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Pseudo-Anosov extensions and degree one maps between hyperbolic surface bundles

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Abstract Let F', F be any two closed orientable surfaces of genus $g' > g \geq 1$, and $f : F \to F$ be any pseudo-Anosov map. Then we can "extend" f to be a pseudo-Anosov map $f' : F' \to F'$ so that there is a fiber preserving degree one map $M(F', f') \to M(F, f)$ between the hyperbolic surface bundles. Moreover the extension f' can be chosen so that the surface bundles $M(F', f')$ and $M(F, f)$ have the same first Betti numbers.

Mathematics Subject Classification (2000) Pseudo-Anosov extension · Degree-one maps · Hyperbolic surface bundles

1 Introduction

All surfaces are oriented and all automorphisms on surfaces are orientation preserving.

Let F be an oriented closed surface of genus $g \geq 1$, and $f : F \to F$ be an automorphism. We denote the surface bundle with fiber F and monodromy f by $M(F, f)$.

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Definition 1 Suppose G is a compact surface of genus $g \geq 1$. A circle c on G is *essential* if c is not contractible or boundary parallel. An automorphism f on G is *pseudo-Anosov* if $f^n(c)$ is not isotopic to c for any essential circle $c \subset G$ and any integer n. (Note in the case G is a torus, the term "pseudo-Anosov" we define here is usually known as "Anosov".)

Remark 1 Our definition of pseudo-Anosov maps is slightly different from the more standard definition in the literature. Pseudo-Anosov maps in our sense should be considered as "maps isotopic to a pseudo-Anosov map" in the standard sense.

Profound theories of Nielsen-Thurston and of Thurston in 2- and 3-dimensional topology tell us that pseudo-Anosov is the most important class of surface automorphisms, and when $\chi(F) < 0$, $M(F,f)$ is a hyperbolic 3-manifold if and only if f is pseudo-Anosov.

Theorem 1 *Let F_s, F_t be closed orientable surfaces of genus g_s, g_t respectively, where $g_s > g_t \geq 1$, and $f_t : F_t \to F_t$ be a pseudo-Anosov map. Then*

(1) *There is a hyperbolic 3-manifold $M(F_s,f_s)$, such that there is a fiber preserving degree one map $P : M(F_s,f_s) \to M(F_t,f_t)$. (Here the subscript s means "source", and the subscript t means "target".)*
(2) *Moreover the map f_s in (1) can be chosen so that $M(F_s,f_s)$ and $M(F_t,f_t)$ have the same first Betti numbers.*

Motivation for Theorem 1 is from [1], where the following facts were proved:

(1) For each 3-manifold M, there is a degree one map $f : M(F_s,f_s) \to M$ such that $M(F_s,f_s)$ is hyperbolic and $\beta_1(M(F_s,f_s)) = \beta_1(M) + 1$.
(2) If there is a degree one map $f : M(F_s,f_s) \to M$ with $\beta_1(M(F_s,f_s)) = \beta_1(M)$ and M is irreducible, then M is a surface bundle and f can be homotoped to a fiber preserving one.

It is natural to wonder how to find fiber preserving degree one maps between non-homeomorphic hyperbolic surface bundles (of the same first Betti numbers). In Sect. 2, we will prove Theorem 1 (1). In Sect. 3, by modifying the proof in Sect. 2, we will prove Theorem 1 (2).

The proof of Theorem 1 relies on an extension process from the pseudo-Anosov map f_t on F_t to a pseudo-Anosov map f_s on F_s, which is delicate and somewhat complicated.

We will outline this process, i.e., for given $M(F_t,f_t)$ and F_s, how to find f_s. In this outline we assume that $g_t \geq 2$. This process in Section 2 is divided into three steps.

Step 1. Fix a disk $D \subset F_t$ and let $V = F_t - \text{int}(D)$. We can assume that $f_t|D = \text{id}|D$ up to isotopy. Then as a restriction of a pseudo-Anosov map, $f_t|V : V \to V$ is a pseudo-Anosov map (Lemma 2).

Step 2. We will construct two embeddings $e_0, e_1 : V \to F_s$ such that (1) $e_0(\partial V)$ and $e_1(\partial V)$ are not homotopic in F_s, (2) two pinches $p_0, p_1 : F_s \to F_t$ (see Definition 2) defined by $p_j \circ e_j = \text{id}_V : V \to V$ are homotopic (Lemma 4).

Step 3. The two embeddings e_1 and e_2 in step 2 also provide a homeomorphism $\bar{f}_t := e_1 \circ f_t| \circ e_0^{-1} : e_0(V) \to e_1(V)$. With properties of e_1 and e_2 described in Step 2, we will be able to extend $\bar{f}_t$ to a pseudo-Anosov map $f_s : F_s \to F_s$ (Proposition 1).

