Journal of Topology 4 (2011) 799–816
doi:10.1112/jtopol/jtr020

Dehn surgeries on knots in product manifolds

Yi Ni

Abstract

We show that if a surgery on a knot in a product sutured manifold yields the same product sutured manifold, then this knot is a 0- or 1-crossing knot. The proof uses techniques from sutured manifold theory.

1. *Introduction*

An interesting problem on Dehn surgery is: when does a surgery on a knot yield a manifold homeomorphic to the original ambient manifold? The most famous result in this direction is the Knot Complement Theorem proved by Gordon and Luecke [**9**]: when the ambient manifold is S^3, only the unknot admits surgeries that yield S^3.

In this paper, we are going to study this problem for knots in surfaces times an interval. Our main result is as follows.

Theorem 1.1. *Suppose that F is a compact surface and that $K \subset F \times I$ is a knot. Suppose that α is a non-trivial slope on K, and that $N(\alpha)$ is the manifold obtained from $F \times I$ via the α-surgery on K. If the pair $(N(\alpha), (\partial F) \times I)$ is homeomorphic to the pair $(F \times I, (\partial F) \times I)$, then one can isotope K such that its image on F under the natural projection*

$$p\colon\ F \times I \longrightarrow F$$

has either no crossing or exactly 1-crossing.

The slope α can be determined as follows. Let λ_b be the 'blackboard' frame of K associated with the previous projection; namely, λ_b is the frame specified by the surface F. When the projection has no crossing, $\alpha = 1/n$ for some integer n with respect to λ_b; when the minimal projection has exactly 1-crossing, $\alpha = \lambda_b$.

It is easy to see that the surgeries in the statement of Theorem 1.1 do not change the homeomorphism type of the pair $(F \times I, (\partial F) \times I)$. In fact, when K is a 0-crossing knot, it is clear that the $1/n$-surgery preserves the homeomorphism type of the pair. When K is a 1-crossing knot, we can add a 1-handle to $F \times \frac{1}{2}$ near the crossing to get a Heegaard surface F' for $F \times I$. K can be embedded into F' as in Figure 1. The Heegaard surface F' splits $F \times I$ into two parts U_0 and U_1, where U_0 is $F \times [0, \frac{1}{2}]$ with a 1-handle added to $F \times \frac{1}{2}$, and U_1 is $F \times [\frac{1}{2}, 1]$ with a 1-handle added to $-F \times \frac{1}{2}$. The embedding of K can be chosen such that K goes through each of the two 1-handles exactly once. Now the blackboard frame λ_b is the frame specified by F', and the λ_b-surgery on F' cancels each 1-handle with a 2-handle. Hence, the new pair is still homeomorphic to $(F \times I, (\partial F) \times I)$.

Received 27 January 2010; revised 20 June 2011; published online 23 September 2011.

2010 *Mathematics Subject Classification* 57M99, 57R30.

The author was partially supported by an AIM Five-Year Fellowship and NSF grant number DMS-0805807.

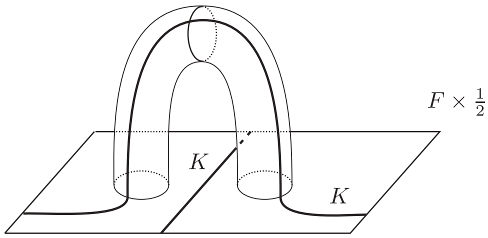

FIGURE 1. *A local picture of the crossing.*

DEFINITION 1.2. Notation is as in the previous theorem. Fix a product structure on $(\partial F) \times I$. Up to an isotopy relative to $(\partial F) \times I$, this product structure uniquely extends to a product structure $\mathcal{P}$ on $F \times I$ and a product structure $\mathcal{P}_\alpha$ on $N(\alpha)$. (This fact can be proved using Alexander's trick.) Identify F with $F \times 1$. Let $i, i_\alpha\colon F \times 0 \to F \times 1$ be the natural identity maps with respect to $\mathcal{P}$ and $\mathcal{P}_\alpha$, respectively. We call

$$\varphi_\alpha = i \circ i_\alpha^{-1}\colon F \longrightarrow F$$

the map induced by the α-surgery. This map φ_α fixes ∂F pointwise, and is unique up to an isotopy relative to ∂F. Hence, φ_α can be viewed as an element in the mapping class group $\mathcal{MCG}(F, \partial F)$.

The definition of the map φ_α is justified by the following lemma.

LEMMA 1.3. *Let $Y(\alpha)$ be the manifold obtained from $F \times S^1$ by α-surgery on K. Then $Y(\alpha)$ can be obtained from $F \times I$ by identifying $(x, 0)$ with $(\varphi_\alpha(x), 1)$ for any $x \in G$.*

Proof. The manifold $F \times S^1$ is obtained from $F \times I$ by identifying y with $i(y)$ for each $y \in F \times 0$. Let $y = (x, 0)$ with respect to the product structure $\mathcal{P}_\alpha$ on $N(\alpha)$; then $i_\alpha(y) = (x, 1)$ with respect to $\mathcal{P}_\alpha$. We then have

$$i(y) = \varphi_\alpha(x, 1) = (\varphi_\alpha(x), 1),$$

as we identify F with $F \times 1$ in the above definition. Hence, $(x, 0)$ is identified with $(\varphi_\alpha(x), 1)$ in $Y(\alpha)$ for each $x \in F$. □

The following proposition is a consequence of Lemma 5.4.

PROPOSITION 1.4. *Notation is as in Theorem* 1.1. *When the projection of K has no crossing and $\alpha = 1/n$,*

$$\varphi_\alpha = \tau^n,$$

where τ is the right-hand Dehn twist along $K \subset F$. When the minimal projection of K has exactly 1-crossing, let a, b, c be the simple closed curves obtained by resolving the crossing in two different ways as in Figure 2 and let τ_a, τ_b, τ_c be the right-hand Dehn twists along a, b, c, respectively. Then

$$\varphi_\alpha = \tau_a^2 \tau_b^2 \tau_c^{-1}$$

when the crossing is positive, and $\varphi_\alpha = \tau_a^{-2} \tau_b^{-2} \tau_c$ when the crossing is negative.

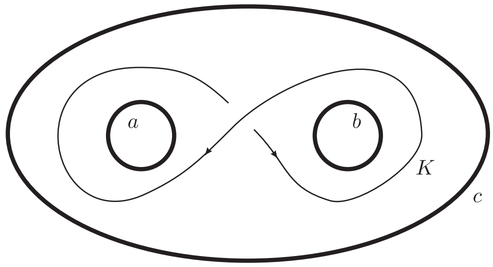

FIGURE 2. *A* 1*-crossing knot.*

This paper can be compared with Ni [**10**]. In fact, [**10**, Theorem 1.4] can be stated in a form similar to Theorem 1.1.

THEOREM 1.5. *Suppose that F is a compact surface, that $K \subset F \times I$ is a knot and that α is a slope on K. Let $N(\alpha)$ be the manifold obtained by the α-surgery on K. If $F \times \{0\}$ is not Thurston norm minimizing in $H_2(N(\alpha), (\partial F) \times I)$, then there is an ambient isotopy of $F \times I$ that takes K to a curve in $F \times \{\frac{1}{2}\}$. Moreover, α is the frame on K specified by $F \times \{\frac{1}{2}\}$.*

The proof of Theorem 1.5 uses Gabai's sutured manifold theory [**2**, **4**, **5**] and an argument due to Ghiggini [**8**]. Using a different method, Scharlemann and Thompson [**12**] get the same conclusion of Theorem 1.5 under the assumption that $F \times \{0\}$ is compressible in $N(\alpha)$.

This paper is organized as follows. In Section 2, we give some preliminaries on sutured manifold theory and foliations, as well as a characterization of 1-crossing knot projections. In Section 3, we study some warm-up cases. In Section 4, we use the argument in the proof of Theorem 1.5 to reduce our problem to the case where F is a pair of pants. In Section 5, we study this case by analyzing the map induced by surgery and using a variant of the argument in Ni [**10**].

This article is dedicated to the memory of Professor Andrew Lange.

2. *Preliminaries*

In this section, we are going to review the sutured manifold theory introduced by Gabai [**2**]. We also state a uniqueness result for the Euler classes of taut foliations of fibered manifolds. In addition, we define 'double primitive' knots in $F \times I$ and show that they are exactly the knots with a projection consisting of only 1-crossing.

