on a hyperbolic knot with the same knot Floer homology as $T_{4,3}$ yields the same manifold. Conjecturally on a hyperbolic knot K in S^3 if a nontrivial $\frac{p}{q}$-surgery yields a manifold of finite fundamental group, then $|p| \geq 17$, a bound which can be realized on the $(-2, 3, 7)$-pretzel knot.

Our proof of Theorem 1.1, given in the next section, is mainly based on our earlier work [6] and a recent paper of Baldwin–Hu–Sivek [15].

Characterizing slopes for general torus knots have been studied in [6, 16–18].

2. Proof of Theorem 1.1

The strategy of the proof of Theorem 1.1 is as follows. Suppose that $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ for some slope $\frac{p}{q}$ in the range stated in the theorem. Using our earlier work [6], we can conclude that the knot Floer homology $\widehat{HFK}(K)$ of K is the same as the knot Floer homology of $T_{5,2}$. Hence K is a fibered knot by [19]. By a recent paper of Baldwin–Hu–Sivek [15], the monodromy of K is the lift of a map of the disk to the two-fold branched cover of the disk branching over five points. Hence K has an order-2 symmetry, which allows us to use results on Dehn surgeries on periodic knots to determine K.

For a knot K in S^3, $\Delta_K(T)$ denotes the symmetric Alexander polynomial of K. The following theorem and remark are [6, Theorem 4.1] and the remark after the proof.

Theorem 2.1. *Suppose that $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ for a knot $K \subset S^3$ and a nontrivial slope $\frac{p}{q}$. Then one of the following two cases happens*:

(1) *K is a genus $(n+1)$ fibered knot for some $n \geq 1$ with*

$$\Delta_K(T) = (T^{n+1} + T^{-(n+1)}) - 2(T^n + T^{-n}) + (T^{n-1} + T^{1-n}) + (T + T^{-1}) - 1. \tag{2.1}$$

(2) *K is a genus 1 knot with $\Delta_K(T) = 3T - 5 + 3T^{-1}$.*

Moreover, if

$$\frac{p}{q} \in \left\{\frac{p}{q} > 1\right\} \cup \left\{\frac{p}{q} < -6, |q| \geq 2\right\},$$

then the number n *in the first case must be* 1.

Remark 2.2. We have the following addendum to Theorem 2.1:

(a) If p is even, then (Case 2) of Theorem 2.1 cannot happen and in (Case 1) of Theorem 2.1, the number n must be odd.
(b) If p is divisible by 3, then (Case 2) cannot happen and in (Case 1), the number n is not divisible by 3.

The following result is implicitly contained in [6, Sec. 4.1]. Background information about Heegaard Floer homology can be found in [6, Sec. 3]. In particular, we will use the invariants $V_i(K)$ which are nonnegative integers capturing the $\mathbb{Z}[U]$-module structure of the knot Floer chain complex. They are related to another sequence of integers

$$t_i(K) = \sum_{j=1}^{\infty} a_{|i|+j},$$

where a_j are the coefficients of $\Delta_K(T) = \sum a_j T^j$.

Proposition 2.3. *Suppose that* $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ *for a knot* $K \subset S^3$ *and a nontrivial slope* $\frac{p}{q} > 1$. *Then*

$$\widehat{HFK}(S^3, K) \cong \widehat{HFK}(S^3, T_{5,2}), \tag{2.2}$$

as a bigraded group.

Proof. In [6, Sec. 4.1], it is proved that $\Delta_K = \Delta_{T_{5,2}}$ and $V_0(K) = V_1(K) = 1$. By [6, Proposition 4.2],

$$t_s(K) = V_s(K) + \text{rank} H_{\text{red}}(A_s^+).$$

Since $t_0(K) = t_1(K) = 1$ and $t_s(K) = 0$ when $s > 1$, we have

$$H_{\text{red}}(A_s^+) = 0, \quad \text{whenever } s \geq 0.$$

So $H_*(A_s^+) \cong \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$ for every $s \geq 0$. By the large surgery formula [20, Theorem 4.4], K admits an L-space surgery when the surgery slope is sufficiently large, so K is an L-space knot. Since $\Delta_K = \Delta_{T_{5,2}}$, using the fact that the knot Floer homology of an L-space knot is completely determined by its Alexander polynomial [21, Theorem 1.2], we get (2.2). □

Remark 2.4. In [6, Proposition 4.7], it is proved that if $\frac{p}{q} < -6$ and $|q| \geq 2$, then $\Delta_K = \Delta_{T_{5,2}}$. However, we cannot conclude that (2.2) holds. Algebraically, it is possible that $\widehat{HFK}(S^3, K, 1)$ has rank 3, while $HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(T_{5,2}))$.