Then clearly $p_1 \circ f_s = f_t \circ p_0$, hence there exists a fiber preserving degree one map $P : M(F_s, f_s) \to M(F_t, f_t)$ (Lemma 1). This finishes the proof of Theorem 1 (1).

Now we give more detailed outline of the extension process in Step 3, on which the proof of Theorem 1 (2) is based.

Let $\bar{f}_s : F_s \to F_s$ be any extension of $\bar{f}_t : e_0(V) \to e_1(V)$ (Lemma 3). Let $W_1 = F_s - \text{int}e_1(V)$ and $h : W_1 \to W_1$ be any pseudo-Anosov map. Let $\mathcal{A}_1$ be any maximal independent set of disjoint circles on W_1 (see Definition 3), let $\tau(\mathcal{A}_1)$ be a composition of Dehn twists along all components in $\mathcal{A}_1$. Then $f_s = \tau^l(\mathcal{A}_1) \circ h^k \circ \bar{f}_s$ is pseudo Anosov for large integers k and l (Lemmas 5, 6, 8, 9, 7).

In Sect. 3, we choose $\bar{f}_s$, h and $\mathcal{A}_1$ carefullly so that Theorem 1 (2) is proved (Lemmas 10–12).

We end the introduction by a comment on a related work [9]. The main result in [9] is that for an orientable closed surface F with $\chi(F) < 0$ and two non-isotopic circles c and c' on F, if $g(c) = c'$ for some automorphism g on F, then $f(c) = c'$ for some pseudo-Anosov map f on F. Some arguments in Proving Lemmas 8 and 9 were modified from the arguments in [9]. Indeed [9] was produced in a rather earlier stage of understanding the present project.

2 Homotopic pinches and pseudo-Anosov extensions

Definition 2 Let D be a fixed disc in F_t and $V = F_t - \text{int}(D)$. A degree one map $p : F_s \to F_t$ is a *pinch* if $p| : p^{-1}(V) \to V$ is a homeomorphism.

It has been known since Nielsen and Kneser that every degree one map between surfaces is homotopic to a pinch, see [2] for a reference.

Notation 1 *In the rest of this paper,* $r = s, t$ *and* $j = 0, 1$.

Recall that $M(F_r, f_r) = F_r \times [0,1]/f_r'$, *where* $f_r' : F_r \times \{0\} \to F_r \times \{1\}$ *is given by* $f_r'(x, 0) = (f_r(x), 1)$. *Let* $q_r : F_r \times [0,1] \to F_r$ *be the projection defined by* $q_r(x, u) = x$, *and* $e_{r,j} : F_r \to F_r \times \{j\} \subset F_r \times [0,1]$ *be the homeomorphism given by* $e_{r,j}(x) = (x, j)$. *Let* $o_r : F_r \times [0,1] \to M(F_r, f_r)$ *be the quotient map and* $F_r' = o_r(F_r \times 0) = o_r(F_r \times 1)$. *Then*

$$q_r \circ f_r' \circ e_{r,0} = f_r, \quad e_{r,j} \circ q_r = id_{F_r \times \{j\}}. \tag{$*$}$$

Lemma 1 *There exists a fiber preserving degree one map* $P : M(F_s, f_s) \to M(F_t, f_t)$ *if and only if there are homotopic pinches* $p_0, p_1 : F_s \to F_t$ *such that* $p_1 \circ f_s = f_t \circ p_0$.

Proof Suppose first that $P : M(F_s, f_s) \to M(F_t, f_t)$ is a fiber preserving degree 1 map. Up to homotopy we may assume that $P^{-1}(F_t') = F_s'$ and $P| : F_s' \to F_t'$ is a pinch. Moreover we may assume that the induced degree one map on S^1 is orientation preserving. Then by cutting $M(F_r, f_r)$ along F_r', P provides a proper degree one map

$$\bar{P} : (F_s \times [0,1], F_s \times \{0\}, F_s \times \{1\}) \to (F_t \times [0,1], F_t \times \{0\}, F_t \times \{1\})$$

with the property $\bar{P}|_{F_s \times \{1\}} \circ f_s' = f_t' \circ \bar{P}|_{F_s \times \{0\}}$.