2.1. *Sutured manifold decompositions*

DEFINITION 2.1. A *sutured manifold* (M, γ) is a compact oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. The core of each component of $A(\gamma)$ is a *suture*, and the set of sutures is denoted by $s(\gamma)$.

Every component of $R(\gamma) = \partial M - \text{int}(\gamma)$ is oriented. Define $R_+(\gamma)$ or $R_-(\gamma)$ to be the union of those components of $R(\gamma)$ whose normal vectors point out of or into M, respectively. The orientations on $R(\gamma)$ must be coherent with respect to $s(\gamma)$, hence every component of $A(\gamma)$ lies between a component of $R_+(\gamma)$ and a component of $R_-(\gamma)$.

As an example, let S be a compact oriented surface, $M = S \times I$, $\gamma = (\partial S) \times I$, $R_-(\gamma) = S \times 0, R_+(\gamma) = S \times 1$; then (M, γ) is a sutured manifold. In this case, we say that (M, γ) is a *product sutured manifold*.

Definition 2.2. Let S be a compact oriented surface with connected components $S_1, \ldots, S_n$. We define

$$x(S) = \sum_i \max\{0, -\chi(S_i)\}.$$

Let M be a compact oriented 3-manifold and A be a compact codimension-0 submanifold of ∂M. Let $h \in H_2(M, A)$. The *Thurston norm* $x(h)$ of h is defined to be the minimal value of $x(S)$, where S runs over all the properly embedded surfaces in M with $\partial S \subset A$ and $[S] = h$.

Definition 2.3. Let (M, γ) be a sutured manifold, and S be a properly embedded surface in M, such that no component of ∂S bounds a disk in $R(\gamma)$ and no component of S is a disk with boundary in $R(\gamma)$. Suppose that, for every component λ of $S \cap \gamma$, one of the following holds:

(1) λ is a properly embedded non-separating arc in γ;
(2) λ is a simple closed curve in an annular component A of γ in the same homology class as $A \cap s(\gamma)$;
(3) λ is a homotopically non-trivial curve in a toral component T of γ, and if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $H_1(T)$.

Then S is called a *decomposing surface*, and S defines a *sutured manifold decomposition*

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma'),$$

where $M' = M - \mathrm{int}(\mathrm{Nd}(S))$ and

$$\begin{aligned} \gamma' &= (\gamma \cap M') \cup \mathrm{Nd}(S'_+ \cap R_-(\gamma)) \cup \mathrm{Nd}(S'_- \cap R_+(\gamma)), \\ R_+(\gamma') &= ((R_+(\gamma) \cap M') \cup S'_+) - \mathrm{int}(\gamma'), \\ R_-(\gamma') &= ((R_-(\gamma) \cap M') \cup S'_-) - \mathrm{int}(\gamma'), \end{aligned}$$

where S'_+ or S'_- is that component of $\partial \mathrm{Nd}(S) \cap M'$ whose normal vector points out of or into M', respectively.

Definition 2.4. A sutured manifold (M, γ) is *taut*, if M is irreducible and $R(\gamma)$ is Thurston norm minimizing in $H_2(M, \gamma)$.

Suppose that S is a decomposing surface in (M, γ) and that S decomposes (M, γ) to (M', γ'). The surface S is *taut* if (M', γ') is taut. In this case, we also say that the sutured manifold decomposition is *taut*.

Definition 2.5. Suppose

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$$

is a taut decomposition; by Gabai [**2**] we can extend this decomposition to a sutured manifold hierarchy of (M, γ), from which we can construct a taut foliation $\mathscr{F}$ of M, such that $R(\gamma)$ consists of compact leaves of $\mathscr{F}$. We then call $\mathscr{F}$ a *foliation induced by* S. Moreover, when $R_+(\gamma)$ is homeomorphic to $R_-(\gamma)$, from M we can obtain a manifold Y with boundary consisting of tori by gluing $R_+(\gamma)$ to $R_-(\gamma)$ via a homeomorphism. The foliation $\mathscr{F}$ then becomes a taut foliation $\mathscr{F}_1$ of Y. We also say that $\mathscr{F}_1$ is a *foliation induced by* S.

DEFINITION 2.6. A decomposing surface is called a *product disk* if it is a disk that intersects $s(\gamma)$ in exactly two points. A decomposing surface is called a *product annulus* if it is an annulus with one boundary component in $R_+(\gamma)$ and the other boundary component in $R_-(\gamma)$. The sutured manifold decomposition associated to a product disk or product annulus is called a *product decomposition*.

By definition, a product annulus is always a decomposing surface, so none of its components can bound a disk in $R(\gamma)$. This rules out some trivial cases.

The following lemma is a standard fact (see Gabai [**3**, Lemmas 2.2 and 2.5]).

LEMMA 2.7. *Suppose* $(M, \gamma) \stackrel{S}{\rightsquigarrow} (M', \gamma')$ *is a product decomposition. Then* (M, γ) *is a product sutured manifold if and only if* (M', γ') *is a product sutured manifold.*

We recall the main result in Gabai [**4**], which has been intensively used in Ni [**10**]. Note that the result is not stated in its original form, but it is contained in the argument in [**4**]. See also [**10**, Theorem 2.8] for a sketch of the proof.

DEFINITION 2.8. An *I-cobordism* between closed connected surfaces T_0 and T_1 is a compact 3-manifold V such that $\partial V = T_0 \cup T_1$ and, for $i = 0, 1$, the induced maps $j_i\colon\ H_1(T_i) \to H_1(V)$ are injective.

DEFINITION 2.9. Suppose that M is a 3-manifold and that T is a toral component of ∂M. If all tori in M that are I-cobordant to T in M are parallel to T, then we say that M is *T-atoroidal*.

THEOREM 2.10 (Gabai). *Let* (M, γ) *be a taut sutured* 3*-manifold. Suppose that* T *is a toral component of* γ*, and that* S *is a decomposing surface such that* $S \cap T = \emptyset$ *and the decomposition*

$$(M, \gamma) \stackrel{S}{\rightsquigarrow} (M_1, \gamma_1)$$

is taut. Suppose that M *is* T*-atoroidal*; *then, for any slope* α *on* T *except at most one slope, the decomposition after Dehn filling*

$$(M(\alpha), \gamma \backslash T) \stackrel{S}{\rightsquigarrow} (M_1(\alpha), \gamma_1 \backslash T)$$

is taut.

A special case of the above theorem is the case $\gamma = \partial M$, which is the original form in [**4**].

2.2. *Euler classes of foliations*

We shall need the Euler classes of foliations from Thurston [**13**, Section 3].

DEFINITION 2.11. Suppose that Y is a compact 3-manifold with ∂Y consisting of tori. We denote by $\mathscr{P}$ an oriented plane field transverse to ∂Y. Let $T(\partial Y)$ be the tangent plane field of ∂Y. The line field $\mathscr{P} \cap T(\partial Y)$ has a natural orientation induced by the orientations of $\mathscr{P}$ and $T(\partial Y)$, thus it has a nowhere vanishing section $v \subset \mathscr{P}|_{\partial Y}$. Then one can define the *relative*

Euler class

$$e(\mathscr{P}) \in H^2(Y, \partial Y)$$

of $\mathscr{P}$ to be the obstruction to extending v to a nowhere vanishing section of $\mathscr{P}$. When $\mathscr{F}$ is a foliation of Y that is transverse to ∂Y, let $T\mathscr{F}$ be the tangent plane field of $\mathscr{F}$ and let $e(\mathscr{F}) = e(T\mathscr{F})$.

Definition 2.12. Suppose that C is a properly embedded curve in a compact surface F. We say C is *efficient* in F if

$$|C \cap \delta| = |[C] \cdot [\delta]|, \quad \text{for each boundary component } \delta \text{ of } F.$$

Suppose that S is a properly embedded surface in a compact 3-manifold Y with boundary consisting of tori. We say S is *efficient* in Y if $S \cap T$ consists of coherently oriented parallel essential curves for each boundary component T of Y.

