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A note on knot Floer homology of links

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Ozsváth and Szabó proved that knot Floer homology determines the genera of knots in S^3. We will generalize this deep result to links in homology 3–spheres, by adapting their method. Our proof relies on a result of Gabai and some constructions related to foliations. We also interpret a theorem of Kauffman in the world of knot Floer homology, hence we can compute the top filtration term of the knot Floer homology for alternative links.

57R58, 53D40; 57M27, 57R30

1 Introduction

The central problem in knot theory is the classification of knots and links. In order to study this problem, people have introduced a lot of invariants. Among them, knot Floer homology seems to be particularly interesting. This invariant was introduced by Ozsváth and Szabó in [14], and independently by Rasmussen in [18], as part of Ozsváth and Szabó's Heegaard Floer theory.

One remarkable feature of knot Floer homology is that it detects the genus in the case of classical knots, Ozsváth and Szabó [13, Theorem 1.2], namely, the genus of a classical knot is the highest nontrivial filtration level of the knot Floer homology. A direct corollary is: knot Floer homology detects the unknot. Even this corollary has its own interest.

In this paper, we will generalize this deep result to links in homology 3–spheres. Our main theorem is the following.

Theorem 1.1 *Suppose L is an oriented link in a closed 3–manifold Z with $\mathrm{H}_1(Z)=0$. $|L|$ denotes the number of components of L, and $\chi(L)$ denotes the maximal Euler characteristic of the Seifert surface bounded by L. Then*

$$\frac{|L|-\chi(L)}{2}=\max\{i \mid \widehat{HFK}(Z,L,i)\neq 0\}.$$

In particular, knot Floer homology detects trivial links in homology 3–spheres.

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Remark 1.2 When $Z = S^3$, the Euler characteristic of $\widehat{HFK}(S^3, L)$ gives rise to the Alexander–Conway polynomial $\Delta_L(t)$ (see Ozsváth–Szabó [14]). But knot Floer homology certainly contains more information than $\Delta_L(t)$. For example, when L is a boundary link, Δ_L is 0, while our theorem implies $\widehat{HFK}(S^3, L)$ is nontrivial.

Our paper is organized as follows:

In Section 2 we give some background on Heegaard Floer theory, especially some useful remarks about links.

In Section 3 we will state a result of Gabai in [7], which allows us to prove the main theorem for knots in homology spheres immediately. But in order to study links, we need some constructions related to foliations. As a consequence of the constructions, we can embed a certain 3–manifold into a symplectic 4–manifold with a "good" symplectic structure. Results in this section have their own interest.

Section 4 will be devoted to the proof of our main theorem for links. Our argument is not much different from Ozsváth and Szabó's original one.

In Section 5, we interpret a theorem of Kauffman on alternative links, using the language of knot Floer homology. Hence we can compute the top filtration term of the knot Floer homology for alternative links. This section is independent of our main result.

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2 Preliminaries

2.1 Heegaard Floer theory

We will briefly include some background on Heegaard Floer theory here, a survey of Heegaard Floer theory can be found in Ozsváth–Szabó [12].

For any closed oriented 3–manifold Y, and a Spin^c structure $\mathfrak{s}$ on Y, Ozsváth and Szabó associate a package of Floer homologies: $HF^\infty(Y,\mathfrak{s})$, $HF^+(Y,\mathfrak{s})$, $HF^-(Y,\mathfrak{s})$, $\widehat{HF}(Y,\mathfrak{s})$, $HF_{\mathrm{red}}(Y,\mathfrak{s})$. These are topological invariants of $(Y,\mathfrak{s})$. Furthermore, the homologies (except $\widehat{HF}(Y,\mathfrak{s})$) admit an action U, which lowers degree by 2. So these groups are $\mathbb{Z}[U]$–modules. When $\mathfrak{s}$ is torsion, one can equip an absolute $\mathbb{Q}$–grading to these groups.

This theory is functorial. Namely, if we have a cobordism W from Y_1 to Y_2, $\mathfrak{r}$ is a Spin^c structure on W, $\mathfrak{s}_i$ is the restriction of $\mathfrak{r}$ on Y_i, then there is a homomorphism

$$F^{\circ}_{W,\mathfrak{r}}\colon HF^{\circ}(Y_1,\mathfrak{s}_1)\to HF^{\circ}(Y_2,\mathfrak{s}_2). \tag{1}$$

Here HF° denotes any one of the five homologies. When $\mathfrak{s}_1,\mathfrak{s}_2$ are both torsion, $F^{\circ}_{W,\mathfrak{r}}$ shifts degree by

$$\frac{c_1(\mathfrak{r})^2-2\chi(W)-3\sigma(W)}{4}. \tag{2}$$

Moreover, suppose W_1 is a cobordism from Y_1 to Y_2, W_2 is a cobordism from Y_2 to Y_3, $W=W_1\cup_{Y_2}W_2$, $\mathfrak{r}_i$ is a Spin^c structure on W_i. Then we have the composition law:

$$F^{\circ}_{W_2,\mathfrak{r}_2}\circ F^{\circ}_{W_1,\mathfrak{r}_1}=\sum_{\mathfrak{r}\in\mathrm{Spin}^c(W),\,\mathfrak{r}|W_i=\mathfrak{r}_i}\pm F^{\circ}_{W,\mathfrak{r}} \tag{3}$$

The above $F^{\circ}_{W,\mathfrak{r}}$ is an invariant of cobordisms. In particular, when X is a closed 4–manifold with $b_2^+>1$, $\mathfrak{r}\in\mathrm{Spin}^c(X)$, Ozsváth and Szabó can define a mixed invariant $\Phi_{X,\mathfrak{r}}\in\mathbb{Z}$. Φ behaves like, and is conjectured to agree with, the Seiberg–Witten invariant.