Let $p_j = q_t \circ \bar{P}|_{F_s \times \{j\}} \circ e_{s,j} : F_s \to F_t$. Then p_j is a pinch and $q_t \circ \bar{P} : F_s \times [0,1] \to F_t$ is a homotopy from p_0 to p_1. Moreover $\bar{P}|_{F_s \times \{1\}} \circ f_s' = f_t' \circ \bar{P}|_{F_s \times \{0\}}$ and $(*)$ imply that

$$\begin{aligned} p_1 \circ f_s &= q_t \circ P|_{F_s \times \{1\}} \circ e_{s,1} \circ q_s \circ f_s' \circ e_{s,0} \\ &= q_t \circ \bar{P}|_{F_s \times \{1\}} \circ f_s' \circ e_{s,0} \\ &= q_t \circ f_t' \circ \bar{P}|_{F_s \times \{0\}} \circ e_{s,0} \\ &= q_t \circ f_t' \circ e_{t,0} \circ q_t \circ \bar{P}|_{F_s \times \{0\}} \circ e_{s,0} \\ &= f_t \circ p_0. \end{aligned}$$

Suppose then there are two homotopic pinches $p_0, p_1 : F_s \to F_t$ such that $p_1 \circ f_s = f_t \circ p_0$. Let $P' : F_s \times [0,1] \to F_t$ be a homotopy from p_0 to p_1. Then P' provides a proper degree one map

$$\bar{P} : (F_s \times [0,1], F_s \times \{0\}, F_s \times \{1\}) \to (F_t \times [0,1], F_t \times \{0\}, F_t \times \{1\})$$

defined by $\bar{P}(x,u) = (P'(x,u), u)$. Clearly $\bar{P}$ is fiber preserving and $p_j = q_t \circ \bar{P}|_{F_s \times \{j\}} \circ e_{s,j}$. Then $p_1 \circ f_s = f_t \circ p_0$ and $(*)$ implies that

$$q_t \circ P|_{F_s \times \{1\}} \circ e_{s,1} \circ q_s \circ f_s' \circ e_{s,0} = q_t \circ f_t' \circ e_{t,0} \circ q_t \circ \bar{P}|_{F_s \times \{0\}} \circ e_{s,0},$$

hence

$$q_t \circ P|_{F_s \times \{1\}} \circ f_s' \circ e_{s,0} = q_t \circ f_t' \circ \bar{P}|_{F_s \times \{0\}} \circ e_{s,0}.$$

Since $q_t|_{F_t \times \{1\}}$ and $e_{s,0}$ are invertible, we have

$$P|_{F_s \times \{1\}} \circ f_s' = f_t' \circ \bar{P}|_{F_s \times \{0\}}.$$

Hence $\bar{P}$ is able to induce a fiber preserving degree one map $P : M(F_s, f_s) \to M(F_t, f_t)$.

By Lemma 1, to prove Theorem 1 (1), we need only to find two homotopic pinches $p_0, p_1 : F_s \to F_t$ and a pseudo-Anosov map $f_s : F_s \to F_s$ such that $p_1 \circ f_s = f_t \circ p_0$.

For $D \subset F_t$ and $V = F_t - \text{int}(D)$ given in Definition 2, we can assume that $f_t|D = \text{id}$ up to isotopy.

Lemma 2 *If $f_t : F_t \to F_t$ is a pseudo-Anosov map, then $f_t|V : V \to V$ is also a pseudo-Anosov map.*

Proof Suppose there is a non-contractible circle c on V such that $f_t|^n(c) \sim c$ on V for some $n > 0$, then $f_t^n(c) \sim c$ on F_t. Since f_t is pseudo-Anosov, c is contractible on F_t. Hence c bounds a disc D^* in F_t and $D \subset D^*$. It follows that $c = \partial D^*$ is parallel to $\partial D = \partial V$. Hence $f_t|V : V \to V$ is pseudo-Anosov by definition.

Let $p_0, p_1 : F_s \to F_t$ be two pinches. Then the pull-back of V into F_s provides embeddings $e_j : V \hookrightarrow F_s$. Let $V_j = e_j(V)$, $W_j = F_s - \text{int}(V_j)$, $(j = 0, 1)$. The following lemma is clear.

Lemma 3

$$\bar{f}_t := e_1 \circ f_t \circ e_0^{-1} : V_0 \to V_1$$

is a homeomorphism. Moreover, $\bar{f}_t$ can be extended to a homeomorphism $f_s : F_s \to F_s$, such that $f_t \circ p_0 = p_1 \circ f_s$.

A necessary condition to guarantee the extension f_s in Lemma 3 to be pseudo-Anosov is that $e_0(\partial D)$ is **not** homotopic to $e_1(\partial D)$.

Now with Lemma 1 and Lemma 3, Theorem 1 (1) follows from the following Lemma 4 and Proposition 1.