Proposition 2.13. *Suppose that Y is a compact 3-manifold that fibers over S^1. Let G be a fiber of the fibration $\mathscr{E}$. Suppose that $\mathscr{F}$ is a taut foliation of Y that is transverse to ∂Y such that G is a leaf of $\mathscr{F}$. Then*

$$e(\mathscr{F}) = e(\mathscr{E}) \in H^2(Y, \partial Y)/\mathrm{Tors}.$$

Proof. This result follows easily from the fact that the Floer homology of a fibered manifold is 'monic', namely, the topmost term of the Floer homology is one-dimensional. Using this approach, one can even prove that the two Euler classes are equal in $H^2(Y, \partial Y)$. Here we present a more geometric proof.

To prove the desired result, we only need to show that

$$\langle e(\mathscr{F}), h\rangle = \langle e(\mathscr{E}), h\rangle, \tag{1}$$

for any $h \in H_2(Y, \partial Y)$. When $h = [G]$, we have

$$\langle e(\mathscr{F}), [G]\rangle = \langle e(\mathscr{E}), [G]\rangle = \chi(G). \tag{2}$$

In general, suppose $\overline{U} \subset Y$ is a proper surface representing h such that $\overline{U} \pitchfork G$. We can choose the representative $\overline{U}$ such that $\overline{U}$ is efficient in Y. Then $\overline{U} \cap G$ can also be made efficient in G. Cutting Y open along G, we get $G \times I$. Let $U \subset G \times I$ be the proper surface obtained by cutting $\overline{U}$ open along $C = \overline{U} \cap G$. Let $C_0, C_1 \subset G$ be proper oriented curves such that

$$-C_0 \times 0 = (\partial U) \cap (G \times 0), \quad C_1 \times 1 = (\partial U) \cap (G \times 1).$$

As C_0 and C_1 are homologous efficient curves in G relative to ∂G, as in the proof of Gabai [**4**, Lemma 0.6], we can find compact subsurfaces $V_1, V_2, \ldots, V_n$ and efficient curves

$$C_0 = \gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n = C_1$$

in G, such that

$$\overline{\partial V_i \backslash (\partial G)} = \gamma_i \cup (-\gamma_{i-1}).$$

Let $W_i = \overline{G \backslash V_i}$. Perturbing the surface

$$\bigcup_{i=1}^{n} \left(\left(-\gamma_{i-1} \times \left[\frac{i-1}{n}, \frac{i}{n} \right] \right) \cup \left(V_i \times \frac{i}{n} \right) \right) \tag{3}$$

slightly, we get a proper surface $V \subset G \times I$, such that

$$(\partial V) \cap (G \times 0) = -C_0 \times 0, \quad (\partial V) \cap (G \times 1) = C_1 \times 1.$$

Similarly, perturbing the surface

$$\bigcup_{i=1}^{n}\left(\left(\gamma_{i-1}\times\left[\frac{i-1}{n},\frac{i}{n}\right]\right)\cup\left(W_i\times\frac{i}{n}\right)\right) \tag{4}$$

slightly, we have a proper surface $W\subset G\times I$, such that

$$(\partial W)\cap(G\times 0)=C_0\times 0,\quad (\partial W)\cap(G\times 1)=-C_1\times 1.$$

Let $\overline{V}\subset Y$ be the proper surface obtained from V by identifying $C_0\times 0$ and $C_1\times 1$ with $C\subset G\subset Y$. Similarly, define $\overline{W}\subset Y$. Note that

$$[\overline{V}]-[\overline{U}]=[V\cup(-U)]\in H_2(Y,\partial Y).$$

Perturbing $V\cup(-U)$ slightly, we get a properly immersed surface in Y, which is disjoint from the fiber G. So $[V\cup(-U)]=m[G]$ for some integer m. Using (2), in order to check (1) for $h=[\overline{U}]$, we only need to check it for $h=[\overline{V}]$.

As $\mathscr{F}$ is taut, by Thurston [**13**, Corollary 1] we have

$$\chi(\overline{V})\leqslant\langle e(\mathscr{F}),[\overline{V}]\rangle,$$
$$\chi(\overline{W})\leqslant\langle e(\mathscr{F}),[\overline{W}]\rangle.$$

Adding the two inequalities together, we get

$$\chi(\overline{V})+\chi(\overline{W})\leqslant\langle e(\mathscr{F}),[\overline{V}]+[\overline{W}]\rangle. \tag{5}$$

By the constructions (3) and (4), the result of doing oriented cut-and-pastes to $\overline{V}$ and $\overline{W}$ is n copies of G. So the left-hand side of (5) is $n\chi(G)$, whereas the right-hand side is $\langle e(\mathscr{F}),n[G]\rangle=n\chi(G)$. So the equality holds. In particular, we should have

$$\chi(\overline{V})=\langle e(\mathscr{F}),[\overline{V}]\rangle.$$

The same argument shows that

$$\chi(\overline{V})=\langle e(\mathscr{E}),[\overline{V}]\rangle,$$

so (1) holds for $h=[\overline{V}]$. As we have checked (1) for all elements $h\in H_2(Y,\partial Y)$, $e(\mathscr{F})$ is equal to $e(\mathscr{E})$ up to a torsion element in $H^2(Y,\partial Y)$. □

2.3. *A characterization of* 1*-crossing knot projections*

In this subsection, we give a characterization of 1-crossing knot projections in terms of double primitive knots. This fact is not used in the current paper, but it is useful to bear this in mind.

DEFINITION 2.14. Let $F'\subset F\times I$ be a connected surface of genus $g(F)+1$, and $\partial F'=(\partial F)\times\frac{1}{2}$. Suppose that F' is a Heegaard surface; namely, F' splits $F\times I$ into two parts U_0 and U_1, such that U_0 is homeomorphic to $(F\times[0,\frac{1}{2}])\cup H_1$, and U_1 is homeomorphic to $(F\times[\frac{1}{2},1])\cup H_2$, where H_1 is a 1-handle with feet on $F\times\frac{1}{2}$ and H_2 is a 1-handle with feet on $-F\times\frac{1}{2}$. A knot $K\subset F\times I$ is a *double primitive* knot if it is isotopic to a curve on F' that goes through each of H_1,H_2 exactly once.

LEMMA 2.15. *A knot $K\subset F\times I$ is double primitive if and only if it has a projection that has only* 1*-crossing.*

Proof. If a knot has a 1-crossing projection, then it is double primitive as shown in Section 1. Now assume that K is double primitive; then K is embedded into a Heegaard surface F' as in the above definition.

We claim that F' is stabilized; namely, there is a compressing disk $D_0 \subset U_0$ and a compressing disk $D_1 \subset U_1$ such that $|(\partial D_0) \cap (\partial D_1)| = 1$. When F is closed, this follows from the theorem of Scharlemann and Thompson [**11**] that the Heegaard splittings of $F \times I$ are standard. When F is not closed, let R be the torus with one hole. We can glue a copy of R to each component of ∂F; then F becomes a closed surface G and F' becomes a Heegaard surface G' in $G \times I$. Using Scharlemann and Thompson's theorem, G' is stabilized, hence there are compressing disks D_0 and D_1 in the two compression bodies separated by G', such that $|(\partial D_0) \cap (\partial D_1)| = 1$. Using standard arguments, we can isotope D_0 and D_1 to be disjoint from the copies of $R \times I$, so $D_0 \subset U_0$ and $D_1 \subset U_1$; thus our claim follows.

As $g(F') = g(F) + 1$, after compressing F' along D_0, we get a surface homeomorphic to F (and hence parallel to $F \times 0$ in $F \times I$). In other words, if we cut U_0 open along D_0, then we get a submanifold isotopic to $F \times [0, \frac{1}{2}]$. Thus, we can think of D_0 as the cocore of a 1-handle attaching to $F \times [0, \frac{1}{2}]$. Let $D_0^+, D_0^- \subset F \times [\frac{1}{2}]$ be the attaching regions of this 1-handle.

We claim that D_0 is the unique non-separating compressing disk in U_0 up to isotopy. Otherwise, suppose that D_0' is another non-separating compressing disk. A standard outermost-disk argument enables us to isotope D_0' to be disjoint from D_0. As $F \times \frac{1}{2}$ is incompressible and $F \times [0, \frac{1}{2}]$ is irreducible, $\partial D_0'$ bounds a disk Δ_0 in $F \times \frac{1}{2}$ and $D_0' \cup \Delta_0$ bounds a 3-ball. As D_0' is a compressing disk in U_0, Δ_0 must contain at least one of the attaching regions D_0^+, D_0^-. On the other hand, Δ_0 can contain at most one of D_0^+, D_0, as D_0' is non-separating. Hence, D_0' is isotopic to D_0.