When K is a null-homologous knot in Y, one can define a $\mathbb{Z}\oplus\mathbb{Z}$–filtered chain complex $CFK^{\infty}(Y,K)$. From this chain complex, we can get some knot invariants, one of which is called the knot Floer homology, denoted by $\widehat{HFK}(Y,K)$. Given a Seifert surface of K, one can associate a filtration to $\widehat{HFK}(Y,K)$. The summand of $\widehat{HFK}(Y,K)$ at filtration level i is denoted by $\widehat{HFK}(Y,K,i)$.

2.2 A few remarks about links

In [14], Ozsváth and Szabó gave a well-defined correspondence from links to knots, and the invariant for links is defined to be the invariant for the corresponding knots.

The construction is as follows: given a null-homologous oriented n–component link L in Y, choose two points p,q on different components of L. Remove two balls at p,q, then glue in a 3–dimensional tube $S^2\times I$. (In [14], this $S^2\times I$ is called a 1–handle, but we would rather call it a tube.) Inside the tube, there is a band, along which we can perform a connected sum of the two components of L containing p and q. We choose the band so that the connected sum respects the original orientation on L. Now we have a link in $Y^3\#S^2\times S^1$, with one fewer components. Repeat this construction until we get a knot. The new knot is denoted by $\kappa(L)$, and the new manifold is denoted by $\kappa(Y) = Y\#(n-1)(S^2 \times S^1)$. Ozsváth and Szabó proved that this correspondence $(Y, L) \mapsto (\kappa(Y), \kappa(L))$ is well-defined.

At first sight, the above construction seems rather artificial. But as we will see, it is pretty reasonable. As for now, we are satisfied to justify the construction by giving two folklore propositions.

Proposition 2.1 (Adjunction Inequality) *$L \subset Y$ is an oriented link. If $\widehat{HFK}(Y, L, i) \neq 0$, then, for each Seifert surface F for L, we have that*

$$i \leq \frac{|L| - \chi(F)}{2}.$$

Proof Let $\kappa(F)$ be the Seifert surface of the knot $\kappa(L)$, obtained by adding $|L|-1$ bands to F. Then $\chi(\kappa(F)) = \chi(F) - (|L|-1)$, hence $\frac{|L|-\chi(F)}{2} = g(\kappa(F))$. Our result holds by the usual adjunction inequality for knots. □

Proposition 2.2 *Let $L \subset Y$ be an oriented fibred link, ie, $Y - L$ fibers over the circle, and the fiber is a Seifert surface F of L. Then*

$$\widehat{HFK}(Y, L, \frac{|L| - \chi(F)}{2}) \cong \mathbb{Z}.$$

Proof Suppose $p, q \in L$ are two points in different components. Add a tube $R = S^2 \times I$ to Y with feet at p, q, and perform the band-connected sum operation to L, we get a new link $L' \subset Y\#S^2 \times S^1$. Its Seifert surface F' is obtained by adding a band B to F.

F is connected, hence we can find an arc $\gamma \subset F$ connecting p to q. A neighborhood of γ in F is a rectangle D, we choose D so that $B \cup D$ is an annulus A. $N = \mathrm{Nd}(R \cup \gamma)$ is a punctured $S^2 \times S^1$, we can isotope F' slightly so that $F' \cap \partial N = D$, $F' \cap N = A$. Cap off the boundary sphere of N by a ball, the new manifold is denoted by $\widehat{N}$. Now ∂A is a fibred link in $\widehat{N}$, with fiber A, and $(Y\#S^2 \times S^1, L')$ is obtained by plumbing the two fibred links (Y, L), $(\widehat{N}, \partial A)$. It is well-known that $(Y\#S^2 \times S^1, L')$ is also fibred. This fact was first proved by Stallings in [20], using algebraic method. A geometric proof can be found in Gabai [6], where it is also shown that if L' is fibred then so is L.

The above proof shows that $(\kappa(Y), \kappa(L))$ is a fibred knot, now we use Ozsváth–Szabó [17, Theorem 1.1]. □

2.3 Kauffman states for links

The material in this subsection will be used in Section 5, not in the proof of our main theorem.

There is no known algorithm to compute the invariants in Heegaard Floer theory. But one can find all the generators of the chain complex in a purely combinatorial way. In [11], Ozsváth and Szabó described such a method for classical knots.

Suppose $L \subset S^3$ is an oriented link. Choose a generic projection of L, we get a 4–valent graph $\mathcal{D}$ on S^2. $\mathcal{D}$ is called a *diagram* for L. We always choose $\mathcal{D}$ so that it is *connected*. If we fix an edge in $\mathcal{D}$, then $\mathcal{D}$ is called a *decorated diagram*.

Suppose a decorated diagram $\mathcal{D}$ has m vertices, then $\mathcal{D}$ divides S^2 into $m+2$ regions. Suppose A, B are the two regions sharing the distinguished edge. A *Kauffman state* is a correspondence x, which associates one of the four corners at v to each vertex v of $\mathcal{D}$, so that the corners are not in A, B, and any two corners are not in the same region.

One can also associate filtration level and grading to a Kauffman state. The local filtration level contribution $\mathrm{Fil}(x(v))$ at v is shown in Figure 1, and the local grading contribution $\mathrm{Gr}(x(v))$ at v is shown in Figure 2. We define

$$\mathrm{Fil}(x) = \frac{|L|-1}{2} + \sum_{v \in \mathrm{Vert}(\mathcal{D})} \mathrm{Fil}(x(v)),$$

$$\mathrm{Gr}(x) = \frac{|L|-1}{2} + \sum_{v \in \mathrm{Vert}(\mathcal{D})} \mathrm{Gr}(x(v)).$$

When L is a knot, one can choose the chain complex $\widehat{CFK}(S^3, L)$, so that the generators are precisely the Kauffman states with filtration level $\mathrm{Fil}(x)$ and grading $\mathrm{Gr}(x)$ [11].