Lemma 4 *With the notation above, there exist two pinches $p_0, p_1 : F_s \to F_t$ such that*

(i) *p_0 and p_1 are homotopic;*
(ii) *$e_0(\partial D)$ is not homotopic to $e_1(\partial D)$.*

Proof We will find two essential circles $\gamma_0, \gamma_1 \subset F_s$ such that

(1) γ_0 is not homotopic to γ_1,
(2) γ_j separates F_s into 1-punctured surfaces V_j and W_j, where V_0, V_1 have genus g_t.

Then we define the pinch $p_j : F_s \to F_t$ such that W_j is the pinched part.

Case 1 $g_t \geq 2$. W_j, V_j, are shown in Fig. 1.

Let $p_j : F_s \to F_t$ be a pinch which sends W_j to $D_j \subset F_t$ such that the restrictions $p_0|, p_1| : F_s \setminus (W_0 \cup W_1) \to F_t \setminus (D_0 \cup D_1)$ are identical homeomorphisms. This requirement can be reached if we consider $p_j : F_s \to F_t$ as a quotient map which is the identity on $F_s \setminus W_j$ and pinches W_j to D_j.

Note $W_0 \cup W_1$ is a compact surface with two boundary components and $D_0 \cup D_1$ is annulus. Now the restrictions $p_0|, p_1| : W_0 \cup W_1 \to D_0 \cup D_1$ are degree one maps which are the identity on the boundary. It is also easy to see that there is an proper arc α of $W_0 \cup W_1$ connecting the different boundary components so that the pinches p_0 and p_1 can be homotoped relative to the boundaries so that $p_0(\alpha) = p_1(\alpha)$ and $p_0(\alpha)$ is a proper arc connecting the different boundary components of the annulus $D_0 \cup D_1$. Now it follows from a classical argument that $p_0|, p_1| : W_0 \cup W_1 \to D_0 \cup D_1$ are homotopic relative to the boundary, and furthermore $p_0, p_1 : F_s \to F_t$ are homotopic.

Case 2 $g_t = 1$. Then $\pi_1(F_t) = \mathrm{H}_1(F_t)$ is abelian and for each map $p : F_s \to F_t$, $p_\pi : \pi_1(F_s) \to \pi_1(F_t)$ is a composition of $\sigma : \pi_1(F_s) \to \mathrm{H}_1(F_s)$ and $p_\# : \mathrm{H}_1(F_s) \to \mathrm{H}_1(F_t)$, where σ is the abelianizing map, $p_\#$ is the map on homology. So the homotopy class of p is determined by $p_\#$ by elementary homotopy theory (see [3]).

Using this fact, we can construct γ_0 and γ_1 as following: choose essential curves α, β_0 and β_1 on F_s, see Fig. 2, such that

(1) β_0 and β_1 are in the same homology class, but not in the same homotopy class;
(2) $|\alpha \cap \beta_0| = |\alpha \cap \beta_1| = 1$.

Fig. 1

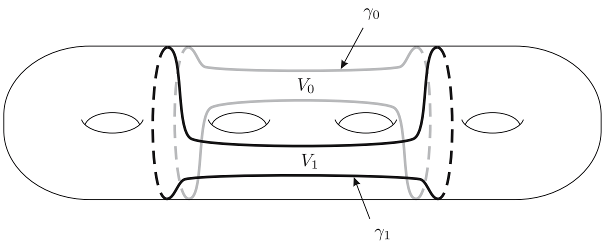
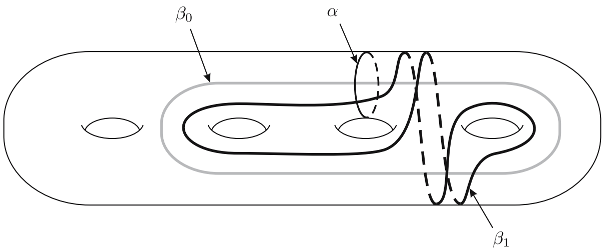


Fig. 2



Let $\gamma_j = \partial N(\alpha \cup \beta_j)$, $V_j = N(\alpha \cup \beta_j)$. It is easy to check that $\gamma_0 \nsim \gamma_1$ and $p_{0\#} = p_{1\#} : \mathrm{H}_1(F_s) \to \mathrm{H}_1(F_t)$.

Proposition 1 *With the notation as above, once the two pinches $p_0, p_1 : F_s \to F_t$ are chosen to meet* (i) *and* (ii) *in Lemma 4, the extension f_s in Lemma 3 can be chosen to be pseudo-Anosov.*

Suppose the two pinches $p_0, p_1 : F_s \to F_t$ are chosen to meet (i) and (ii) in Lemma 4, and $f_s : (F_s, V_0) \to (F_s, V_1)$ is an extension of $\bar{f}_t = e_1 \circ f_t| \circ e_0^{-1} : V_0 \to V_1$ with the condition $p_1 \circ f_s = f_t \circ p_0$.