The last paragraph shows that there is (up to isotopy) only one 1-handle H_0 in U_0 that can be viewed as attached to $F' \times [0, \frac{1}{2}]$. The disk D_1 is attached to U_0 such that ∂D_1 goes through the 1-handle exactly once. Similarly, D_1 is the cocore of the unique 1-handle H_1 in U_1.

Consider a neighborhood of $D_0 \cup D_1$. The local picture of F' in this neighborhood looks exactly like in Figure 1. The knot K goes through the 1-handle H_0 once and intersects ∂D_1 once, so there is a crossing near D_1 and no crossing elsewhere. □

3. *Warm-up cases*

In this section, we are going to prove some easy cases of our main theorem. When F is a disk or sphere, our result follows from Gordon and Luecke's Knot Complement Theorem [**9**]. When F is an annulus, we have the following lemma.

Lemma 3.1. *Theorem* 1.1 *is true when F is an annulus.*

Proof. Let $\mathcal{M}$ be the meridian of the solid torus $V = F \times I$, and $\mathcal{L}$ be the frame of V specified by ∂F. By Gabai [**6**], if K is non-trivial, then K is a 0- or 1-bridge braid in $F \times I$.

Capping off one boundary component of F with a disk, we get a disk D. Let λ be the Seifert frame of K in $D \times I$ and μ be the meridian of K.

If K is the core of V, then the surgery preserves the homeomorphism type of $(F \times I, (\partial F) \times I)$ if and only if the slope is $\mu + n\lambda$ for some integer n.

From now on we assume that the braid index of K is greater than 1.

If K is a 0-bridge braid, then K is isotopic to $p\mathcal{L} + q\mathcal{M}$ on ∂V for some $p, q \in \mathbb{Z}$. Let Λ be the frame on K specified by ∂V; then $\Lambda = pq\mu + \lambda$. A surgery on K yields a solid torus if and only if the slope α of the surgery satisfies $\Delta(\alpha, \Lambda) = 1$, namely, when the slope α is $\mu + n\Lambda$ for some integer n. Now $p\alpha = p\mu + pn\Lambda$ is homologous to $\mathcal{M} + pn(p\mathcal{L} + q\mathcal{M})$ in $V \backslash K$, so the meridian of the new ambient solid torus after surgery is $(1 + pqn)\mathcal{M} + p^2n\mathcal{L}$. As the surgery preserves the homeomorphism type of the pair $(F \times I, (\partial F) \times I)$, we must have $\Delta((1 + pqn)\mathcal{M} + p^2n\mathcal{L}, \mathcal{L}) = 1$, thus $1 + pqn = \pm 1$. As $p > 1, n \neq 0$, we have $(p, q, n) =$

$(2, 1, -1)$ or $(2, -1, 1)$. When $(p, q) = (2, 1)$, the slope α on K is

$$\mu + n(pq\mu + \lambda) = (1 + pqn)\mu + n\lambda,$$

which is 1 with respect to the frame λ, and the meridian of the new ambient solid torus is $\mathcal{M} + 4\mathcal{L}$; when $(p, q) = (2, -1)$, the slope α on K is -1, and the meridian of the new ambient solid torus is $\mathcal{M} - 4\mathcal{L}$. As K is a $(2, \pm 1)$-cable in V, the projection of K on F has exactly 1-crossing. We can check that α is the blackboard frame.

If K is a 1-bridge braid, then K is determined by three parameters ω, b, t by Gabai [7]. Here $\omega > 0$ is the braid index, $1 \leqslant b \leqslant \omega - 2$, $t \equiv r \pmod{\omega}$ for some integer r with $1 \leqslant r \leqslant \omega - 2$. As the α-surgery yields a solid torus, by Gabai [7, Lemma 3.2] the slope of the surgery is $\lambda - (t\omega + d)\mu$, where $d \in \{b, b + 1\}$. So $t\omega + d = \pm 1$, which is impossible for any ω, b, t satisfying the previous restrictions. □

Lemma 3.2. *In the above lemma, let $\varphi_\alpha \in \mathcal{MCG}(F, \partial F)$ be the map induced by the α-surgery. If K is the core of $F \times I$ and $\alpha = 1/n$, then $\varphi_\alpha = \tau^n$, where τ is the right-hand Dehn twist in F; if K is the $(2, \pm 1)$-cable in $F \times I$, then $\varphi_\alpha = \tau^{\pm 4}$.*

Proof. When K is the core of $F \times I$, the conclusion is well known. When K is the $(2, \pm 1)$-cable, then, from the proof of the previous lemma, we know that the meridian of the new ambient solid torus is $\mathcal{M} \pm 4\mathcal{L}$, hence the conclusion follows from the first case. □

The following lemma easily follows from Lemma 2.7.

Lemma 3.3. *Suppose $(C \times I) \subset (F \times I)$ is a product disk or product annulus, $(C \times I) \cap K = \emptyset$. Let F_1 be the surface obtained from F by cutting F open along C, and let N_1 be the manifold obtained from $N = (F \times I)\backslash \mathrm{int}(\mathrm{Nd}(K))$ by cutting N open along $C \times I$. Then the pair $(N(\alpha), (\partial F) \times I)$ is homeomorphic to $(F \times I, (\partial F) \times I)$ if and only if the pair $(N_1(\alpha), (\partial F_1) \times I)$ is homeomorphic to $(F_1 \times I, (\partial F_1) \times I)$.*

Lemma 3.4. *Theorem 1.1 is true when F is a torus.*

Proof. Let $C \subset F$ be a simple closed curve such that K is homologous to a multiple of C. Consider the homology class $[C \times I] \in H_2(F \times I, \partial(F \times I))$; then $[C \times I] \cdot [K] = 0$. It follows that $[C \times I]$ is also a homology class in $H_2((F \times I)\backslash K, \partial(F \times I))$.

Let $(S, \partial S) \subset ((F \times I)\backslash K, \partial(F \times I))$ be a taut surface representing $[C \times I]$. By Theorem 2.10, S remains taut in at least one of the original $F \times I$ and $N(\alpha) \cong F \times I$. Hence, S must be a product annulus. Cutting $F \times I$ open along S, K becomes a knot in (annulus $\times I$). Now we can apply Lemmas 3.1 and 3.3 to get our conclusion. □

Lemma 3.5. *If the conclusion of Theorem 1.1 holds for all knots whose exteriors are $\partial(\mathrm{Nd}(K))$-atoroidal, then the conclusion holds for all knots in $F \times I$.*

Proof. By assumption, we only need to consider the case where there is a torus in $N = F \times I\backslash \mathrm{int}(\mathrm{Nd}(K))$, which is I-cobordant but not parallel to $\partial \mathrm{Nd}(K)$. Let R be an 'innermost' such torus.

By Ni [**10**, Lemma 3.1], R bounds a solid torus U in $F\times I$, such that $K\subset U$. As R is innermost in N, if a torus in $(F\times I)\backslash\mathrm{int}(U)$ is I-cobordant to $\partial U=R$, then this torus is parallel to R. Let V be the manifold obtained from U by α-surgery on K.

By Gabai [**6**], one of the following cases holds.

The manifold (1) $V=D^2\times S^1$. In this case, K is a 0-bridge or 1-bridge braid in U, and the core K' of the surgery is also a 0-bridge or 1-bridge braid in V. Moreover, K and K' have the same braid index ω.

The manifold (2) $V=(D^2\times S^1)\#W$, where W is a closed 3-manifold and $1<|H_1(W)|<\infty$.

The manifold (3) V is irreducible and ∂V is incompressible.

As $V\subset N(\alpha)\cong F\times I$, Cases (2) and (3) cannot occur, so the only possible case is (1). Thus, the core of U is a knot such that a surgery on the knot yields the pair $(N(\alpha),(\partial F)\times I)$, which is homeomorphic to $(F\times I,(\partial F)\times I)$. Moreover, $N\backslash\mathrm{int}(U)$ is ∂U-atoroidal. By our assumption, the core of U is a 0-crossing or 1-crossing knot in $F\times I$.