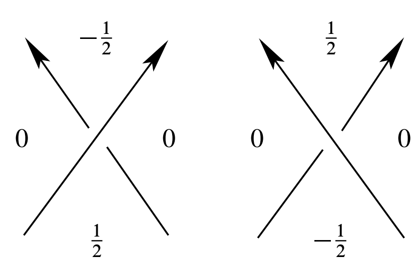

Figure 1: Local filtration level contribution $\mathrm{Fil}(x(v))$

The reader should note that, when L is a link, the Kauffman states do not correspond to the generators of $\widehat{CFK}(S^3, L)$ in a natural way. So the notions "filtration level" and

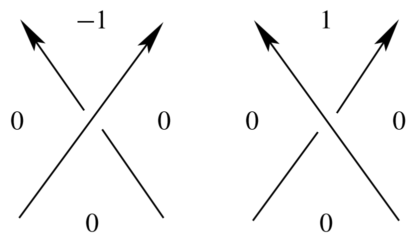

Figure 2: Local grading contribution $\mathrm{Gr}(x(v))$

"grading" do not make sense for links. In Section 5, we will show that, for alternative links, the Kauffman states at the top filtration level are in one-to-one correspondence with the generators of the corresponding knot Floer homology group, and the filtration level and grading of the Kauffman states are equal to the filtration level and grading of the knot Floer homology at the top filtration level.

Remark 2.3 During the course of this work, Khovanov and Rozansky posted a paper [9], in which they associate a triply-graded cohomology to a classical link. Euler number of this cohomology gives the HOMFLYPT polynomial. It is not known whether their work can eventually lead to a purely combinatorial account of knot Floer homology for classical links.

3 A few constructions related to taut foliations

Suppose M is a 3–manifold, $S \subset M$ is an embedded sphere. We say S is an *essential sphere* if S does not bound a ball in M. M is *irreducible* if it does not contain any essential sphere. Otherwise we say M is *reducible*. We will need the following lemma, whose proof only involves standard arguments in 3–dimensional topology.

Lemma 3.1 *Z is a closed oriented 3–manifold, $L \subset Z$ is an n–component null-homologous oriented link, $(Y, K) = (\kappa(Z), \kappa(L))$. If $Z - L$ is irreducible, then $Y - K$ is irreducible.*

Proof Suppose G is a Seifert surface of K with minimal genus. $R_1, \dots, R_{n-1}$ are the tubes added to Z, $P_1, \dots, P_{n-1}$ are their belt spheres, ie, $S^2 \times point$ in the tube $S^2 \times I$. Assume S is an essential sphere in $Y - K$.

We can assume P_k, G, S are mutually transverse. $P_k \cap K$ consists of two points, hence $P_k \cap G$ consists of an arc and some circles. If C is an innermost circle in $P_k - arc$, then C bounds a disk D. Cut G open along C, glue in two copies of D, and possibly throw away a closed component, we get a new Seifert surface G'. $G'\cap P_k$ has fewer components than $G\cap P_k$, and $\chi(G')\geq\chi(G)$. Hence we can assume $P_k\cap G$ is an arc.

$S\cap G$ consists of circles, each circle bounds a disk in G since G is a Seifert surface with minimal genus. Now assume C is an innermost circle in such a disk, C bounds D. Using D to do surgery to S, we get two spheres S_1, S_2. S is essential in $Y-K$, hence one of S_1, S_2 must be essential, and the intersection of this sphere with G has fewer components. Hence we can assume S is disjoint from G.

By the same argument, we can assume S is disjoint from all P_k's, hence S lies in $Y-\cup_{k=1}^{n-1}R_k$, which is a submanifold of Z. Hence $S\subset Z-L$. Since $Z-L$ is irreducible, S bounds a 3–ball, a contradiction. □

Remark 3.2 Suppose F is a Seifert surface of L, with maximal Euler number. The second paragraph in the above proof shows that, a minimal genus Seifert surface of K can be obtained by adding $n-1$ bands to F.

In [13], Ozsváth and Szabó proved that, for classical knots, genus is the highest nontrivial filtration level of the knot Floer homology. Their proof relies on the existence of a "nice" taut foliation of knot complement, which was proved by Gabai in [7]. In the same paper, Gabai also gave another existence result for taut foliations:

Theorem 3.3 [7, Theorem 8.9] *K is a null-homologous knot in Y. Y is reducible and $Y-K$ is irreducible. $\mathrm{H}_1(Y;\mathbb{Z})$ is torsion free. If G is a minimal genus Seifert surface of K, then there exists a taut finite depth foliation $\mathscr{F}$ of $Y-\mathrm{int}(\mathrm{Nd}(K))$, such that $\mathscr{F}|\partial\mathrm{Nd}(K)$ is a foliation by circles which are longitudes, and G is a leaf of $\mathscr{F}$. Moreover, when G has genus >1, one can arrange $\mathscr{F}$ to be smooth.*

Remark 3.4 We briefly describe how Gabai can extend his result for classical knots to this case. In S^3, there is a Heegaard sphere. Gabai introduced the notion of "thin position", hence he could use the Heegaard sphere like an essential sphere. When Y is reducible, there is already given an essential sphere, so Gabai's argument can proceed.

"$\mathrm{H}_1(Y)$ is torsion free" is a technical condition, to eliminate some bad cases like the existence of a Scharlemann cycle.

Gabai did not mention smoothness in the statement of [7, Theorem 8.9]; however, one can obtain the smoothness conclusion when genus >1 just as in the proof of the main result of that paper.

Having Theorem 3.3, we can immediately prove our main theorem in the case of knots, namely, the following result.

