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Uniqueness of PL Minimal Surfaces

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Abstract Using a standard fact in hyperbolic geometry, we give a simple proof of the uniqueness of PL minimal surfaces, thus filling in a gap in the original proof of Jaco and Rubinstein. Moreover, in order to clarify some ambiguity, we sharpen the definition of PL minimal surfaces, and prove a technical lemma on the Plateau problem in the hyperbolic space.

Keywords PL minimal surface, hyperbolic geometry, Plateau problem
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PL minimal surfaces were introduced by Jaco and Rubinstein in [1], where some fundamental properties of PL minimal surfaces we are established, including the existence and uniqueness. In this paper, by using a standard result in hyperbolic geometry, we give a simple proof of the uniqueness of PL minimal surfaces.

The readers are referred to [1] for basic definitions and notations.

Suppose M is a 3-manifold with triangulation $\mathscr{T}$. In [1], the PL area of a normal surface is defined to be the pair (ω, l), lexicographically ordered, where the weight ω measures the intersection of the normal surface with 1-simplices, and the length l measures the intersection of the normal surface with 2-simplices. This definition does not involve the interior of 3-simplices, so logically one may arbitrarily deform the surface in the interior of 3-simplices, so as to get surfaces with the same PL area. Hence it is not clear what the "uniqueness" means. In order to get rid of this ambiguity, we should make the choice of the intersection of F with the 3-simplices "canonical". We hence sharpen the definition of PL minimal surfaces as follows:

Definition 1 *A normal surface $f : F \to M$ is called PL minimal if its length l is stationary with respect to small variations of F, and for any 3-simplex Δ, each component of $F \cap \Delta$ is a minimal surface in the usual sense in differential geometry. Here Δ is viewed as a regular ideal tetrahedron in the hyperbolic space.*

As in Remark 3 of [1], when l is stationary, the intersection between F and the 2-simplices consists of geodesic segments. Now suppose Δ is a 3-simplex, B is a component of $F \cap \Delta$. When B is of triangle type, the area of B reaches the infimum exactly when B is the totally geodesic triangle passing through ∂B. Generally, we have the following technical lemma:

Lemma 2 *Let Γ be a triangle or quadrilateral on the boundary of Δ, so that the sides of Γ are geodesic segments. Then Γ bounds exactly one embedded minimal surface of disk type in Δ, and this minimal surface is free of branch points.*

Assuming this lemma, we will prove our main result:

Theorem 3 (Theorem 2 in [1]) *There is exactly one PL minimal surface in a normal homotopy class $\mathscr{N}(f)$ for any normal surface $f : F \to M$ which is not a multiple of a linking 2-sphere.*

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Proof The proof of the existence of PL minimal surfaces is the same as for [1, Theorem 1], modulo Lemma 2. We only need to prove the uniqueness.

Suppose $f : F \to M$ is a normal surface. Let ω denote the weight of f, in other words, $f(F) \cap \mathscr{T}^{(1)}$ consists of ω points (counted with multiplicity) $x_1, \ldots, x_\omega$. By the definition of normal homotopy, each x_i varies on a fixed edge of $\mathscr{T}$.

We assume $f(F) \cap \mathscr{T}^{(2)}$ consists of geodesic segments. Then the length $l(f)$ of $f(F)$ is the sum of terms in the form $d(x_i, x_j)$, where $d(\cdot, \cdot)$ is the hyperbolic distance.

Recall a fact in hyperbolic geometry:

Fact *α, β are two geodesic lines in hyperbolic space, parametrized by arc length. $x, y \in \mathbb{R}$ denote points varying on α, β, respectively. Then $d(x, y)$ is a strictly convex function on $\mathbb{R}^2$.*

The readers are referred to [2, Theorem 2.5.8] for a precise statement of this fact. A more geometric proof can be found in [3, Proposition 29, Chapitre 3].

By this fact, each summand in the expression of $l(f)$ is strictly convex in two variables. Their sum is also strictly convex, because every variable x_i appears in the expression of $l(f)$. So $l(f) = l(x_1, \ldots, x_\omega)$, as a strictly convex function of $\mathbb{R}^\omega$, has at most one critical point, which, if it exists, is a minimum. Hence there is at most one choice for $(x_1, \ldots, x_\omega)$. Once $(x_1, \ldots, x_\omega)$ is given, $f(F) \cap \mathscr{T}^{(2)}$ consists of the geodesic segments connecting x_i's. Then by Lemma 2, there is a unique choice for $f(F)$ in the interior of 3-simplices. Hence the PL minimal surface is unique.

The idea in the following remark was told to the author by David Gabai:

Remark 4 In [1], a physical model was provided for finding the unique PL minimal surface. One can think of $\mathscr{T}^{(1)}$ as a collection of wires, the endpoints of the arcs of $f(F) \cap \mathscr{T}^{(2)}$ as beads which are free to slide along the wires, and the arcs of $f(F) \cap \mathscr{T}^{(2)}$ as rubber bands connecting the beads. Then the PL minimal surface is given by the unique equilibrium (minimum energy) position for the configuration. This physical model is compelling, but incorrect. The point is, if we change the initial length of the rubber bands, we can certainly get a different equilibrium position. The following analysis to the argument in [1] is guided by this idea.