Lemma 5 (1) *If c is an essential circle in V, then c is essential in F_t.*
(2) *No non-trivial circle $c \subset V_j$ can be isotoped into $W_{j'}$, $j, j' \in \{0, 1\}$, $j \neq j'$.*

Proof (1) Otherwise c would bound a disk D^* in F_t with $\partial V \subset D^*$, hence c is parallel to ∂V in V, a contradiction.
(2) Otherwise say $c \subset V_0$ is a non-trivial circle, which is isotopic to a circle $c' \subset W_1$ in F_s.

First suppose that c is essential in V_0. By $p_0 \sim p_1$ we have $p_0(c) \sim p_1(c')$. On one hand c is essential in V_0 implies that $p_0(c)$ is essential in V_0, and then $p_0(c)$ is essential in F_t by (1). But on the other hand, $c' \subset W_1$ implies that $p_1(c')$ is homotopically trivial. We reach a contradiction.

Then suppose that ∂W_1 can be isotoped into W_0. Then one of the two components V_1 and W_1 must be contained in W_0. Since W_0 and W_1 are homeomorphic, if $W_1 \subset W_0$, we must have ∂W_0 is parallel to ∂W_1, a contradiction. Hence $V_1 \subset W_0$, which implies that $\pi_1(V_1) \subset \ker p_{0\pi} = \ker p_{1\pi}$, which clearly is impossible.

So what remains to us is to modify $f_s|_{W_0}$.

Definition 3 [9] A set of mutually disjoint circles $\mathcal{C} = \{c_1, \ldots, c_m\}$ on a compact surface F is an *independent set*, if the circles in $\mathcal{C}$ are essential and mutually non-parallel.

Lemma 6 *Let $h : W_1 \to W_1$ be a pseudo-Anosov map which is the identity in ∂W_1. We extend h by the identity to an automorphism h of F_s. $\mathcal{A}$ is a maximal independent set of circles in W_0. $f = f_s : F_s \to F_s$ is an extension of $\bar{f}_t : V_0 \to V_1$.*

Then when k is sufficiently large, for any $\alpha \in \mathcal{A}$, $h^k f(\alpha)$ is not isotopic to any circle in $\mathcal{A}$.

Proof Suppose $k_1 < k_2, \alpha \in \mathcal{A}$. We claim that $h^{k_1} f(\alpha) \nsim h^{k_2} f(\alpha)$ in F_s. In fact, $h^{k_1} f(\alpha)$ is an essential curve in W_1, and any two curves in W_1,which are homotopic in F_s, must be homotopic in W_1. But $h|_{W_1}$ is a pseudo-Anosov automorphism on W_1, so $h^{k_2-k_1}(h^{k_1}f(\alpha))$ is not isotopic to $h^{k_1}f(\alpha)$.

Hence for any $\alpha \in \mathcal{A}$, there are only finitely many k, such that $h^k f(\alpha)$ is homotopic to a circle in $\mathcal{A}$. Hence the conclusion holds.

From now on we replace f by $h^k f$.

Let $\mathcal{A}_0$ be a maximal independent set of circles on W_0, $\mathcal{A}_1$ be its image under f_s. Let $V'_j = e_{s,j}(V_j)$, $W'_j = e_{s,j}(W_j)$, $\mathcal{A}'_j = e_{s,j}(\mathcal{A}_j)$. Let $L = o_s(\mathcal{A}' \bigcup \partial W'_0)$, $V' = o_s(V'_j)$, $W' = o_s(W'_j)$, $F' = o_s(F_s \times \{j\})$; and $X = M(F_s, f_s) - \mathrm{int}(N(L))$, $F^* = F' \cap X$. Then

$$f'_s : (F_s \times \{0\}, V'_0, W'_0, \mathcal{A}_0) \to (F_s \times \{1\}, V'_1, W'_1, \mathcal{A}_1)$$

is a homeomorphism.

Suppose that we already know X is hyperbolic. Let $L = \{\alpha_1, \ldots, \alpha_m\}$, let T_l be the torus $\partial N(\alpha_l)$ on ∂X, $l = 1, \ldots, m$. Denote by τ_c the right hand Dehn twist along a circle c on F_s. Pick a meridian-longitude pair for each T_l, with longitude a component of $F' \cap T_l$. q_l is a slope on T_l, define $X(q_1, \ldots, q_m)$ to be the manifold obtained by q_l Dehn filling on T_l. The following lemma recalls a well-known relationship between Dehn fillings and Dehn twists, which has been used in some papers, say [4] and [9].