For any hyperbolic knot K in S^3, a result of Gabai and Mosher [22] states that the complement of K contains an essential lamination λ which has an associated degeneracy locus $d(\lambda)$ in the form of $d(\lambda) = \frac{m}{n}$, $(m, n) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$, such that if

$$\Delta\left(\frac{p}{q}, d(\lambda)\right) := |pn - qm| \geq 2,$$

then λ remains an essential lamination in $S^3_{p/q}(K)$. Since $S^3_{1/0}(K) = S^3$ does not contain any essential lamination, it follows that $d(\lambda) = m/0$ or $m/1$. Furthermore we have the following two facts.

Fact (i). If $\Delta(\frac{p}{q}, d(\lambda)) \geq 3$, then $S^3_{p/q}(K)$ is an irreducible, atoroidal and non-Seifert fibered manifold [23, Theorem 2.5].

Fact (ii). If K is fibered and $d(\lambda) = m/1$ where λ is the stable lamination of K, then $|m| \geq 2$ [24, Theorem 8.8]. (Another proof of this was given in [25]).

We remark in passing that when K is a hyperbolic fibered knot, λ is the stable lamination of K, and $d(\lambda) = m/n$, then

$$\frac{n/\gcd(m, n)}{m/\gcd(m, n)},$$

is the fractional Dehn twist coefficient of the monodromy of K [26, Sec. 2].

Proposition 2.5. *Suppose that* $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$, $|\frac{p}{q}| < 1$ *and* $\frac{p}{q} \notin \{0, \pm\frac{1}{2}, \pm\frac{1}{3}\}$. *Then* $K = T_{5,2}$.

Proof. When K is a torus knot, Proposition 2.5 holds by [17, Lemma 4.2].

Next suppose that K is a satellite knot. Let K' be a companion knot of K. Since $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ is atoroidal and irreducible (since $p/q \neq 10$) [8], the work of Gabai [27] implies that K is a 0-bridge or 1-bridge braid in a tubular neighborhood V of K' (the definition of 0- and 1-bridge braids can be found in [27]). If K is a 0-bridge braid, then K is a (r, s)-cable of K' and the surgery slope p/q must be $(qrs \pm 1)/q$ [28]. If K is a 1-bridge braid, then it follows from [29, Lemma 3.2] that $\frac{p}{q} \in \mathbb{Z}$. In both cases we get a contradiction with the assumption that $|p/q| < 1$.

So K is hyperbolic. By Theorem 2.1, either K is fibered or $g(K) = 1$. If K is fibered and $d(\lambda) = \frac{m}{0}$, since $|q| \geq 3$, $\Delta(d(\lambda), \frac{p}{q}) \geq 3$. Hence $S^3_{p/q}(K)$ is not Seifert fibered by Fact (i) above, which contradicts the assumption $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$. So we may assume that when K is fibered, $d(\lambda) = m/1$ for the stable lamination λ of K, and therefore $|m| \geq 2$ by Fact (ii) above. Since $|\frac{p}{q}| < 1$ and $|m| \geq 2$, $\Delta(d(\lambda), \frac{p}{q}) \geq 3$. Again by Fact (i), we get a contradiction with the assumption $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$.

So K is hyperbolic and $g(K) = 1$. By [30, Theorem 1.5] $0 < |p| \leq 3$ since $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ is a Seifert fibered space. By Remark 2.2, $|p| = 1$. Since $\frac{p}{q} \notin \{0, \pm\frac{1}{2}, \pm\frac{1}{3}\}$, we again have $\Delta(d(\lambda), \frac{p}{q}) \geq 3$ (whether $d(\lambda) = \frac{m}{0}$ or $\frac{m}{1}$), which leads to a contradiction as in the preceding paragraph. ∎

Proposition 2.6. *Suppose that $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ for a nontrivial slope $\frac{p}{q}$ with $\frac{p}{q} > 1$. Then $K = T_{5,2}$.*

Proof. To get a contradiction we assume that $K \neq T_{5,2}$.