Proposition 3.5 *Suppose K is a knot in a closed 3–manifold Z, $\mathrm{H}_1(Z)=0$. Then*

$$g(K)=\max\{i\in\mathbb{Z}\,|\,\widehat{HFK}(Z,K,i)\neq 0\}.$$

Proof If $Z-K$ is reducible, then $Z-K=(Z'-K)\#Z''$, where $Z'-K$ is irreducible, and Z', Z'' are homology spheres. Now we have

$$\widehat{HFK}(Z,K)\cong\widehat{HFK}(Z',K)\otimes\widehat{HF}(Z'').$$

$\widehat{HF}(Z'')$ is nontrivial since its Euler characteristic is 1, so we reduce our problem to (Z',K).

From now on, we assume $Z-K$ is irreducible, and Z is not S^3. Consider the knot $K\#K$ in $Z\#Z$. We find that the conditions in Theorem 3.3 hold, namely, $Y=Z\#Z$ is reducible, the complement of $K\#K$ is irreducible, and $\mathrm{H}_1(Y)$ is torsion free. Hence there exists a taut smooth foliation $\mathscr{F}$ of $Y-\mathrm{int}(\mathrm{Nd}(K\#K))$, such that $\mathscr{F}|\partial\mathrm{Nd}(K\#K)$ is a foliation by longitudes.

Now we can prove our proposition for $K\#K$, the proof is exactly same as the proof of [13, Theorem 1.2], modulo the following Lemma 3.6. The reader can find the argument in [13], or a more delicate but essentially same argument in the proof of our main theorem.

By the Künneth formula for connected sums [14, Theorem 7.1], we get the desired result for K. □

Lemma 3.6 (Compare [14, Corollary 4.5]) *Let K be a knot in a homology 3–sphere Y, Y_p be the manifold obtained by p–surgery on K. Let d be an integer satisfying $\widehat{HFK}(Y,K,i)=0$ when $i\geq d$, and suppose that $d>1$. Then*

$$HF^+(Y_0,d-1)=0.$$

Proof Consider the bigraded chain complex $C=CFK^\infty(Y,K)$. We have the short exact sequence

$$0\to C\{i<0\text{ and }j\geq d-1\}\to C\{i\geq 0\text{ or }j\geq d-1\}\xrightarrow{\Psi} C\{i\geq 0\}\to 0,$$

hence a long exact sequence relating their homologies. In $C\{i<0\text{ and }j\geq d-1\}$, all summands lie at filtration levels $\geq d$. By our assumption,

$$\mathrm{H}_*(C\{i<0\text{ and }j\geq d-1\})=0.$$

Hence Ψ induces an isomorphism ψ on the level of homology.

By [14, Theorem 4.4] (also compare [10, Theorem 2.3]), when $p > 0$ is a sufficiently large integer,

$$\mathrm{H}_*(C\{i \geq 0 \text{ or } j \geq d-1\}) \cong HF^+(Y_p, d-1),$$

and the map Ψ coincides with the map induced by the cobordism W from Y_p to Y, endowed with the Spin^c structure $\mathfrak{r}$ with

$$\langle c_1(\mathfrak{r}), [\widetilde{F}]\rangle = 2d - 2 - p.$$

Here F is a Seifert surface of K, and $\widetilde{F}$ is a closed surface in W, obtained from F by capping off ∂F in W.

According to [15, Theorem 9.19], there is an long exact sequence

$$\cdots \to HF^+(Y_0, d-1) \to HF^+(Y_p, [d-1]) \xrightarrow{f} HF^+(Y) \to \cdots$$

We will compare ψ and f.

Suppose $\mathfrak{r}' \in \mathrm{Spin}^c(W)$ nontrivially contributes to f, then $\mathfrak{r}' - \mathfrak{r}$ must be $k\mathrm{PD}([\widetilde{F}])$ for some integer k, where PD is the Poincaré duality map. It is not hard to find

$$c_1^2(\mathfrak{r}') = -\frac{(2d-2-p+2kp)^2}{p}, \quad k \in \mathbb{Z}.$$

Now by the degree shifting formula (2), f has the form $\psi + \iota$, where ι is a sum of homogeneous maps which have lower orders than ψ. Since ψ is an isomorphism, it is clear that $\psi + \iota$ is also an isomorphism, hence $HF^+(Y_0, d-1) = 0$. □

Remark 3.7 Now we can prove our main theorem for links which have connected minimal Seifert surfaces, by a pretty trivial argument sketched as follows. If F is a connected minimal Seifert surface of $L \subset Z$, then we can perform Murasugi sums of F with Hopf bands, to get a knot K in Z. Using the skein exact sequence and the adjunction inequality, we can reduce our problem to K, for which the problem is already solved in Proposition 3.5.

In order to prove our main theorem for general links, we will do some constructions related to foliations, contact structures and symplectic structures.

Remark 3.8 We recall the procedures in [2; 3; 4] of passing from taut foliations to symplectic structures. Suppose $\mathcal{G}$ is a C^2 taut foliation of M, then there exists a closed 2–form ω on M, so that ω does not vanish on $\mathcal{G}$. Let $W = M \times I$, define a closed 2–form $\Omega' = p^*\omega + \varepsilon \mathrm{d}(t\alpha)$. Here p is the projection of W onto M, α is the 1–form defining $\mathcal{G}$. Hence (W, Ω') weakly symplectically semi-fills $(M, \mathcal{G})$. The plane field $\mathscr{G}$ can be slightly perturbed to a contact structure, which is also weakly symplectically semi-fillable.

By [2, Theorem 1.3], (W,Ω') can be symplectically embedded into a closed symplectic manifold (X,Ω).

We also need the following basic operation of modifying foliations, which preserves tautness.