Lemma 7 *Let* $\tilde{f}(k_1, \ldots, k_m) = \tau_{\alpha_1}^{k_1} \circ \cdots \circ \tau_{\alpha_m}^{k_m} \circ f$. *Then*

$$M(F_s, \tilde{f}(k_1, \ldots, k_n)) = X(1/k_1, \ldots, 1/k_m)$$

for all $k_l \in \mathbb{Z}$.

Then we can use Thurston's hyperbolic surgery theorem to conclude that $\tilde{f}(k_1, \ldots, k_m)$ is pseudo-Anosov when $k_1, \ldots, k_m$ are sufficiently large.

Now we are going to prove that X is hyperbolic. We first have

Lemma 8 (1) *$\mathcal{A}'_j$ is a maximal independent set of circles on W'_j and each non-trivial circle in F^* is either essential in V'_j or parallel to a component of $\partial W'_j \cup \mathcal{A}'_j$.*

(2) *No component of $\partial W_0 \cup \mathcal{A}_0$ is homotopic to a component of $\partial W_1 \cup \mathcal{A}_1$, in F_s.*

Proof (1) follows directly from the definitions and the constructions.

(2) ∂W_1 (resp. ∂W_0) can not be isotoped into W_0 (resp. W_1) by Lemma 5 (2). No component of $\mathcal{A}_1 = f_s(\mathcal{A}_0)$ is isotopic to a component of $\mathcal{A}_0$ by Lemma 6. Hence (2) follows.

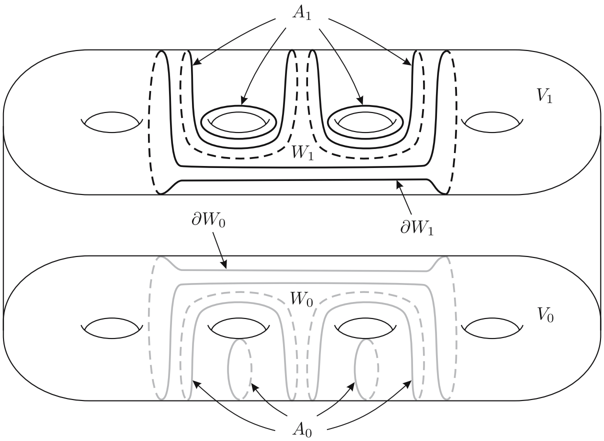
Figure 3 is what happens in $F_s \times [0,1]$, and clearly illustrate the meaning of Lemma 8.

Lemma 9 *X is atoroidal.*

Proof Suppose that T is an essential torus in X. We assume that T has been isotoped in X so that $|T \cap F^*|$ is minimal. Then $T \cap F^* = \mathcal{C}^*$ consists of π_1-injective circles on both T and F^*. Note $\mathcal{C}^* \neq \emptyset$, otherwise T would be an incompressible torus in $F_s \times [0,1]$, which is impossible.

Cutting X along F^*, we get a manifold $X' \subset F_s \times [0,1]$ and $T \backslash \mathcal{C}$ is a collection of annuli $A_1, A_2, \ldots, A_n \subset X'$. In the case A_k is vertical, we denote the component of ∂A_k in $F_s \times \{j\}$ by $c_{k,j}$, and $q_s(c_{k,0}) = q_s(c_{k,1}) \subset F_s$ by c_k.

Fig. 3



Now we claim that if one component of ∂A_k is essential in V'_j, then A_k is vertical. Because otherwise $\partial A_k \subset V'_j$, and we can push A_k across V' in X to reduce $|T \cap F^*|$, a contradiction.

By Lemma 8 (1), there are two cases:

Case 1 Some component of ∂A_k is an essential circle on V'_j.

By the claim above, A_k is vertical. Then $f'_s(c_{k,0})$ is a component of ∂A_l. ($l = k-1$ or $k+1$.) Since $f'_s(c_{k,0})$ is essential in V'_1, it follows by the claim A_l is vertical and $f'_s(c_{k,0}) = c_{l,1}$. Then $f_s(c_k) = c_l$ and c_l is essential in V_1. Hence c_l can not be isotoped into W_0 by Lemma 5 (2). Since $c_{l,0}$ is disjoint from $\partial W'_0$, $c_l \subset V_0$. Clearly c_l is still an essential circle in V_0, and therefore $c_{l,0}$ is essential in V'_0. Since T is connected, repeat the same argument finitely many times, we get that

(1) all A_k are vertical;
(2) all $c_{k,0}$ are essential in V'_0 and all $c_{k,1}$ are essential in V'_1;
(3) $f'_s(c_{k,0}) = c_{k+1,1}$ (re-indexing A_k if needed, and the subscript k is considered mod n). Hence each c_k is essential in both V_0 and V_1, and $f_s(c_k) = c_{k+1}$.

