*Acta Mathematica Sinica, English Series*
Dec., 2021, Vol. 37, No. 12, pp. 1841–1846
Published online: December 15, 2021
https://doi.org/10.1007/s10114-021-0408-4
http://www.ActaMath.com

**Acta Mathematica Sinica, English Series**


# A Characterization of $T_{2g+1,2}$ among Alternating Knots

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**Abstract** Let $K$ be a genus $g$ alternating knot with Alexander polynomial $\Delta_K(T) = \sum_{i=-g}^{g} a_i T^i$. We show that if $|a_g| = |a_{g-1}|$, then $K$ is the torus knot $T_{2g+1,\pm 2}$. This is a special case of the Fox Trapezoidal Conjecture. The proof uses Ozsváth and Szabó's work on alternating knots.

**Keywords** Alternating knots, Alexander polynomial, strongly quasipositive fibered knots

**MR(2010) Subject Classification** 57M25

## 1 Introduction

Alternating knots have many good properties. For example, the information from the Alexander polynomial of an alternating knot $K$ determines the genus of $K$ and whether $K$ is fibered [2, 11]. Even so, there are still some open problems about alternating knots. One of these problems is the following conjecture made by Fox [3, Problem 12].

**Conjecture 1.1** (Fox Trapezoidal Conjecture) *Let $K$ be an alternating knot with normalized Alexander polynomial*

$$\Delta_K(T) = \sum_{i=-g}^{g} a_i T^i, \tag{1.1}$$

*where $g$ is the genus of $K$. Then*

$$|a_i| \le |a_{i-1}| \quad \text{when } 0 < i \le g.$$

*Moreover, if $|a_i| = |a_{i-1}|$ for some $i$, then $|a_j| = |a_i|$ whenever $0 \le j \le i$.*

This conjecture was known for 2–bridge knots [9] and alternating arborescent knots [12]. Using Heegaard Floer homology, Ozsváth and Szabó [14] proved the first part of the conjecture for $i = g$. See (2.2) for the precise inequality. As a result, they proved the conjecture for genus–2 knots.

In this paper, we will prove the second part of Conjecture 1.1 for $i = g$. In this case, we will get a stronger conclusion.

**Theorem 1.2** *Let $K$ be an alternating knot with normalized Alexander polynomial given by* (1.1)*, where $g$ is the genus of $K$. If $|a_g| = |a_{g-1}|$, then $K$ or its mirror is the torus knot $T_{2g+1,2}$.*

Received July 31, 2020, accepted May 31, 2021
Supported by NSF (Grant No. DMS-1811900)

Our proof uses Ozsváth and Szabó's work [14].

This paper is organized as follows. In Section 2, we prove that if a knot $K$ has thin knot Floer homology, and $|a_g| = |a_{g-1}|$, then $K$ is a strongly quasipositive fibered knot. In Section 3, we prove that strongly quasipositive fibered alternating knots are connected sums of torus knots of the form $T_{2n+1,2}$. Hence we get a proof of Theorem 1.2.

## 2 Thin Knots with $|a_g| = |a_{g-1}|$

Let $K \subset S^3$ be a knot with knot Floer homology [16, 18]

$$\widehat{HFK}(S^3, K) = \bigoplus_{i,j \in \mathbb{Z}^2} \widehat{HFK}_j(S^3, K, i).$$

We say the knot Floer homology is *thin*, if it is supported in the line

$$j = i - \tau,$$

where $\tau = \tau(K)$ is the concordance invariant defined in [15].

By work of Hedden [10], we will make the following definition of strongly quasipositive fibered knots. We do not need the original definition of strong quasipositivity in [19].

**Definition 2.1** *A strongly quasipositive fibered knot is a fibered knot $K \subset S^3$, such that the open book with binding $K$ supports the tight contact structure on $S^3$.*

Now we can state the main result we will prove in this section.

**Proposition 2.2** *Let $K \subset S^3$ be a knot with thin knot Floer homology. Let the normalized Alexander polynomial be given by* (1.1). *If $|a_g| = |a_{g-1}|$, then $K$ or its mirror is a strongly quasipositive fibered knot.*

Let $S^3_0(K)$ be the manifold obtained by 0-surgery on $K$. Ozsváth and Szabó proved that if $\widehat{HFK}(S^3, K)$ is thin and $\tau(K) \geq 0$, then

$$HF^+(S^3_0(K), s) \cong \mathbb{Z}^{b_s} \oplus (\mathbb{Z}[U]/U^{\delta(-2\tau, s)}) \tag{2.1}$$

for $s > 0$, where

$$\delta(-2\tau, s) = \max\left\{0, \left\lceil \frac{|\tau| - |s|}{2} \right\rceil\right\}$$

and

$$(-1)^{s-\tau} b_s = \delta(-2\tau, s) - t_s(K)$$

with

$$t_s(K) = \sum_{j=1}^{\infty} j a_{s+j}.$$

See [14, Theorem 1.4] and the paragraph after it.