Operation 3.9 (Creating or killing holonomy) Suppose $\mathscr{G}$ is a foliation of M, $S^1\times I$ is an annulus in M, so that $\mathscr{G}$ restricts to $S^1\times t$'s on $S^1\times I$. $f\colon I\to I$ is a homeomorphism supported in $\mathrm{int}(I)$. We can cut open M along $S^1\times I$, then reglue by $\mathrm{id}\times f$, thus get a new foliation $\mathscr{G}'$. If $\mathscr{G}$ and f are smooth, then $\mathscr{G}'$ is also smooth.

In order to prove our main theorem, it seems that Theorem 3.3 and Remark 3.8 are not enough. We will also need a "good" symplectic structure. The next lemma gives a way to construct such symplectic structures.

Lemma 3.10 *Let $\mathscr{G}$ be a smooth taut foliation of M. $T_1,\dots,T_m$ are closed surfaces in M. Suppose there exist simple closed curves $c_1,\dots,c_m$, each curve is contained in a leaf of $\mathscr{G}$, and*

$$[c_i]\cdot[T_j]=\delta_{ij}.$$

If for each curve c_i, the holonomy along c_i is trivial, then M can be embedded into a closed symplectic 4–manifold (X,Ω), so that $X=X_1\cup_M X_2$, $b_2^+(X_j)>0$, and

$$\int_{T_i}\Omega=0.$$

Proof Since the holonomy along c_i is trivial, we can choose a neighborhood $\mathrm{Nd}(c_i)=S^1\times[-1,1]\times[-1,1]$, with coordinates (x,y,t), so that restriction of $\mathscr{G}$ to $\mathrm{Nd}(c_i)$ consists of $S^1\times[-1,1]\times t$'s.

Let u be a bump function supported in $(-1,1)$, so that $\int_{-1}^{1}u(t)\mathrm{d}t=1$. Hence

$$\varphi_i=u(y)u(t)\mathrm{d}y\wedge\mathrm{d}t$$

is a closed 2–form supported in $\mathrm{Nd}(c_i)$. We can extend φ_i by 0 to the whole M, then φ_i restricts to 0 on $\mathscr{G}$, and $\int_T\varphi_i=[c_i]\cdot[T]$ for any closed surface $T\subset M$.

Now we perturb ω by multiples of $\varphi_1,\dots,\varphi_m$, so as to get a closed 2–form ω', which never vanishes on $\mathscr{G}$, and satisfies $\int_{T_i}\omega'=0$ for any i.

As in Remark 3.8, the form

$$\Omega' = p^*\omega' + \varepsilon \mathrm{d}(t\alpha)$$

is a symplectic form on $M \times I$. Moreover, $\int_{T_i} \Omega' = 0$ for any i. Hence the result holds by Eliashberg and Etnyre's theorem. □

Notation 3.11 In the rest of this section and Section 4, we will use the following notation. Let L be an n–component link in a homology sphere Z, $n > 1$, $(Y, K) = (\kappa(Z), \kappa(L))$. G is a minimal genus Seifert surface of K. By Remark 3.2, we can assume G is obtained by adding $n-1$ bands $B_1, \dots, B_{n-1}$ to a Seifert surface F of L with maximal Euler characteristic. Hence $\chi(G) = \chi(F) - (n-1)$. Let $\widehat{G}$ be the extension of G in Y_0 obtained by gluing a disk to G.

Y is obtained from Z by adding $n-1$ (3–dimensional) tubes $R_1, \dots, R_{n-1}$. Suppose P_i is the belt sphere of the tube R_i. The knot K intersects P_i in exactly 2 points, we can remove two disks from P_i at these two points, then glue in a long and thin (2–dimensional) tube along an arc in K, so as to get a torus T_i. T_i is homologous to P_i, but disjoint from K. T_i will play an important role in our proof.

Proposition 3.12 *Notation is as above. We suppose that $Z - L$ is irreducible, and genus$(G) > 1$. After doing connected sum with some fibred knots in S^3, we get a new link L^*. We consider $(Y^*, K^*) = (\kappa(Z), \kappa(L^*))$, and the 0–surgered space Y_0^*. The conclusion is: for a suitably chosen L^*, Y_0^* can be embedded into a closed symplectic 4–manifold (X, Ω), so that $X = X_1 \cup_{Y_0^*} X_2$, $b_2^+(X_j) > 0$, and*

$$\int_{T_i^*} \Omega = 0$$

for all i. Moreover,

$$\langle c_1(\mathfrak{k}(\Omega)), [\widehat{G^*}]\rangle = 2 - 2g(\widehat{G^*}).$$

Proof We first consider the case that L consists of two components K_1, K_2. We apply Theorem 3.3 to get a smooth taut foliation $\mathscr{F}$ on $Y - \mathrm{int}(\mathrm{Nd}(K))$. According to Gabai [7; 5], each leaf of $\mathscr{F}$ intersects $\partial \mathrm{Nd}(K)$.

P_1 is the belt sphere of the (3–dimensional) tube R_1. $P_1 - \mathrm{int}(\mathrm{Nd}(K))$ is an annulus. A standard argument (see [21, Theorem 4]) shows that one can deform P_1 so that $\mathscr{F}|(P_1 - \mathrm{int}(\mathrm{Nd}(K)))$ is a product foliation consisting of arcs. Note that $Y - \mathrm{int}(\mathrm{Nd}(K)) - \mathrm{int}(\mathrm{Nd}(P_1))$ is diffeomorphic to $Z - \mathrm{int}(\mathrm{Nd}(L))$, hence $\mathscr{F}$ induces a foliation $\mathscr{E} = \kappa^{-1}(\mathscr{F})$ of $Z - \mathrm{int}(\mathrm{Nd}(L))$. Conversely, $\mathscr{F}$ is obtained by adding bands to leaves of $\mathscr{E}$.