Now both $p_0(c_k)$ and $p_1(c_k)$ are essential circles in V, and therefore essential in F_t by Lemma 5 (1). Since $f_t \circ p_0 = p_1 \circ f_s$, we have that $f_t \circ p_0(c_k) = p_1 \circ f_s(c_k) = p_1(c_{k+1})$. Since p_0 and p_1 are homotopic, we have $p_0(c_k) \sim p_1(c_k)$. Then up to isotopy $f_t^n \circ p_0(c_k) = p_0(c_k)$, which contradicts to the fact that f_t is a pseudo-Anosov map on F_t.

Case 2 Each component of ∂A_k is parallel to a component of $\partial W'_0 \cup \mathcal{A}'_0$. By Lemma 8 (2), no A_k is vertical, hence both components of ∂A_k are parallel to a component c of $\partial W'_0 \cup \mathcal{A}'_0$. So A_k can be rel ∂A_k isotoped into $N(c)$ in X'. Hence back to X the torus T can be isotoped into $N(o_s(c))$. This means that T is boundary parallel in X, contrary to our assumption.

It is easy to see that X is irreducible: a reducing sphere S would bound a ball B in $M(F_s, f_s)$, because $M(F_s, f_s)$, as a surface bundle over circle, is irreducible. Hence B would contain some component of L. This is impossible because each component of L is essential in $M(F_s, f_s)$.

X is not a Seifert fibered space: it contains $q(F_s \times \frac{1}{2})$, a non-separating, hyperbolic, closed incompressible surface. No such surface exists in a Seifert fibered space with boundary, because an essential surface in such a manifold is either horizontal (hence bounded) or vertical (hence a torus or an annulus).

Now the geometrization theorem of Thurston for Haken manifolds [8] leads us to the following:

Corollary 1 *X is a hyperbolic manifold.*

Proof (Proof of Proposition 1) By Corollary 1, X is a hyperbolic manifold, therefore, by the hyperbolic surgery theorem of Thurston [7], $X(1/k_1,\ldots,1/k_m)$ is hyperbolic for sufficiently large k_l. The previous lemma implies that

$$X(1/k_1,\ldots,1/k_m) = M(F_s,\tilde{f}(k_1,\ldots,k_m)).$$

The theorem now follows from Thurston's theorem that $M(F_s,\tilde{f})$ is hyperbolic if and only if $\tilde{f}$ is isotopic to a pseudo-Anosov map [5,8]).

3 Adjusting Betti numbers

Now we pay attention to the Betti numbers of the surface bundles. Using HNN extension one can calculate directly that

$$\mathrm{H}_1(M(F,f);\mathbb{Z}) = \mathrm{H}_1(F,\mathbb{Z})/\ker(I_{2g} - f_\#)\ \oplus \mathbb{Z}$$

where $g = g(F)$, I_{2g} is the unit matrix in $SL_{2g}(\mathbb{Z})$.

By abuse of notation, denote the image of $\mathrm{H}_1(V_j)$ in $\mathrm{H}_1(F_s)$ by $\mathrm{H}_1(\widehat{V}_j)$. Similarly, define $\mathrm{H}_1(\widehat{W}_j)$. We have

$$\mathrm{H}_1(F_s) = \mathrm{H}_1(\widehat{V}_0) \oplus \mathrm{H}_1(\widehat{W}_0) = \mathrm{H}_1(\widehat{V}_1) \oplus \mathrm{H}_1(\widehat{W}_1).$$

Lemma 10

$$\mathrm{H}_1(\widehat{V}_0) = \mathrm{H}_1(\widehat{V}_1),$$
$$\mathrm{H}_1(\widehat{W}_0) = \mathrm{H}_1(\widehat{W}_1),$$

as subgroups of $\mathrm{H}_1(F_s)$.

Proof The second equation is easy, because $\mathrm{H}_1(\widehat{W}_j) = \ker p_{j\#}$, and $p_0 \sim p_1$ implies $p_{0\#} = p_{1\#}$.

Now we will prove the first equation. Suppose $c_0 \subset V_0$ is a circle, then $c = p_0(c_0)$ is a circle in V, thus $c_1 = p_1^{-1}(c)$ is a circle in V_1. Since $p_0 \sim p_1$, we have a map $H : F_s \times [0,1] \to F_t,\ H(\cdot,j) = p_j(\cdot)$.