Using (2.1), one can deduce the following inequality as in [14]:

$$|a_{g-1}| \geq 2|a_g| + \begin{cases} -1 & \text{if } |\tau| = g, \\ 1 & \text{if } |\tau| = g - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

*Proof of Proposition* 2.2 It follows from [17] that $a_g \neq 0$. If $|a_g| = |a_{g-1}|$, then by (2.2) we must have

$$|a_g| = 1, \quad |\tau| = g.$$

By [6, 13], $K$ is fibered. Replacing $K$ with its mirror if necessary, we may assume $\tau = g$. It follows from [14, Corollary 1.7] that the open book with binding $K$ supports the tight contact structure. □

## 3 Strongly Quasipositive Fibered Alternating Knots

Suppose that $K$ is a fibered alternating link. Let $\mathcal{D} \subset S^2$ be a reduced connected alternating diagram of $K$. Applying Seifert's algorithm to $\mathcal{D}$, we can get a Seifert surface $F$ which is a union of disks and twisted bands corresponding to the crossings in $\mathcal{D}$. We call the disks *Seifert disks* with boundary *Seifert circles*, and call the twisted bands *Seifert bands*. By [5, Theorem 5.1], $F$ is a fiber of the fibration of $S^3 \setminus K$ over $S^1$.

Following [7], we say a Seifert circle is *nested*, if each of its complementary regions contains another Seifert circle. It is well-known that $F$ decomposes as a Murasugi sum of two surfaces along a nested Seifert circle $C$ [11, 20]. More precisely, let $D_1, D_2$ be the two disks bounded by $C$. Let $\mathcal{B}_i$ be the union of Seifert bands connecting $C$ to Seifert circles in $D_i$, $i = 1, 2$. We cut $F$ open along $\mathcal{B}_{3-i} \cap C$ to get a disconnected surface. Let $F_i$ be the component such that the projection of $\partial F_i$ is supported in $D_i$. Then $F$ is a Murasugi sum of $F_1$ and $F_2$. Gabai [4] proved that $F$ is a fiber of a fibration of $S^3 \setminus K$ if and only if each $F_i$ is a fiber of a fibration of $S^3 \setminus \partial F_i$, $i = 1, 2$.

**Definition 3.1** *If a diagram contains no nested Seifert circles, then this diagram is special as defined in* [11].

Suppose that $\mathcal{D} \subset S^2$ is a reduced connected special alternating diagram for a link $K$. Let $S_1, \ldots, S_k$ be the Seifert circles in $\mathcal{D}$. Since $\mathcal{D}$ is special, these Seifert circles bound disjoint disks $D_1, \ldots, D_k$. We color the complementary regions of $\mathcal{D}$ by two colors black and white, so that two regions sharing an edge have different colors. The coloring convention is that the disks $D_1, \ldots, D_k$ have the black color. Clearly, there are no other black regions. We will construct the black graph $\Gamma_B$ and the white graph $\Gamma_W$ as usual. Namely, the vertices in $\Gamma_B$ (or $\Gamma_W$) are the black (or white) regions, and the edges correspond to the crossings. These two graphs are embedded in $S^2$ as a pair of dual graphs. We also construct the reduced black graph $\Gamma_B^r$ by deleting all but one edges connecting two vertices $v_i$ and $v_j$ if there is any edge connecting them.

The following proposition can be found in [1, Propositions 13.24 and 13.25].

**Proposition 3.2** *Suppose that $\mathcal{D} \subset S^2$ is a reduced connected special alternating diagram for a fibered link $K$, then all but one vertices in $\Gamma_W$ have valence* 2. *As a result, $K$ is a connected sum of torus links*

$$K = \#_{i=1}^{\ell} T_{k_i,2}.$$

From Proposition 3.2, it is not hard to get the following characterization of $\mathcal{D}$ in terms of $\Gamma_B^r$.