In [1], the uniqueness is proved via the variational method. Basically, when using the variational method, one considers a functional on an infinite-dimensional space $\mathcal{M}$. For example, in the problem of finding a closed geodesic in a manifold M, $\mathcal{M} = \{\varphi : S^1 \to M\}$. The critical points of the length functional $l(\varphi)$ correspond to the φ's whose image is a closed geodesic. The critical points of the energy functional $E(\varphi)$ correspond to the φ's whose image is a geodesic, **and** the parametrization is proportional to the arc length. By abuse of notation, we say that the critical points of l one-to-one correspond to the critical points of E.

In case of the proof in [1], $F \cap \mathscr{T}^{(2)}$ is homeomorphic to a graph G. $\mathcal{M}$ is not specified in the proof, but we can fix a metric on G, and choose $\mathcal{M}$ to be $\{\varphi : G \to M\}$. Under the assumption that the arcs (of $F \cap \mathscr{T}^{(2)}$) α_i's are parametrized by arc length, it is shown that the first variation of the length l is equal to the first variation E. A consequence should be, when the parametrization on $F \cap \mathscr{T}^{(2)}$ is uniformly proportional to the arc length, the critical points of $E(\varphi)$ correspond to the critical points of $l(\varphi)$. But now the relative proportion of the lengths of the arcs $\alpha_1, \alpha_2, \ldots, \alpha_m$ are determined by the metric on the graph G. Certainly we can vary φ in $\mathcal{M}$ to change the relative proportion of the arc lengths. Therefore the argument in [1] did not clearly show that the critical points of E and l are the same.

The rest of this paper is devoted to the proof of Lemma 2. This part is not the major part of our paper, we need it to justify our enhanced definition of PL minimal surfaces. A reader with a casual interest could just skip it.

We note that Lemma 2 is almost a special case of the main theorem in [4], except that in [4] Γ has certain smoothness everywhere, while in our case Γ is piecewise smooth (in fact piecewise straight). Our task is to adapt the method in [4] to our case.

Our situation is as follows: D is the unit open disk, $f : \overline{D} \to \Delta$ is a map, so that $f|\partial D$ is a homeomorphism onto Γ. $f \in \mathcal{M} = C^2(D, \Delta) \cap C^0(\overline{D}, \Delta)$. The minimal surfaces (of disk type) bounded by Γ correspond to the critical points of the area functional $\mathrm{Area}(f(D))$.

The existence of such minimal surfaces is guaranteed by Theorem 1 in [5], which also shows that the solutions are embedded. For our convenience, we cite Theorem 2 in [5], which ensures that there is no boundary branch point.

Lemma 5 (Gauss–Bonnet Formula) *Suppose $f : \overline{D} \to \Delta$ is a map as above, B is the image of f. f has no branch points on the boundary, $\{w_\alpha\}$ are the branch points of f in D, $k(w_\alpha)$ is the branching order of w_α. Then we have*

$$1 + \sum_\alpha k(w_\alpha) = \frac{1}{2\pi}\left(\sum \theta_i + \int_\Gamma \kappa_g + \iint_B K\right),$$

where κ_g is the geodesic curvature of Γ on B, K is the Gauss curvature of B, θ_i are the external angles of Γ on B.

Proof For any sufficiently large integer n, let

$$G_n = \left\{ w \in D : |w| < 1, |w - w_\alpha| > \frac{1}{n} \right\}.$$

By the standard Gauss–Bonnet formula (see 4–5 of [6]), we have

$$1 - \sum_\alpha 1 = \frac{1}{2\pi}\left(\sum \theta_i + \int_{\partial f(G_n)} \kappa_g + \iint_{f(G_n)} K\right).$$

Letting $n \to \infty$, the same argument as in the proof of [Lemma 3, 7] gives our result.

Lemma 6 *Any solution f of the Plateau problem spanned by Γ is strictly stable and free of branch points.*

We omit the proof here, since it is almost the same as the one in Lemma 1 of [4]. One thing we need to check is:

$$\int_\Gamma \kappa_g + \sum \theta_i < 4\pi.$$

This is obvious, because the sides of Γ are geodesic, and there are 3 or 4 θ_i's, each being less than π. We also point out that we use our Lemma 5 instead of the Gauss–Bonnet formula in [7]. Moreover, in [4], in order to prove the first eigenvalue $\lambda_1 > 0$, Li-Jost shows that λ_1 has an eigenfunction, and the eigenspace is 1-dimensional. This is a standard fact in ellipic PDE [8, Theorem 8.38].

Now we can prove the uniqueness of minimal surfaces, thus finishing the proof of Lemma 2.

Proof of Uniqueness Suppose there are at least two minimal surfaces. By Lemma 6, they are strictly stable. By Morse theory, there is at least one unstable solution to the Plateau problem, which contradicts Lemma 6.

Remark 7 One can check that all the properties of PL minimal surfaces with the definition in [1] are enjoyed by PL minimal surfaces with our enhanced definition.

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