Make H transverse to c, then $H^{-1}(c)$ is a submanifold of $F_s \times [0,1]$ with boundary $c_0 \cup c_1$. So c_0 and c_1 represent the same homology class in $\mathrm{H}_1(F_s \times [0,1]) = \mathrm{H}_1(F_s)$. Hence the generators of $\mathrm{H}_1(\widehat{V}_0)$ is the same as the ones of $\mathrm{H}_1(\widehat{V}_1)$, our conclusion holds.

Choose a basis of $\mathrm{H}_1(\widehat{V}_0)$ and a basis of $\mathrm{H}_1(\widehat{W}_0)$ to make up a basis of $\mathrm{H}_1(F_s)$. Under this basis, $f_{s\#}$ will be represented by a matrix of the form:

$$\begin{pmatrix} f_{t\#} & 0 \\ 0 & A \end{pmatrix}.$$

Lemma 11 *The map f_s can be chosen so that the matrix $I - A$ is non-degenerate.*

Proof Let $\delta = g_s - g_t$. Choose curves $\alpha_1, \ldots, \alpha_\delta, \beta_1, \ldots, \beta_\delta \subset W_1$, such that they are mutually disjoint, except that α_l intersects β_l in a single point transversely. These 2δ curves form a symplectic basis of $\mathrm{H}_1(\widehat{W}_1)$. Under this basis, the intersection form of $\mathrm{H}_1(\widehat{W}_1)$ is $\begin{pmatrix} 0 & I_\delta \\ -I_\delta & 0 \end{pmatrix}$. So if $f : (F_s, W_0) \to (F_s, W_1)$ is a homeomorphism, $f_\#|_{\mathrm{H}_1(\widehat{W}_0)}$ will be represented by a symplectic matrix F.

We choose a map $\eta : W_1 \to W_1$, such that it fixes the points on ∂W_1, and it induces F^{-1} on the homology.

When $\delta > 1$, by Theorem 2 in [6], every symplectic matrix of rank 2δ can be represented by a pseudo-Anosov map on a closed surface of genus δ. So there is a pseudo-Anosov map $h : W_1 \to W_1$, such that it fixes the points on ∂W_1, and induces $-I_{2\delta}$ on the homology. Extend η, h to maps on F_s, with the points in V_1 fixed.

Let $\gamma_l = \partial N(\alpha_l \cup \beta_l)$, which is a separating circle in W_1 (see those separating circles in $\mathcal{A}_1$, Figure 3). Now we extend $\{\alpha_1, \ldots, \alpha_\delta, \gamma_1, \ldots, \gamma_\delta\}$ to a maximal independent set $\mathcal{A}$ on W_0. Then every curve in $\mathcal{A} - \{\alpha_1, \ldots, \alpha_\delta\}$ is homologous to 0. Let $L = \mathcal{A} \cup \{\partial W_1\}$ So the only Dehn twists along circles in L, which act nontrivially on the homology group, are $\tau_{\alpha_1}, \ldots, \tau_{\alpha_\delta}$. The action of products of these twists on $\mathrm{H}_1(\widehat{W}_1)$ is represented by a upper-triangular matrix T, whose diagonal elements are all 1. By Lemma 6, $h^{2k+1}\eta f$ does not send any curve in L into L when k is sufficiently large. The matrix of $h^{2k+1}\eta f$, when restricted on $\mathrm{H}_1(\widehat{W}_0)$, is $-I_{2\delta}$. Now replace f by the composition of Dehn twists along L and $h^{2k+1}\eta f$, we have

$$I_{2\delta} - A = I_{2\delta} + T$$

is non-degenerate.

When $\delta = 1$, we give a direct construction. Let α, β be a symplectic basis of W_0. Choose a map $h' : W_1 \to W_1$, with matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $h'\eta f$ meets the conclusion of Lemma 6. τ is the Dehn twist along $\eta f(\alpha)$, then the matrix of τ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. τ_0 is the Dehn twist along ∂W_1. Now the matrix A of $\tau_0^m \tau^k h' \eta f$ is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2+k & 1+k \\ 1 & 1 \end{pmatrix}.$$

One can check $I - A$ is non-degenerate when $k \geq 0$.

To prove Theorem 1 (2), we need only to show the following lemma.

Lemma 12 *With the notation as above, the extension f_s in Proposition 1 can be chosen so that*

$$rank\ (\mathrm{H}_1(F_s; \mathbb{Q}) / \ker(I_{2g_s} - f_{s\#})) = rank\ (\mathrm{H}_1(F_t; \mathbb{Q}) / \ker(I_{2g_t} - f_{t\#})).$$

Proof The conclusion follows from the formula of computing $H_1(M(F,f))$ and Lemma 11.

Proof (Proof of Theorem 1) This theorem follows from Proposition 1 and Lemma 12.

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