If the core of U is isotopic to $\eta\times\{\frac{1}{2}\}$ for some simple closed curve $\eta\subset F$, let $G\subset F$ be a tubular neighborhood of η, and then K lies in $G\times I$ after an isotopy. Let $M=(G\times I)\backslash\mathrm{int}(\mathrm{Nd}(K))$. By Lemma 3.3, $(M(\alpha),(\partial G)\times I)$ is homeomorphic to $(G\times I,(\partial G)\times I)$. Applying Lemma 3.1, we find that K is the $(2,\pm 1)$-cable of the core of $G\times I$, and the slope α is the blackboard frame λ_b.

If the core of U is a 1-crossing knot, then the blackboard frame λ_b' on ∂U is the meridian of V, so λ_b' cobounds a punctured disk with ω oriented copies of α in $U\backslash\mathrm{int}(\mathrm{Nd}(K))$. Moreover, the meridian μ' on ∂U cobounds a punctured disk with ω oriented copies of μ in $U\backslash\mathrm{int}(\mathrm{Nd}(K))$. As $[\lambda_b']\cdot[\mu']=1$, considering the intersection of the two punctured disks, we conclude that $\omega=1$. Hence, ∂U is parallel to $\partial\mathrm{Nd}(K)$, which is a contradiction. □

In light of the above lemma, from now on we assume that the exterior of the knot K is $\partial(\mathrm{Nd}(K))$-atoroidal.

4. *Comparing Euler classes of foliations*

Suppose $K\subset F\times I$ is a knot satisfying the hypothesis of Theorem 1.1. Let E be a maximal (up to isotopy) compact essential subsurface of F, such that K can be isotoped in $F\times I$ to be disjoint from $E\times I$. Let $G=\overline{F\backslash E}$.

The goal of this section is to prove the following proposition.

PROPOSITION 4.1. *The subsurface G is either an annulus or a pair of pants.*

Let $T=\partial(\mathrm{Nd}(K))$, $\gamma=((\partial G)\times I)\cup T$.

Let $N=(F\times I)\backslash\mathrm{int}(\mathrm{Nd}(K))$, $M=(G\times I)\backslash\mathrm{int}(\mathrm{Nd}(K))$.

Then the sutured manifold (M,γ) contains no product disks or product annuli. For a proper surface $S\subset M$, let $\partial_i(S)=S\cap(G\times i)$, $i=0,1$.

Let $X=(G\times S^1)\backslash\mathrm{int}(\mathrm{Nd}(K))$ be the manifold obtained from M by gluing $G\times 1$ to $G\times 0$ via the identity map of G. Suppose that ξ is a slope on K. Let $N(\xi),M(\xi),X(\xi)$ be the manifolds obtained from N,M,X by ξ-filling on T, respectively. Let $K(\xi)\subset M(\xi)$ be the core of the new solid torus.

By Lemma 3.3, $X(\xi)$ is a surface bundle over S^1 with fiber G when $\xi=\infty$ or α. We then let $\mathscr{E}(\xi)$ be the fibration of $X(\xi)$.

LEMMA 4.2. *Let $K\subset F\times I$ be as in Theorem 1.1. Suppose that N is T-atoroidal. Let $\overline{S}\subset X$ be the surface obtained from S by gluing ∂_0S to ∂_1S via the identity map. Let $\mathscr{F}$ be a* *taut foliation of X induced by S. Then*

$$\langle e(\mathscr{F}), [\overline{S}]\rangle = \langle e(\mathscr{E}(\xi)), [\overline{S}]\rangle = \chi(\overline{S}),$$

for some $\xi \in \{\infty, \alpha\}$.

Proof. By Theorem 2.10, S remains taut in $M(\xi)$ for some $\xi \in \{\infty, \alpha\}$. Let $\mathscr{F}'$ be a taut foliation of $X(\xi)$ induced by S. By Proposition 2.13,

$$e(\mathscr{F}') = e(\mathscr{E}(\xi)) \in H^2(X(\xi), \partial X(\xi); \mathbb{Q}).$$

As both $\mathscr{F}$ and $\mathscr{F}'$ are induced by S, we have

$$\chi(\overline{S}) = \langle e(\mathscr{F}), [\overline{S}]\rangle = \langle e(\mathscr{F}'), [\overline{S}]\rangle = \langle e(\mathscr{E}(\xi)), [\overline{S}]\rangle.$$ □

Proposition 4.3. *Let $K \subset F \times I$ be as in Theorem 1.1. Suppose that N is T-atoroidal.*

$$i_*\colon\ H_1(K;\mathbb{Q}) \longrightarrow H_1(G;\mathbb{Q}).$$

Let

$$\mathcal{V} = \{v \in H_1(G, \partial G; \mathbb{Q}) |\ v \cdot i_*[K] = 0\}.$$

Then the dimension of $\mathcal{V}$ is at most 1.

Let

$$\rho_\xi\colon\ H^2(X, \partial X; \mathbb{Q}) \longrightarrow H^2(X(\xi), \partial X(\xi); \mathbb{Q})$$

be the map induced by the map of pairs

$$(X(\xi), \partial X(\xi)) \longrightarrow (X(\xi), (\partial X(\xi)) \cup K(\xi)).$$

Lemma 4.4. *Notation is as in Proposition 4.3. If the dimension of $\mathcal{V}$ is greater than 1, then there exists a properly embedded surface $H \subset X$ such that the following conditions are satisfied:*

(1) *$[H]$ is not a multiple of $[G]$;*
(2) *$H \cap T = \emptyset$;*
(3) *for any two elements $\varepsilon_\infty \in \rho_\infty^{-1}(e(\mathscr{E}(\infty))), \varepsilon_\alpha \in \rho_\alpha^{-1}(e(\mathscr{E}(\alpha)))$, we have*

$$\langle \varepsilon_\infty, [H]\rangle = \langle \varepsilon_\alpha, [H]\rangle.$$

Proof. There is a natural injective map

$$\sigma\colon\ H_1(G, \partial G) \longrightarrow H_2(G \times S^1, \partial G \times S^1)$$

defined via multiplying with the S^1 factor. Moreover, all elements in $\sigma(\mathcal{V})$ are represented by surfaces that are disjoint from K, hence $\sigma|\mathcal{V}$ induces an injective map

$$\widetilde{\sigma}\colon\ \mathcal{V} \longrightarrow H_2(X, \partial X; \mathbb{Q}).$$

We pick two elements $\varepsilon'_\infty \in \rho_\infty^{-1}(e(\mathscr{E}(\infty)))$ and $\varepsilon'_\alpha \in \rho_\alpha^{-1}(e(\mathscr{E}(\alpha)))$. If $\dim \mathcal{V} > 1$, then there exists a non-zero integral element $h \in \widetilde{\sigma}(\mathcal{V})$ such that

$$\langle \varepsilon'_\infty, h\rangle = \langle \varepsilon'_\alpha, h\rangle.$$

Let $H \subset X$ be a proper surface representing h such that $H \cap T = \emptyset$. We claim that this H is what we need. We only need to check (3) as the first two conditions are obvious.

From the Mayer–Vietoris sequence

$$H^1(K(\xi)) \longrightarrow H^2(X, \partial X) \xrightarrow{\rho_\xi} H^2(X(\xi), \partial X(\xi)),$$

and the fact that $h \cdot [T] = 0$, we conclude that $\langle \varepsilon_\xi, h\rangle$ does not depend on the choice of $\varepsilon_\xi \in \rho_\xi^{-1}(e(\mathscr{E}(\xi)))$. Hence, (3) holds. □

Assume that the dimension of $\mathcal{V}$ is greater than 1, let H be a surface as in Lemma 4.4 and suppose $H \pitchfork G$. Without loss of generality, we can assume that no component of $C = H \cap G$ is null-homologous in $H_1(G, \partial G)$, and H is efficient in $G \times S^1$, hence we can also assume $H \cap G$ is efficient in G.

Let $p \in G \backslash C$ be a point. Performing cut-and-paste of H with copies of G or $-G$ if necessary, we may assume that

$$[H] \cdot [p \times S^1] = 0.$$

Let $\mathcal{S}_m(+C)$ be the set of properly embedded oriented surfaces $S \subset G \times I$, such that $S \cap K = \emptyset$, $\partial_0 S = -C \times 0$, $\partial_1 S = C \times 1$, and the algebraic intersection number between S and $p \times I$ is m. Similarly, let $\mathcal{S}_m(-C)$ be the set of properly embedded surfaces $S \subset G \times I$, such that $S \cap K = \emptyset$, $\partial_0 S = C \times 0$, $\partial_1 S = -C \times 1$, and the algebraic intersection number of S with $p \times I$ is m. As $[C] \cdot i_*([K]) = 0$, it follows that $\mathcal{S}_m(\pm C) \neq \emptyset$.