By Proposition 2.3, we have $\widehat{HFK}(S^3, K) \cong \widehat{HFK}(S^3, T_{5,2})$. In [15, Sec. 3], Baldwin–Hu–Sivek proved that if $\widehat{HFK}(S^3, K) \cong \widehat{HFK}(S^3, T_{5,2})$ and $K \neq T_{5,2}$, then K is a hyperbolic, doubly-periodic (i.e., periodic of order 2), genus two fibered knot, the degeneracy locus of the stable lamination of K is 4, and moreover there exists a pseudo-Anosov 5-braid β whose closure $B = \hat{\beta}$ is an unknot with braid axis A, such that K is the lift of A in the branched double cover $\Sigma(S^3, B) \cong S^3$. Let V be the exterior of A in S^3 and M_K the exterior of K in S^3. Then V is a solid torus and M_K is a double branched cover of V with B as the branched set in V. Let τ be the corresponding covering involution on M_K and U the branched set in M_K. Then U is the fixed point set of τ which is a knot disjoint from ∂M_K, and $(M_K, U)/\tau = (V, B)$. The restriction of τ on ∂M_K is a free action — an order two rotation along the longitude factor of ∂M_K.

We also use $M_K(p/q)$ to denote the surgery manifold $S^3_{p/q}(K)$ and similarly $V(p/q)$ for $S^3_{p/q}(A)$. Note that the involution τ on M_K extends to an involution $\tau_{p/q}$ on $M_K(p/q)$. In fact if we let $N_{p/q}$ denote the filling solid torus in forming $M_K(p/q) = M_K \cup N_{p/q}$ and let $K_{p/q}$ be the center circle of $N_{p/q}$, then the fixed point set of $\tau_{p/q}$ is

$$\mathrm{Fix}(\tau_{p/q}) = \begin{cases} U, & \text{if } p \text{ is odd,} \\ U \cup K_{p/q}, & \text{if } p \text{ is even} \end{cases} \tag{2.3}$$

and

$$M_K(p/q)/\tau_{p/q} = \begin{cases} V(p/2q), & \text{if } p \text{ is odd,} \\ V((p/2)/q), & \text{if } p \text{ is even.} \end{cases} \tag{2.4}$$

To see how Eqs. (2.3) and (2.4) are obtained, let $f : (M_K, U) \to (V, B)$ be the corresponding double branched covering, and let $\mu, \lambda \subset M_K$ be the meridian and longitude of K, and $\bar{\mu}, \bar{\lambda} \subset \partial V$ the meridian and longitude of A. Then the restriction $f : \partial M_K \to \partial V$ is a free double covering such that $f(\lambda) = \bar{\lambda}^2$ and $f(\mu) = \bar{\mu}$. When p is odd, $f(\mu^p\lambda^q) = \bar{\mu}^p\bar{\lambda}^{2q}$, which means a connected simple closed essential curve of slope p/q in ∂M_K is homeomorphic, under the map f, to a connected simple closed essential curve of slope $p/2q$ in ∂V. This means that the involution τ will send a connected simple closed essential curve of slope p/q in ∂M_K to a disjoint parallel in ∂M_K. Hence the free action of τ on ∂M_K extends naturally to a free action on $N_{p/q}$ (through the meridian disks of $N_{p/q}$). When p is even, $f(\mu^p\lambda^q) = \bar{\mu}^p\bar{\lambda}^{2q} = (\bar{\mu}^{p/2}\bar{\lambda}^q)^2$, which means a connected simple closed essential curve of slope p/q in ∂M_K double covers, under the map f, a connected simple closed essential curve of slope $(p/2)/q$ in ∂V. This means that the involution τ will send a connected simple closed essential curve of slope p/q in ∂M_K to itself and the quotient curve in ∂V has slope $(p/2)/q$.

Also when we extend the involution τ naturally over the meridian disks of $N_{p/q}$, the center points of these disks become fixed points of the extended involution, that is, the core curve $K_{p/q}$ of $N_{p/q}$ becomes a component of the fixed point set of the extended involution $\tau_{p/q}$.

Let Y be the exterior of U in M_K and W the exterior of B in V. Then Y is a free double cover of W. Since B is the closure of a pseudo-Anosov braid in V, W is hyperbolic. Hence Y is also hyperbolic.