**Lemma 3.3** *Under the same assumptions as in Proposition* 3.2, *the graph $\Gamma_B^r$ is a tree.*

*Proof* Since $D$ is connected, $\Gamma_B^r$ is also connected. If $\Gamma_B^r$ contains only two vertices, there is exactly one edge by the definition of $\Gamma_B^r$, so our conclusion holds. From now on, we assume $\Gamma_B^r$ has at least three vertices. Let $R$ be a complementary region of $\Gamma_B^r$, then it is not a bigon since any two vertices in $\Gamma_B^r$ are connected by at most one edge and $\Gamma_B^r$ has at least three vertices. Let $v$ be the vertex corresponding to $R$ in $\Gamma_W$, then $v$ has valence $> 2$. By Proposition 3.2, $\Gamma_B^r$ has at most one complementary region, which means that $\Gamma_B^r$ is a tree. □

**Lemma 3.4** *Under the same assumptions as in Proposition* 3.2*, if two vertices in* $\Gamma_B$ *are connected by an edge, then they are connected by at least two edges.*

*Proof* Using Lemma 3.3, if $D_i$ and $D_j$ are connected through only one crossing, then $\mathcal{D}$ is not reduced, a contradiction. □

We say two Seifert bands are *parallel* if they connect the same two Seifert disks. The following lemma is well-known. See, for example, [7, Proposition 5.1].

**Lemma 3.5** *If two Seifert bands are parallel, then we can deplumb a Hopf band from $F$. The resulting surface can be obtained by removing one of the bands from $F$.*

**Lemma 3.6** *Let $K$ be a strongly quasipositive fibered alternating knot, and let $\mathcal{D}$ be a reduced connected alternating diagram for $K$. Let $C$ be a nested Seifert circle. If $C$ is connected to two pairs of parallel bands, then these two pairs of bands are on the same side of $C$.*

*Proof* If $C$ is connected to two pairs of parallel bands on different sides of $C$, then we can deplumb a negative Hopf band from $F$. See Figure 1. Hence the open book with page $F$ supports an overtwisted contact structure [8, Lemma 4.1], a contradiction. □

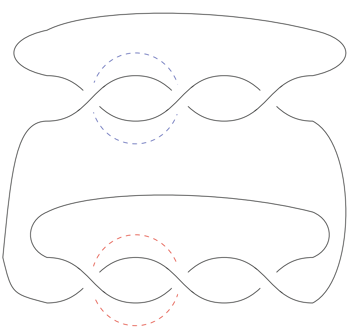

Figure 1 If two collections of parallel bands are on different sides of a nested Seifert circle, we can deplumb a positive Hopf band and a negative Hopf band. The two dashed circles are the cores of the Hopf bands

**Proposition 3.7** *Let $K$ be a strongly quasipositive fibered alternating knot. Then $K$ is a connected sum of torus knots of the form $T_{2n_i+1,2}$ for $n_i > 0$.*

*Proof* If $\mathcal{D}$ is special, by Proposition 3.2, $K$ is a connected sum of torus knots $T_{2n_i+1,2}$. Since $K$ is strongly quasipositive, each $n_i$ must be positive, so our conclusion holds.

Now we assume that $\mathcal{D}$ contains at least one nested Seifert circle. We say a nested Seifert circle is extremal, if one of its complementary regions contains no other nested Seifert circles.

Let $C_1, \dots, C_m$ be a maximal collection of extremal nested Seifert circles in $\mathcal{D}$, and let $R_i$ be the complementary region of $C_i$ which contains no other nested Seifert circles. Then $R_1, \dots, R_m$ are mutually disjoint. Let $\mathcal{D}'$ be the diagram obtained from $\mathcal{D}$ by Murasugi desumming along $C_1 \cup \cdots \cup C_m$. Let $\mathcal{D}_i$ be the part of $\mathcal{D}'$ supported in $R_i$, and let

$$\mathcal{D}^* = \mathcal{D}' \setminus \left( \bigcup_{i=1}^{m} \mathcal{D}_i \right).$$

By [4], $\mathcal{D}^*$ and $\mathcal{D}_i$ are alternating diagrams representing fibered links.

Since $R_i$ contains no other nested Seifert circles, $\mathcal{D}_i$ is special. By Lemma 3.4, $C_i$ is connected to another circle in $R_i$ by at least a pair of parallel bands.