Suppose $S \subset M$ is a properly embedded surface that is transverse to $\partial G \times 0$. For any component S_0 of S we define

$$y(S_0) = \max\left\{\frac{|S_0 \cap (\partial G \times 0)|}{2} - \chi(S_0), 0\right\},$$

and let $y(S)$ be the sum of $y(S_i)$ with S_i running over all components of S. Let $y(\mathcal{S}_m(\pm C))$ be the minimal value of $y(S)$ for all $S \in \mathcal{S}_m(\pm C)$.

Lemma 4.5. *When m is sufficiently large, there exist surfaces $S_1 \in \mathcal{S}_m(+C)$ and $S_2 \in \mathcal{S}_m(-C)$ such that they are taut.*

Proof. Let $x(\cdot)$ be the Thurston norm in $H_2(X, \partial X)$. There exists $N \geqslant 0$, such that if $k > N$, then $x([H] + (k+1)[G]) = x([H] + k[G]) + x(G)$. As in the proof of Gabai [**2**, Theorem 3.13], if $\overline{Q}$ is a Thurston norm-minimizing surface in the homology class $[H] + k[G]$, and $\overline{Q} \cap G$ consists of essential curves in G, then Q gives a taut decomposition of M, where Q is obtained from $\overline{Q}$ by cutting open along $\overline{Q} \cap G$. Moreover, we can assume that $\overline{Q}$ is efficient in X. Hence, $\overline{Q} \cap T = \emptyset$ and for each boundary component δ of $G \times i$, $|\partial Q \cap \delta| = |[\partial Q] \cdot [\delta]|$.

Now we can apply Gabai [**4**, Lemma 0.6] to get a new taut surface Q' such that $\partial_0 Q' = -C \times 0, \partial_1 Q' = C \times 1$. When m is sufficiently large, let S_1 be the surface obtained by doing oriented cut-and-pastes of Q' with $(m - Q' \cdot (p \times I))$ copies of G; then $S_1 \in \mathcal{S}_m(+C)$ is the surface we need. Similarly, we can find the surface $S_2 \in \mathcal{S}_m(-C)$. □

Correction 4.6. After the statement in [**10**, Proposition 3.4], Ni claims that there exists a circle or arc $C \subset G$ such that $[C] \cdot i_*[K] = 0$. This claim is not true. The correct statement should be that there exists an essential efficient curve C in G such that $[C] \cdot i_*[K] = 0$. The proof only needs slight changes: one can make use of the above Lemma 4.5 to find taut surfaces.

The following result is due to Ni [**10**, Lemma 3.6], whose proof uses the assumption that (M, γ) contains no essential product disks or product annuli and an argument of Gabai [**5**].

LEMMA 4.7. *For any positive integers* p, q,

$$y(\mathcal{S}_p(+C)) + y(\mathcal{S}_q(-C)) > (p+q)y(G).$$

Proof of Proposition 4.3. By Lemma 4.5, when m is large, there exist taut surfaces $S_1 \in \mathcal{S}_m(+C)$, $S_2 \in \mathcal{S}_m(-C)$. By Theorem 2.10, S_i remains taut in $M(\xi_i)$ for some $\xi_i \in \{\infty, \alpha\}$, $i = 1, 2$. Let $\mathscr{F}_i$ be a taut foliation of $X(\xi_i)$ induced by S_i.

Let $\overline{S_1}, \overline{S_2} \subset X$ be the surfaces obtained from S_1, S_2 by gluing $C \times 0$ to $C \times 1$. We have

$$[\overline{S_1}] = [H] + m[G], \quad [\overline{S_2}] = -[H] + m[G]$$

in $H_2(X, \partial X)$ and $H_2(X(\alpha), \partial X(\alpha))$.

We have

$$\chi(\overline{S_i}) = \chi(S_i) - |\partial_0 S_i| = -y(S_i),$$

and, by Proposition 2.13,

$$\begin{aligned}\chi(\overline{S_1}) &= \langle e(\mathscr{F}_1), [\overline{S_1}]\rangle \\ &= \langle e(\mathscr{E}(\xi_1)), [H] + m[G]\rangle, \\ \chi(\overline{S_2}) &= \langle e(\mathscr{F}_2), [\overline{S_2}]\rangle \\ &= \langle e(\mathscr{E}(\xi_2)), -[H] + m[G]\rangle.\end{aligned}$$

By Lemma 4.4, $\langle e(\mathscr{E}(\xi_1)), [H]\rangle = \langle e(\mathscr{E}(\xi_2)), [H]\rangle$. So

$$\begin{aligned}\chi(\overline{S_1}) + \chi(\overline{S_2}) &= \langle \mathscr{E}(\xi_1), m[G]\rangle + \langle \mathscr{E}(\xi_2), m[G]\rangle \\ &= 2m\chi(G),\end{aligned}$$

which contradicts Lemma 4.7. □

Proof of Proposition 4.1. By Proposition 4.3, $b_1(G) \leqslant 2$, so G is an annulus, a pair of pants or a genus-1 surface with one boundary component. We only need to show that the last case is not possible.

Suppose $g(G) = 1$ and $|\partial G| = 1$. Let $C \subset G$ be a simple closed curve such that $[C] \cdot i_*[K] = 0$; then there exists a closed taut surface $H \subset X$ such that $[H] = [C \times S^1]$ and $H \cap T = \emptyset$. As M does not contain any product annuli, H is not a torus, hence H is not taut in $X(\infty)$. By Theorem 2.10, H is taut in $X(\alpha)$.

Consider the monodromy φ of $X(\alpha)$; the surface $H \subset X(\alpha)$ forces $\varphi_*[C] = [C]$. As G is a once-punctured torus, $\varphi(C)$ is isotopic to C. Thus, there exists a torus $R \subset X(\alpha)$ such that $R \cap G = C = H \cap G$, which implies that $[H] = [R] + m[G]$ for some integer m. As H is closed, $m = 0$. This contradicts the facts that H is taut in $X(\alpha)$ and that H is not a torus. □

5. *Knots in pants*$\times I$

In this section, we study the case where G is a pair of pants.

The following elementary observation is stated without proof.

LEMMA 5.1. *Suppose $C_1, C_2 \subset G$ are two efficient curves consisting of essential arcs. If they are homologous in $H_1(G, \partial G)$, then they are isotopic.*

Let a, b, c be the three boundary components of G, and let u, v, w be three mutually disjoint oriented arcs in G such that u connects b to c, v connects c to a, and w connects a to b. Then

$$[u] + [v] + [w] = 0 \in H_1(G, \partial G). \tag{6}$$

Lemma 5.2. *None of u, v, w has zero intersection number with $i_*[K]$.*

Proof. The argument is similar to the once-punctured torus case of Proposition 4.1. Assume $[u] \cdot i_*[K] = 0$; then there exists a taut surface $H \subset X$ such that $[H] = [u \times S^1]$. We may assume that H is efficient in X, hence H has two boundary components and $H \cap T = \emptyset$. By Lemma 5.1 we may assume $H \cap G = u$.

As M does not contain any product disks, H is not an annulus, hence H is not taut in $X(\infty) = G \times S^1$. By Theorem 2.10, H is taut in $X(\alpha)$. Let φ be the monodromy of $X(\alpha)$; then H forces $\varphi(u)$ to be homologous and hence isotopic to u by Lemma 5.1. Thus, there exists an annulus $A \subset X(\alpha)$ such that $A \cap G = u = H \cap G$, which implies that $[H] = [A] + m[G]$ for some integers. As H has only two boundary components, $m = 0$. This contradicts the facts that H is taut in $X(\alpha)$ and H is not an annulus. □

Lemma 5.3. *The intersection number of $i_*[K]$ with each of u, v, w is ± 1 or ± 2.*

Proof. Capping off a with a disk, we get an annulus G_a. Let $\widehat{X}$ be the exterior of K in $G_a \times I$. By assumption there exists a homeomorphism

$$f\colon\ (X(\infty), \partial G \times I) \longrightarrow (X(\alpha), \partial G \times I).$$

Postcomposing f with a homeomorphism of $(G \times I, \partial G \times I)$ if necessary, we can assume that f sends $a \times I$ to $a \times I$, and then f extends to a homeomorphism

$$\hat{f}\colon\ (\widehat{X}(\infty), \partial G_a \times I) \longrightarrow (\widehat{X}(\alpha), \partial G_a \times I).$$

Hence, the α-surgery on K does not change the homeomorphism type of the pair $(G_a \times I, (\partial G_a) \times I)$. By the previous lemma, K is non-trivial in $G_a \times I$. By Lemma 3.1, K is the core or the $(2, \pm 1)$-cable in $G_a \times I$, so $i_*[K] \cdot [u]$ is ± 1 or ± 2. The same argument applies to v and w. □

Using the previous two lemmas and (6), we may assume

$$[u] \cdot i_*[K] = [v] \cdot i_*[K] = 1, \quad [w] \cdot i_*[K] = -2, \tag{7}$$

after reversing the orientation of K and renumbering a, b, c, u, v, w if necessary. We give a, b, c the boundary orientation induced from G; then

$$[v] \cdot [a] = [w] \cdot [b] = -[u] \cdot [b] = [u] \cdot [c] = 1. \tag{8}$$

See Figure 3 for the homology class of K.