Note that $M_K(p/q) \cong M_{T_{5,2}}(p/q)$ is a Seifert fibered space whose base orbifold is $S^2(2,5,d)$ with $d = |p - 10q|$ if $p/q \neq 10$, and is a connected sum of two lens spaces of orders 2 and 5 if $p/q = 10$ [8, Propositions 3.1 and 3.4]. Since K is a hyperbolic periodic knot, by [31] $M_K(p/q)$ is irreducible and thus $p/q \neq 10$, by [32] $M_K(p/q)$ is not a lens space, and by [33, Theorem 1.3] $M_K(p/q)$ is not a prism manifold. Thus $M_K(p/q)$ is a Seifert fibered space whose base orbifold is $S^2(2,5,d)$ with $d = |p-10q| > 2$ (Note that $d = 1$ if and only the manifold is a lens space, and $d = 2$ if and only if the manifold is a prism manifold. But we have ruled out these two possibilities). So there is a unique Seifert fibered structure on $M_K(p/q)$ (see e.g., [34, Theorem 2.3]) We may assume that the unique Seifert fibered structure on $M_K(p/q)$ is $\tau_{p/q}$-invariant (see, e.g., [30, Lemma 4.1]), i.e., $\tau_{p/q}$ sends every Seifert fiber to a Seifert fiber preserving the order of singularity.

Since the base orbifold of the Seifert fibered space $M_K(p/q)$ is orientable, the Seifert fibers of $M_K(p/q)$ can be coherently oriented. If $\tau_{p/q}$ preserves the orientations of the Seifert fibers of $M_K(p/q)$, then $\mathrm{Fix}(\tau_{p/q})$ consists of Seifert fibers (see [30, Lemma 4.3]). Since K is hyperbolic, $K_{p/q}$ cannot be a component of $\mathrm{Fix}(\tau_{p/q})$. Thus by Eq. (2.3), p is odd and $\mathrm{Fix}(\tau_{p/q}) = U$, and by Eq. (2.4), $M_K(p/q)/\tau_{p/q} = V(p/2q)$. Moreover, if we let $Y(\partial M_K, p/q)$ denote the Dehn filling of Y along the component ∂M_K of ∂Y with the slope p/q, and similarly define $W(\partial V, p/2q)$, then $Y(\partial M_K, p/q)$ is Seifert fibered and is a free double cover of $W(\partial V, p/2q)$. So the latter manifold $W(\partial V, p/2q)$ is also Seifert fibered. But $W(\partial V, 1/0)$ is a solid torus (it is the exterior of the unknot B in S^3). We get a contradiction with [35, Corollary 15].

Hence $\tau_{p/q}$ reverses the orientations of the Seifert fibers of $M_K(p/q)$. Since $p/q > 1$, we may assume that both p and q are positive. Let μ be the meridian of K, then $[\mu]$ generates $H_1(M_K) \cong \mathbb{Z}$ and $H_1(M_K(p/q)) \cong \mathbb{Z}/p\mathbb{Z}$. Clearly $\tau_*[\mu] = [\mu]$, so

$$\tau_* = \mathrm{id} \text{ on } H_1(M_K), \quad (\tau_{p/q})_* = \mathrm{id} \text{ on } H_1(M_K(p/q)). \tag{2.5}$$

To apply a homological argument, we describe the Seifert fibered structure of $M_{T_{5,2}}(p/q)$ explicitly as follows. Let $V_0 \cup V_1$ be a standard genus one Heegaard splitting of S^3. We may assume that $T_{5,2}$ is embedded in ∂V_0 and is homologous to $5\mathcal{L} + 2\mathcal{M}$, where $\mathcal{L}$ is the canonical longitude of V_0, and $\mathcal{M}$ the meridian of V_0. Let $\mu_0 \subset \partial M_{T_{5,2}}$ be the meridian of $T_{5,2}$, let λ_0 be the canonical longitude of $T_{5,2}$ and let $\mathcal{C}_i$ be the core of V_i, $i = 0, 1$. Then $M_{T_{5,2}}$ is Seifert fibered with C_0 and C_1 as two singular fibers of order 5 and 2, respectively. A regular fiber $\mathcal{F}$ of $M_{T_{5,2}}$ in $\partial M_{T_{5,2}}$ has slope $10\mu_0 + \lambda_0$. If $p/q \neq 10$, the Seifert structure of $M_{T_{5,2}}$ extends to one on