Each leaf of $\mathscr{E}$ intersects either $\partial\mathrm{Nd}(K_1)$ or $\partial\mathrm{Nd}(K_2)$. Let

$$U_i = \{x \in Z - \mathrm{int}(\mathrm{Nd}(L)) \,|\, \text{the leaf containing } x \text{ intersects } \partial\mathrm{Nd}(K_i)\}.$$

U_1, U_2 are open subsets of $Z - \mathrm{int}(\mathrm{Nd}(L))$, $U_1 \cup U_2 = Z - \mathrm{int}(\mathrm{Nd}(L))$. Hence $U_1 \cap U_2 \neq \varnothing$, which means that there exists a leaf $\mathcal{L}_0$ intersecting both $\partial\mathrm{Nd}(K_1)$ and $\partial\mathrm{Nd}(K_2)$.

If there is a band $B \subset R_1$, which connects $\mathcal{L}_0$ to itself, then there is a simple closed curve $c_1 \subset \kappa(\mathcal{L}_0)$, so that $[c_1]\cdot[T_1] = 1$. Here $\kappa(\mathcal{L}_0)$ is the leaf of $\mathscr{F}$ that contains $\mathcal{L}_0$.

Now assume that there is a band $B \subset R_1$ connecting $\mathcal{L}_0$ to a different leaf $\mathcal{L}_1$. We can suppose $B \cap \mathcal{L}_1 \subset \partial\mathrm{Nd}(K_1)$. Perform connected sum of L and a trefoil $Tr_1 \subset S^3$, so that Tr_1 is added to K_1. We get a new link L'. $\mathscr{E}$ can be extended to a foliation $\mathscr{E}'$ of $Z - \mathrm{int}(\mathrm{Nd}(L'))$, so that the part of $\mathscr{E}'$, obtained from $S^3 - Tr_1$, is a fibration over the circle, with fiber a punctured torus. $\mathcal{L}_0$ and $\mathcal{L}_1$ are extended to two leaves $\mathcal{L}'_0$ and $\mathcal{L}'_1$ of $\mathscr{E}'$. Choose a non-separating circle c on the punctured torus, extend it to a vertical annulus $c \times I$ in the fibration part, so that $c \times I$ intersects both $\mathcal{L}'_0$ and $\mathcal{L}'_1$, but $c \times I$ does not intersect the leaf containing F. Then we perform Operation 3.9 along $c \times I$ to $\mathscr{E}'$, so as to get a new foliation $\mathscr{E}''$, where there is a leaf $\mathcal{L}''_0$ satisfying that the band $B \subset R_1$ connects $\mathcal{L}''_0$ to itself. See Figure 3 for an illustration of the local picture.

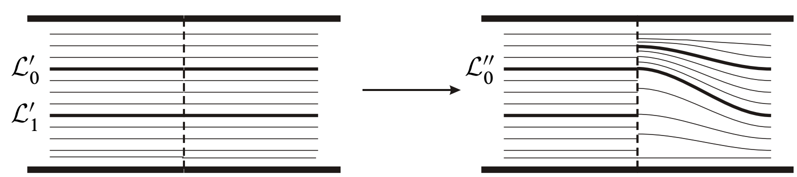

Figure 3: Perform Operation 3.9 to merge $\mathcal{L}'_0$ and $\mathcal{L}'_1$

In any case, we get a simple closed curve $c_1^\circ \subset \kappa(\mathcal{L}_0^\circ)$, c_1° intersects T_1° once. The holonomy along c_1° may be nontrivial. We can perform the connected sum of L° and a trefoil $Tr_2 \subset S^3$ to get a link L^*, and extend the foliation $\mathscr{E}^\circ$ to a foliation $\mathscr{E}^*$ of $Z - \mathrm{int}(\mathrm{Nd}(L^*))$ by a fibration over the circle. Choose two curves d, e on a fiber of the fibration, so that they intersect in one point. Perform connected sum of c_1° and d to get a new curve c_1^* in a leaf of $\kappa(\mathscr{E}^*)$. Do Operation 3.9 along $e \times I$ to cancel the holonomy along c_1°, then the holonomy along c_1^* is trivial.

We note that F^* is a leaf or disjoint union of leaves of the new foliation of $Z - \mathrm{int}(\mathrm{Nd}(L))$, hence G^* is a leaf of the newly induced foliation of $Y - \mathrm{int}(\mathrm{Nd}(K^*))$.

Now apply Lemma 3.10 to get our desired result for the case that L has two components. The general case can be proved by the same method. Indeed, by performing connected sum with fibred knots and creating holonomy, one can get a leaf which intersects every $\partial\mathrm{Nd}(K_i)$. Again, by the same procedure one can get a leaf of $\mathscr{E}$ which is connected to itself by $n-1$ bands in the $n-1$ different tubes $R_1, \dots, R_{n-1}$. Hence there are $n-1$ closed curves in the corresponding leaf of $\kappa(\mathscr{E})$, so that they are dual to $T_1, \dots, T_{n-1}$. Then perform connected sums with fibred knots and kill the holonomy along these curves. The last step is to apply Lemma 3.10. □

4 Proof of the main theorem for links

For simplicity, we only prove the case when L is a two-component link. The reader will find that the proof also works for links with more components.

Suppose $Y = \kappa(Z) = Z \# S^2 \times S^1$, $K = \kappa(L)$. Let $Y_p = Y_p(K)$ be the manifold obtained from Y by p–surgery on K.

The next lemma is an analogue of Lemma 3.6.