Let τ_a, τ_b, τ_c be the right-hand Dehn twists along (parallel copies of) a, b, c. The mapping class group $\mathcal{MCG}(G, \partial G)$ of G is generated by τ_a, τ_b, τ_c. (See, for example, Farb–Margalit [**1**] for preliminaries on the mapping class groups of surfaces with boundary.) As a, b, c are disjoint, $\mathcal{MCG}(G, \partial G) \cong \mathbb{Z}^3$.

Lemma 5.4. *If K is the $(2, 1)$-cable in $G_c \times I$, then the map induced by the α-surgery is*

$$\varphi_\alpha = \tau_a^2 \tau_b^2 \tau_c^{-1}.$$

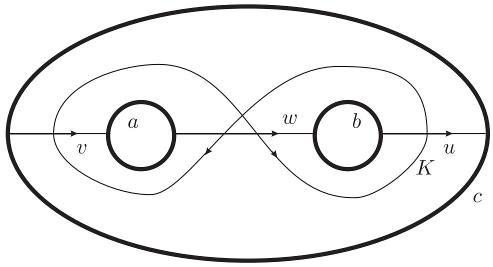

FIGURE 3. *The homology class of K in $G \times I$.*

If K is the $(2, -1)$-cable in $G_c \times I$, then

$$\varphi_\alpha = \tau_a^{-2}\tau_b^{-2}\tau_c.$$

Proof. Capping off a, b with two disks, G becomes a disk G_{ab}. The knot K has a canonical frame λ, which is null-homologous in $(G_{ab} \times I)\backslash K$. Hence, λ is homologous to $l[a] + m[b]$ in M for some integers l, m. By (7) and (8), we conclude that λ is homologous to $a - b$ in M. Hence, λ is also the canonical frame in $G_c \times I$, where G_c is obtained from G by capping off c with a disk.

Suppose $\varphi_\alpha = \tau_a^p\tau_b^q\tau_c^r$. If K is the $(2, 1)$-cable in $G_c \times I$, then by Lemma 3.1, the slope α is 1 with respect to λ.

There is a natural map

$$q_a\colon \mathcal{MCG}(G, \partial G) \to \mathcal{MCG}(G_a, \partial G_a),$$

where $\mathcal{MCG}(G_a, \partial G_a)$ is generated by τ_b. As K is the core in $G_a \times I$ and the slope α is 1, $q_a(\varphi_\alpha)$ must be τ_b by Lemma 3.2. The map q_a sends both τ_b and τ_c to τ_b, and sends τ_a to 1. So $q_a(\varphi_\alpha) = \tau_b^{q+r}$, thus $q + r = 1$. The same argument shows $p + r = 1$.

Now consider the natural map

$$q_c\colon \mathcal{MCG}(G, \partial G) \longrightarrow \mathcal{MCG}(G_c, \partial G_c) = \langle\tau_a\rangle.$$

By Lemma 3.2, $q_c(\varphi_\alpha) = \tau_a^4$. Hence, $p + q = 4$. So we conclude that $p = q = 2, r = -1$. The same argument works when K is the $(2, -1)$-cable in $G_c \times I$. □

Proposition 1.4 follows from the above lemma.

The manifold $G \times S^1$ has a unique product structure. Let $\omega, \omega_\alpha \subset c \times S^1$ be S^1-fibers with respect to the product structures on $X(\infty)$ and $X(\alpha)$, respectively.

LEMMA 5.5. *If K is the $(2, 1)$-cable in $G_c \times I$, then*

$$[\omega_\alpha] = [\omega] + [c].$$

Proof. The manifold $X(\infty)$ is obtained from $G \times I$ by identifying $(x, 0)$ with $(x, 1)$ for each $x \in G$. By Lemma 1.3, $X(\alpha)$ is obtained from $G \times I$ by identifying $(x, 0)$ with $(\varphi_\alpha(x), 1)$ for each $x \in G$. Choose parallel copies of a, b, c in G, denoted by a', b', c', respectively. Let φ_α be supported in the three annuli bounded by $a - a'$, $b - b'$ and $c - c'$. Pick points $p \in c, p' \in c'$; then $p' \times S^1$ is an S^1-fiber of the product structures on both $X(\infty)$ and $X(\alpha)$, whereas $p \times S^1$ is an S^1-fiber of the product structure on $X(\infty)$.

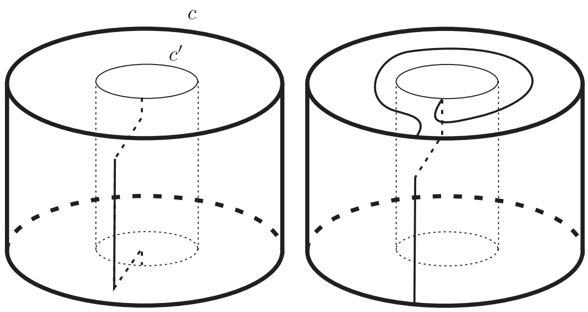

FIGURE 4. *Local pictures of $X(\alpha)$ near $c \times S^1$.*

In $X(\alpha)$, we isotope $p' \times S^1$ such that it becomes a curve $\mathcal{S}$, which is the union of four segments $J, J_\epsilon, J_{1-\epsilon}, J'$, where J is a vertical segment in the interior of $c \times I$, $J_\epsilon \subset G \times \epsilon$, $J_{1-\epsilon} \subset G \times (1-\epsilon)$, and J' is a vertical segment in $c' \times S^1$. See the left-hand side of Figure 4.

We push the previous curve $\mathcal{S}$ down in distance ϵ to get a new curve $\mathcal{S}_-$, then J_ϵ becomes an arc on $G \times 1$. See the right-hand side of Figure 4. Using Lemma 1.3, this new arc is $\varphi_\alpha(J_\epsilon) = \tau_c^{-1}(J_\epsilon)$. The curve $\mathcal{S}_-$ is a fiber of $X(\alpha)$, and it is homologous to $[p \times S^1] + [c \times 1]$. Hence, our conclusion holds. □

LEMMA 5.6. *Let $C = v - u$. Pick a point $p \in c \backslash (\partial C)$; we can then define $\mathcal{S}_m(\pm C)$ as in Section 4. Then there exists a connected surface $S \in S_1(C)$ such that $y(S) = 1$. Moreover, let $S' \subset G \times [0,1]$ be the surface obtained from $-C \times I$ and $G \times 0$ by oriented cut-and-pastes; then S is isotopic to S' in $G \times [0,1]$.*

Proof. For any homology class $h \in H_2(X, (\partial G) \times S^1)$, let $x(h), x_\infty(h), x_\alpha(h)$ denote its Thurston norm in $X, X(\infty), X(\alpha)$, respectively.