$M_{T_{5,2}}(p/q)$ such that the core $\mathcal{C}'$ of the filling solid torus is an order $d = |p - 10q|$ singular fiber if $d > 1$. In $H_1(M_{T_{5,2}}(p/q))$, we have

$$[\mathcal{F}] = 10[\mu_0], \quad [\mathcal{C}_0] = 2[\mu_0], \quad [\mathcal{C}_1] = 5[\mu_0], \quad [\mathcal{C}'] = \pm q'[\mu_0], \tag{2.6}$$

where $q' \in \mathbb{Z}$ satisfies that $qq' \equiv 1 \pmod p$.

Since $\tau_{p/q}$ sends a regular fiber to a regular fiber reversing its orientation, we see, using (2.5) and (2.6), that $10[\mu_0] = -10[\mu_0]$ in $\mathbb{Z}/p\mathbb{Z}$. So $p|20$. We claim that $p = 1$ or 2. Suppose otherwise that $p > 2$. We already know that $M_K(p/q)$ has three singular fibers of orders $2, 5, d = |p - 10q| > 2$. If $d \neq 5$, then $\tau_{p/q}$ sends the order d singular fiber $\mathcal{C}'$ to itself with opposite orientation. Hence $q'[\mu_0] = -q'[\mu_0]$ in $\mathbb{Z}/p\mathbb{Z}$. Since $\gcd(p, q') = 1$, we get $p|2$, which is not possible. So we must have $d = 5$, which, together with the condition $p|20$, implies that $p/q = 5$. By (2.6) the two order 5 singular fibers $\mathcal{C}_0, \mathcal{C}'$ are homologous to $2[\mu_0]$ and $\pm[\mu_0]$, respectively. By (2.5), $\tau_{p/q}$ must send $\mathcal{C}_0$ to itself, and $\mathcal{C}'$ to itself. So $2[\mu_0] = -2[\mu_0]$ in $\mathbb{Z}/5\mathbb{Z}$, which is not possible.

Recall that K has a degeneracy locus 4. Since $\frac{p}{q} \le 2$, we have $\Delta(\frac{p}{q}, 4) \ge 3$ unless $\frac{p}{q} = 2$. By Fact (i), we only need to consider the case $\frac{p}{q} = 2$.

By Eq. (2.3) the fixed point set of τ_2 is $U \cup K_2$ and by Eq. (2.4), $M_K(2)/\tau_2 = V(1)$ which is S^3. Hence the branched set $B \cup K_2^*$ in $M_K(2)/\tau_2 = V(1) = S^3$ is a Montesinos link of two components [36], where K_2^* is the image of K_2 under the map $M_K(2) \to V(1)$, which is also the core of the filling solid torus of $V(1)$. Note that K_2^* is an unknot in S^3 while B is the closure of a 5-braid in the exterior of K_2^* which is a solid torus. Hence the linking number between B and K_2^* is 5 (well defined up to sign).

On the other hand, the involution τ_2 on $M_{T_{5,2}}(2)$ and the quotient space S^3 with the branched set can be explicitly constructed as follows. The knot $T_{5,2}$ is strongly invertible. Figure 2 shows the strong involution of $T_{5,2}$, its axis (colored red), a regular neighborhood N of $T_{5,2}$ in S^3, a pair of canonical longitudes of $T_{5,2}$ in ∂N (colored green), all invariant under the involution. Note that the restriction of the involution on the exterior $M_{T_{5,2}}$ preserves the Seifert structure of $M_{T_{5,2}}$ and reverses the orientations of the Seifert fibers. The quotient of each of these items under the involution is shown in Fig. 3 part (a) (and part (b) after an isotopy). Also the involution on $M_{T_{5,2}}$ obviously extends to an involution on $M_{T_{5,2}}(2)$ preserving its Seifert structure and reversing the orientations its Seifert fiberes. Furthermore the quotient space of $M_{T_{5,2}}(2)$ under the extended involution is S^3 whose branched set can be obtained by replacing the rational 1/0-tangle in Fig. 3(b) with the rational 2-tangle. The resulting branched set is the two-component Montesinos link shown in Fig. 4.