Lemma 4.1 *Let d be an integer satisfying $\widehat{HFK}(Y, K, i) = 0$ for $i \geq d$, and suppose that $d > 1$. Then*

$$HF^+(Y_0, [d-1]) = 0,$$

where

$$HF^+(Y_0, [d-1]) = \bigoplus_{\langle c_1(\mathfrak{s}), [\widehat{G}] \rangle = 2(d-1)} HF^+(Y_0, \mathfrak{s})$$

Proof As in the proof of Lemma 3.6, the chain map

$$\Psi\colon C\{i \geq 0 \text{ or } j \geq d-1\} \to C\{i \geq 0\}$$

induces an isomorphism ψ on the homologies. The map Ψ coincides with the map induced by the cobordism W from Y_p to Y, endowed with the Spin^c structure $\mathfrak{r}$ with

$$\langle c_1(\mathfrak{r}), [\widetilde{G}] \rangle = 2d - 2 - p.$$

Here $\widetilde{G}$ is a closed surface in W, obtained from G by capping off ∂G in W.

We also have the long exact sequence

$$\cdots \to HF^+(Y_0, [d-1]) \to HF^+(Y_p, [d-1]) \xrightarrow{f} HF^+(Y) \to \cdots$$

Apply the adjunction inequality [15, Theorem 7.1] to $T_1 \subset Y_p$, we find that if $\mathfrak{t} \in \mathrm{Spin}^c(Y_p)$ satisfies $HF^+(Y_p, \mathfrak{t}) \neq 0$, then $\mathfrak{t}$ is torsion. Hence if $\mathfrak{r}' \in \mathrm{Spin}^c(W)$

nontrivially contributes to f, $\mathfrak{r}'-\mathfrak{r}$ must be a multiple of $\mathrm{PD}([\widetilde{G}])$, where PD is the Poincaré duality map. It is not hard to find

$$c_1^2(\mathfrak{r}') = -\frac{(2d-2-p+2kp)^2}{p}, \quad k \in \mathbb{Z}.$$

Now by the degree shifting formula (2), f has the form $\psi+\iota$, where ι is a sum of homogeneous maps which have lower orders than ψ. Since ψ is an isomorphism, so is $\psi+\iota$. Hence $HF^+(Y_0,[d-1])=0$. □

Now we can proceed to the proof of our main theorem. The proof is not much different from the proof of [13, Theorem 1.2].

Proof of Theorem 1.1 Suppose L_1, L_2 are links in Z_1, Z_2, respectively. We have

$$\widehat{HFK}(Z_1,L_1)\otimes\widehat{HF}(Z_2)\cong\widehat{HFK}(Z_1\#Z_2,L_1),$$
$$\widehat{HFK}(Z_1\#Z_2,L_1\#L_2)\otimes\widehat{HF}(S^2\times S^1)\cong\widehat{HFK}(Z_1\#Z_2,L_1\sqcup L_2).$$

By the above formulas, we can assume $Z-L$ is irreducible. By doing connected sum with the trefoil, we can assume the genus of K is bigger than 1. Now apply Proposition 3.12 to get a symplectic 4–manifold (X,Ω), $X=X_1\cup_{Y_0^*}X_2$, with $b_2^+(X_j)>0$, $\int_{T_1^*}\Omega=0$, and

$$\langle c_1(\mathfrak{k}(\Omega)),[\widehat{G^*}]\rangle = 2-2g(\widehat{G^*}).$$

By the composition formula (3), the sum

$$\sum_{\eta\in \mathrm{H}^1(Y_0^*)} \Phi_{X,\mathfrak{k}(\Omega)+\delta\eta} \tag{4}$$

is calculated by a homomorphism which factors through $HF^+(Y_0^*,\mathfrak{k}(\Omega)|_{Y_0^*})$.

$\mathrm{H}^1(Y_0^*)\cong\mathbb{Z}\oplus\mathbb{Z}$ is generated by the Poincaré duals of $[T_1^*]$ and $[\widehat{G^*}]$. So the Spin^c structures in (4) are precisely

$$\mathfrak{k}(\Omega)+a\,\mathrm{PD}([T_1^*])+b\,\mathrm{PD}([\widehat{G^*}]) \quad (a,b\in\mathbb{Z}).$$

Here PD is the Poincaré duality map in X. The first Chern classes of these Spin^c structures are

$$c_1(\mathfrak{k}(\Omega))+2a\,\mathrm{PD}([T_1^*])+2b\,\mathrm{PD}([\widehat{G^*}]).$$

The evaluation of $c_1(\mathfrak{k}(\Omega))$ on $[\widehat{G^*}]$ is $2-2g\neq 0$. By the degree shifting formula (2), we conclude that the terms in (4) that have the same degree as $\Phi_{X,\mathfrak{k}(\Omega)}$ are precisely those corresponding to $\mathfrak{k}(\Omega)+a\,\mathrm{PD}([T_1^*])$. By [16, Theorem 1.1] and the fact that

$\int_{T_1^*}\Omega=0$, $\Phi_{X,\mathfrak{k}(\Omega)}$ is the only nontrivial term at this degree. So $HF^+(Y_0^*,\mathfrak{k}(\Omega)|_{Y_0^*})$ is nontrivial. Now apply Lemma 4.1, we get our desired result for L^*.

The result for L holds by the connected sum formula. □

5 Alternative links

In [8], Kauffman defined a class of links called "alternative links". The definition is as follows:

Definition 5.1 Suppose we have a link diagram $\mathcal{D}$ and that $\mathcal{D}$ is connected. We apply Seifert's algorithm (see [19]) to $\mathcal{D}$, and thus get a collection $\mathcal{S}$ of disjoint circles on S^2. The vertices of $\mathcal{D}$ naturally lie in $S^2-\mathcal{S}$. We say $\mathcal{D}$ is an *alternative diagram* if for each component of $S^2-\mathcal{S}$, the vertices in this component have the same sign. A link L is called an *alternative link* if it has an alternative diagram.