Let $U = -[u \times S^1], V = -[v \times S^1] \in H_2(G \times S^1, (\partial G) \times S^1)$. As $(V - U) \cdot [K] = 0$, $V - U$ also represents an element in $H_2(X, (\partial G) \times S^1)$. Note that the Thurston norm of $h \in H_2(G \times S^1, (\partial G) \times S^1)$ is the absolute value of its algebraic intersection number with the S^1-fiber. Consider $V - U + m[G]$ for $m \geqslant 0$. Using Lemma 5.5, we can compute

$$x_\infty(V - U + m[G]) = m,$$
$$x_\alpha(V - U + m[G]) = (V - U + m[G]) \cdot ([\omega] + [c]) = |m - 2|.$$

As $x_\infty(V - U + [G]) = x_\alpha(V - U + [G]) = 1$, Theorem 2.10 implies that $x(V - U + [G]) = 1$. Let $\overline{S} \subset X$ be a taut surface in this homology class such that $\overline{S}$ is efficient in X. Then $\overline{S}$ is disjoint from T. Isotope $\overline{S}$ so that it is transverse to G and its intersection with G contains no trivial loops. Now $\overline{S} \cap G$ is homologous to C. Moreover, $\overline{S} \cap G$ can be made efficient in G. So $\overline{S} \cap G$ is isotopic to C by Lemma 5.1. Without loss of generality, we can assume

$$\overline{S} \cap G = C \quad \text{and} \quad \overline{S} \cap ((\partial C) \times S^1) \subset G.$$

Cutting $\overline{S}$ open along C, we get a surface $S \in \mathcal{S}_1(+C)$ such that $y(S) = 1$. After an isotopy of S, we can assume that the two surfaces $S, C \times [0,1] \subset G \times [0,1]$ are transverse. As $S \cap ((\partial C) \times (0,1)) = \emptyset$, $S \cap (C \times (0,1))$ consists of closed curves that bound disks in $C \times (0,1)$. As S is incompressible and $G \times [0,1]$ is irreducible, we can isotope S such that $S \cap (C \times (0,1)) = \emptyset$, hence $S \cap (C \times [0,1]) = C \times \{0,1\}$. Now we glue S and $C \times [0,1]$ together along $C \times \{0,1\}$ and perturb the resulting surface slightly; then we get a connected surface G' with $x(G') = 1$ and $\partial G'$ is parallel to $(\partial G) \times 0$ in $(\partial G) \times [0,1]$. Hence, G' is parallel to $G \times 0$ in $G \times [0,1]$. It follows that S is isotopic to S' in $G \times [0,1]$. □

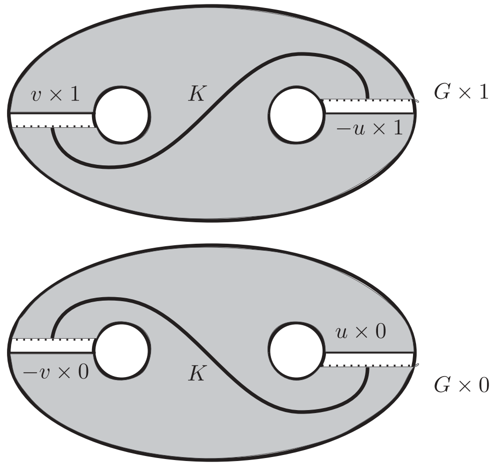

FIGURE 5. *The surface $R_+(\gamma_1)$ containing the knot K.*

LEMMA 5.7. *Let S be the surface obtained in Lemma* 5.6. *Let*

$$G \times I \overset{S}{\rightsquigarrow} (M_1(\infty), \gamma_1)$$

be the sutured manifold decomposition associated with S; then $(M_1(\infty), \gamma_1)$ is a product manifold, and there is an ambient isotopy of $M_1(\infty)$ that takes K to a curve in $R_+(\gamma_1)$ such that the frame of K specified by $R_+(\gamma_1)$ is α.

Proof. By Lemma 5.6, S is obtained from $-C \times I$ and $G \times 0$ by oriented cut-and-pastes. So $(M_1(\infty), \gamma_1)$ is a product sutured manifold and $R_+(\gamma_1)$ is an annulus.

Let $(M_1(\alpha), \gamma_1)$ be the sutured manifold obtained from $M(\alpha)$ by decomposing along S. Then $M_1(\alpha)$ can also be obtained from $M_1(\infty)$ by α-surgery on K.

We claim that $M_1(\alpha)$ is not taut. In fact, let S'' be the surface obtained from S and $G \times 0$ by oriented cut-and-pastes. Let $\overline{S''} \subset X$ be the surface obtained from S'' by gluing $\partial_0 S''$ to $\partial_1 S''$. Then $x(\overline{S''}) = 2$ and $\overline{S''}$ represents $V - U + 2[G]$. We already computed

$$x_\infty(V - U + 2[G]) = 2 > x_\alpha(V - U + 2[G]) = 0,$$

so $\overline{S''}$ is not taut in $X(\alpha)$. Let $M''(\alpha)$ be the non-taut sutured manifold obtained by decomposing $X(\alpha)$ along $\overline{S''}$.

As S'' is obtained from S and $G \times 0$ by oriented cut-and-pastes, and $S \cap (G \times 0) = -C \times 0$ consists of two arcs, there exist two product disks in $M''(\alpha)$ such that the result of decomposing $M''(\alpha)$ along these two disks is $(M_1(\alpha), \gamma_1)$. See the proof of Gabai [**2**, Theorem 3.13] for an explanation of this fact. So $(M_1(\alpha), \gamma_1)$ is not taut by Gabai [**4**, Lemma 0.4].

Now Theorem 1.5 implies our conclusion. □

Proof of Theorem 1.1. By the results in Sections 3 and 4, we only need to consider the case $F = G$ is a pair of pants. By Lemma 5.7, K lies on $R_+(\gamma_1)$, and the frame specified by $R_+(\gamma_1)$ is α.

As $R_+(\gamma_1)$ is an annulus, the only essential curve on it is its core. As in Figure 5, $R_+(\gamma_1)$ can be constructed in the following way. Cut $G \times \{0, 1\}$ open along $(v - u) \times \{0, 1\}$; we get two octagons P_0, P_1. There are two edges of P_0 that are copies of $v \times 0$ with different orientations. We call these two edges $v \times 0, -v \times 0$. Similarly, there are edges $\pm u \times 0, \pm v \times 1, \pm u \times 1$. Now we glue two product disks to P_0, P_1, such that one product disk connects $v \times 0$ to $-v \times 1$ and the other connects $-u \times 0$ to $u \times 1$. The annulus we get is isotopic to $R_+(\gamma_1)$. The core of this annulus is clearly a 1-crossing knot in $G \times I$. The result about the frame also follows as the vertical projection $p\colon R_+(\gamma_1) \to G$ is an immersion. □

Acknowledgement. We are grateful to Danny Calegari for helpful discussions on mapping class groups. This article is dedicated to the memory of Professor Andrew Lange.

References

1. B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series (Princeton University Press, Princeton, NJ), http://www.math.utah.edu/~margalit/primer/.
2. D. Gabai, 'Foliations and the topology of 3-manifolds', *J. Differential Geom.* 18 (1983) 445–503.
3. D. Gabai, 'Detecting fibred links in S^3', *Comment. Math. Helv.* 61 (1986) 519–555.
4. D. Gabai, 'Foliations and the topology of 3-manifolds II', *J. Differential Geom.* 26 (1987) 461–478.
5. D. Gabai, 'Foliations and the topology of 3-manifolds III', *J. Differential Geom.* 26 (1987) 479–536.
6. D. Gabai, 'Surgery on knots in solid tori', *Topology* 28 (1989) 1–6.
7. D. Gabai, '1-bridge braids in solid tori', *Topology Appl.* 37 (1990) 221–235.
8. P. Ghiggini, 'Knot Floer homology detects genus-one fibred knots', *Amer. J. Math.* 130 (2008) 1151–1169.
9. C. Gordon and J. Luecke, 'Knots are determined by their complements', *J. Amer. Math. Soc.* 2 (1989) 371–415.
10. Y. Ni, 'Dehn surgeries that yield fibred 3-manifolds', *Math. Ann.* 344 (2009) 863–876.
11. M. Scharlemann and A. Thompson, 'Heegaard splittings of (surface) $\times I$ are standard', *Math. Ann.* 295 (1993) 549–564.
12. M. Scharlemann and A. Thompson, 'Surgery on a knot in (surface $\times I$)', *Algebr. Geom. Topol.* 9 (2009) 1825–1835.
13. W. Thurston, 'A norm for the homology of 3-manifolds', *Mem. Amer. Math. Soc.* 59 (1986) i–vi and 99–130.

Yi Ni
Department of Mathematics
Caltech, MC 253-37
1200 E. California Blvd
Pasadena, CA 91125
USA

yini@caltech.edu