Recall that $M_K(2) = M_{T_{5,2}}(2)$ has a unique Seifert fibered structure (it restricts to the Seifert fibred structure on $M_{T_{5,2}}$) and the base orbifold of the Seifert fibered structure is $S^2(2, 5, 8)$. Hence it has a unique orientation preserving involution, which preserves the Seifert fibered structure and reverses the orientations of the Seifert fibers (in particular each of the three singular fibers is invariant but with

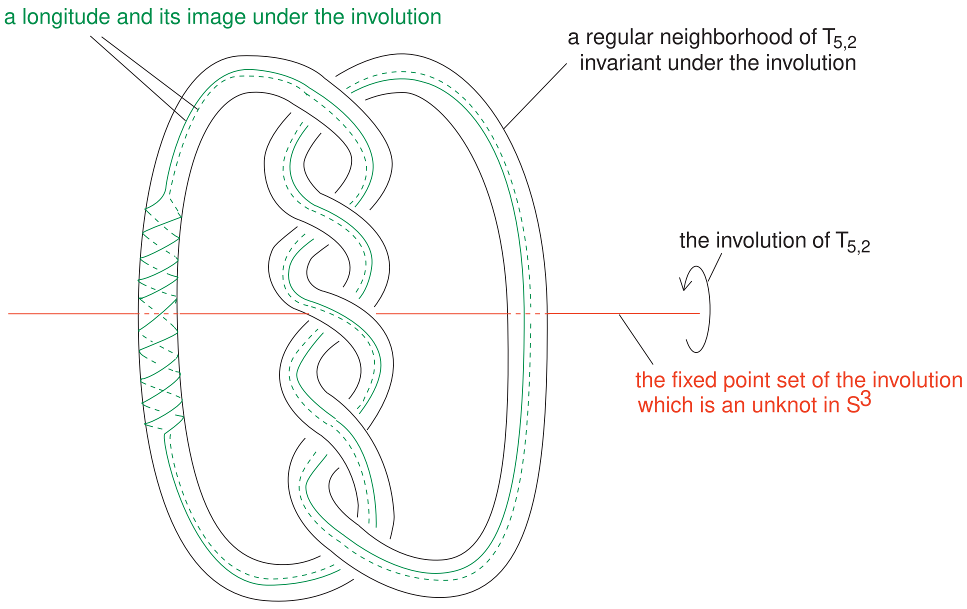

Fig. 2. The strong involution on $(S^3, T_{5,2})$.

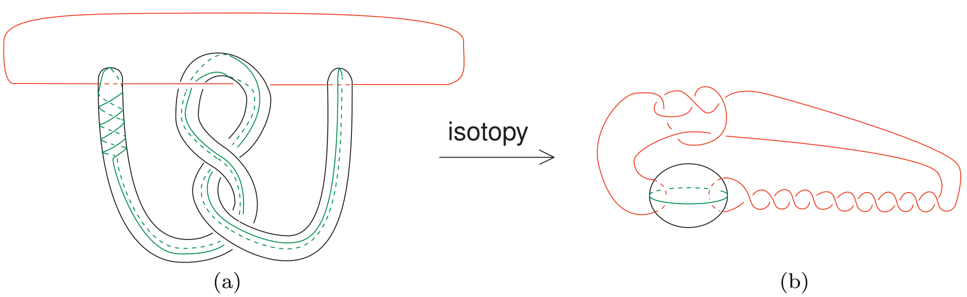

Fig. 3. The quotient.

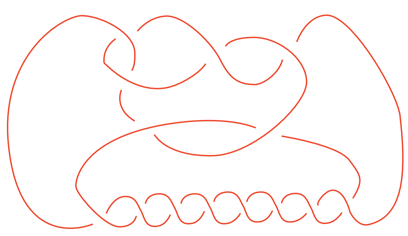

Fig. 4. The branched sets for the 2-surgery on $T_{5,2}$.

orientation reversed under the involution), such that the quotient space is S^3. Therefore the link shown in Fig. 4 should be the link $B \cup K_2^*$. However the linking number between the two components of the link in Fig. 4 is 3 (well defined up to sign), yielding a final contradiction with the early conclusion that the linking number between B and K_2^* is 5 up to sign. □

Remark 2.7. Proposition 2.6 can be also proved without using the degeneracy locus condition.

Now the combination of Propositions 2.5 and 2.6 gives Theorem 1.1.

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