We claim that $\mathcal{D}^*$ is special. Otherwise, let $C$ be an extremal nested Seifert circle, and let $R$ be the complementary region of $C$ which contains no other nested Seifert circles in $\mathcal{D}^*$. Since $C_1, \dots, C_m$ is a maximal collection of extremal nested Seifert circles, $R$ must contain at least one $C_i$. By Lemma 3.4, $C_i$ is connected to another circle in $R \setminus R_i$ (including $C$) by at least a pair of parallel bands. This is a contradiction to Lemma 3.6.

Now $\mathcal{D}^*$ is special. There are at least two Seifert circles in $\mathcal{D}^*$, since $C_1$ is nested in $\mathcal{D}$. By Lemma 3.4, $C_1$ is connected to another Seifert circle in $\mathcal{D}^*$ by at least a pair of parallel bands. We again get a contradiction to Lemma 3.6. Hence $\mathcal{D}$ does not contain any nested Seifert circle. This finishes our proof. □

*Proof of Theorem* 1.2 By [14], $\widehat{HFK}(S^3, K)$ is thin. It follows from Proposition 2.2 that $K$ is strongly quasipositive and fibered. Using Proposition 3.7, $K$ is a connected sum of $T_{2n_i+1,2}$. The condition on the Alexander polynomial forces $K$ to be $T_{2g+1,2}$. □

## References


[1] Burde, G., Zieschang, H.: Knots, Second Edition, De Gruyter Studies in Mathematics, Vol. 5, Walter de Gruyter & Co., Berlin, 2003

[2] Crowell, R.: Genus of alternating link types, *Ann. of Math.* (2), **69**, 258–275 (1959)

[3] Fox, R. H.: Some problems in knot theory, In: Topology of 3-manifolds and Related Topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, 168–176

[4] Gabai, D.: The Murasugi sum is a natural geometric operation, Low-dimensional Topology (San Francisco, Calif., 1981), Contemp. Math., Vol. 20, Amer. Math. Soc., Providence, RI, 1983, 131–143

[5] Gabai, D.: Detecting fibred links in $S^3$, *Comment. Math. Helv.*, **61**(4), 519–555 (1986)

[6] Ghiggini, P.: Knot Floer homology detects genus-one fibred knots, *Amer. J. Math.*, **130**(5), 1151–1169 (2008)

[7] Goda, H., Hirasawa, M., Yamamoto, R.: Almost alternating diagrams and fibered links in $S^3$. *Proc. London Math. Soc.* (3), **83**(2), 472–492 (2001)

[8] Goodman, N.: Overtwisted open books from sobering arcs. *Algebr. Geom. Topol.*, **5**, 1173–1195 (2005)

[9] Hartley, R. I.: On two-bridged knot polynomials. *J. Austral. Math. Soc. Ser. A*, **28**(2), 241–249 (1979)

[10] Hedden, M.: Notions of positivity and the Ozsváth–Szabó concordance invariant. *J. Knot Theory Ramifications*, **19**(5), 617–629 (2010)

[11] Murasugi, K.: On the genus of the alternating knot, I, II. *J. Math. Soc. Japan*, **10**, 94–105, 235–248 (1958)

[12] Murasugi, K.: On the Alexander polynomial of alternating algebraic knots. *J. Austral. Math. Soc. Ser. A*, **39**(3), 317–333 (1985)

[13] Ni, Y.: Knot Floer homology detects fibred knots. *Invent. Math.*, **170**(3), 577–608 (2007)

[14] Ozsváth, P., Szabó, Z.: Heegaard Floer homology and alternating knots. *Geom. Topol.*, **7**, 225–254 (2003)

[15] Ozsváth, P., Szabó, Z.: Knot Floer homology and the four-ball genus. *Geom. Topol.*, **7**, 615–639 (2003)

[16] Ozsváth, P., Szabó, Z.: Holomorphic disks and knot invariants. *Adv. Math.*, **186**(1), 58–116 (2004)

[17] Ozsváth, P., Szabó, Z.: Holomorphic disks and genus bounds. *Geom. Topol.*, **8**, 311–334 (2004)

[18] Rasmussen, J.: Floer homology and knot complements, Thesis (Ph.D.)–Harvard University, ProQuest LLC, Ann Arbor, MI, 2003

[19] Rudolph, L.: Quasipositivity as an obstruction to sliceness, *Bull. Amer. Math. Soc.* (*N.S.*), **29**(1), 51–59 (1993)

[20] Stallings, J.: Constructions of fibred knots and links, In: Algebraic and Geometric Topology, (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., Vol. XXXII, Amer. Math. Soc., Providence, RI, 1978, 55–60