This definition simultaneously generalizes alternating links and positive links. Kauffman proved that for an alternative link, the surface F obtained from Seifert's algorithm has the maximal Euler characteristic among all the Seifert surfaces. Moreover, the rank of $\mathrm{H}_1(F)$ is equal to the degree of the Alexander–Conway polynomial. Kauffman proved his result by showing that all the Kauffman states at the top filtration level have the same grading. The readers are referred to [8, Chapter 9] for a proof. Here, we only state Kauffman's algorithm of getting all the Kauffman states at the top filtration level.

Algorithm 5.2 (Alternative Tree Algorithm, or ATA)

Step 1 Suppose $\mathcal{D}$ is a decorated alternative diagram, choose a checkerboard coloring of the regions. There are two regions R_B, R_W sharing the distinguished edge, with coloring black and white respectively. Now we have the black graph Γ_B, whose vertices are the regions with black color, and the edges correspond to vertices of $\mathcal{D}$. Similarly, there is the white graph Γ_W.

Step 2 For Γ_B, remove all the edges connecting two corners with 0 contribution to the filtration level, see Figure 1. Orient the rest edges so that they point to the corners with contribution $\frac{1}{2}$, hence we get an oriented graph Γ'_B. Γ'_B usually has many components. Similarly, we have Γ'_W.

Step 3 Find a maximal oriented tree $T_B^1\subset\Gamma'_B$, with root at R_B. More precisely, T_B^1 is a maximal tree so that all edges at R_B point away from R_B, and no two edges point to the same vertex. Similarly find a maximal oriented tree T_W^1 with root at R_W.

Step 4 The edges we have chosen correspond to some vertices of $\mathcal{D}$. Choose an edge corresponding to one of the rest vertices, so that it connects a vertex of T_B^1 to a vertex R_B^2 in $\Gamma'_B - T_B^1$. Orient this edge so that it points to R_B^2. Now we choose a maximal oriented tree in Γ'_B with root at R_B^2. The union of the two black trees and the edge connecting them is a larger tree, called T_B^2. Similarly, we get a white tree T_W^2.

Step 5 Repeat the process in the last step to enlarge T_B^2 to a tree T_B^3, T_W^2 to T_W^3. Go on with this process until we get a maximal oriented black tree $T_B \subset \Gamma_B$ and a maximal oriented white tree $T_W \subset \Gamma_W$. Suppose v is a vertex of $\mathcal{D}$, v corresponds to an oriented edge e in T_B or T_W. Associate to v the corner e points to, hence we get a Kauffman state x.

Kauffman proved that there always exist Kauffman states obtained by ATA, and the Kauffman states so obtained are precisely those states at the top filtration level. Moreover, they have the same grading. In fact, we can compute the grading and filtration level as follows: Suppose c_- is the number of components of $S^2 - \mathcal{S}$ in which all the vertices of $\mathcal{D}$ have negative sign, r_- denotes the number of regions in these components. Similarly define c_+, r_+. Then the grading of the Kauffman states obtained by ATA is

$$\mathrm{Gr}_{\max} = \frac{|L|-1}{2} + r_- - c_- \tag{5}$$

and their filtration level is

$$\mathrm{Fil}_{\max} = \frac{|L|-1+r-c}{2} = \frac{|L|+1+m-c}{2} \tag{6}$$

where $r = r_+ + r_-$, $m = r-2$ is the number of vertices in $\mathcal{D}$, $c = c_+ + c_-$.

To conclude, we interpret Kauffman's work in the world of knot Floer homology.

Theorem 5.3 *Use notation as above. Suppose $\mathcal{D}$ is the diagram of an alternative link L, then the maximal nontrivial filtration level of $\widehat{HFK}(L)$ is the $\mathrm{Fil}_{\max}$ as in (6). Moreover, $\widehat{HFK}(L, \mathrm{Fil}_{\max})$ is freely generated by the Kauffman states obtained from ATA, with grading $\mathrm{Gr}_{\max}$ as in (5).*

Proof We prove this theorem by induction on $|L|$. When L is a knot, the result holds by Kauffman's work and the results in Subsection 2.3.

When L is a link, choose two different components which have an intersection in $\mathcal{D}$. Without loss of generality, we can assume this intersection is positive. As in Figure 4, L is the L_0, and there are two alternative links L_+, L_- with fewer components. We have the skein exact sequence

$$\cdots \to \widehat{HFK}(L_-, i) \to \widehat{HFK}(L_0, i) \xrightarrow{f} \widehat{HFK}(L_+, i) \to \cdots,$$

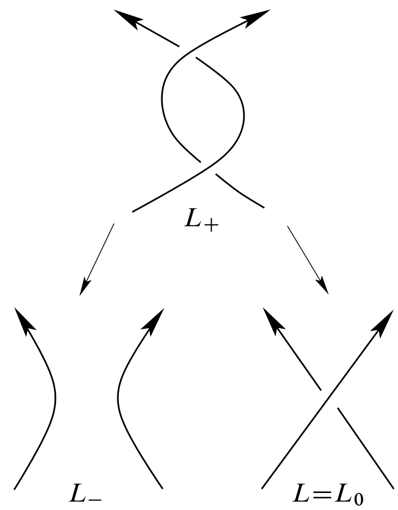

Figure 4: Crossing change

where f drops the grading by $\frac{1}{2}$.

By induction, $\mathrm{Fil}_{\max}$ is the maximal nontrivial filtration level of $\widehat{HFK}(L_+)$, and $\widehat{HFK}(L_+, \mathrm{Fil}_{\max})$ is supported at grading $\mathrm{Gr}_{\max} - \frac{1}{2}$. And $\mathrm{Fil}_{\max} - 1$ is the maximal filtration level of $\widehat{HFK}(L_0)$. Furthermore, it is easy to find a one-to-one correspondence between the Kauffman states of L_+ and Kauffman states of L_0 at the top filtration level. Now our desired result holds. □

Remark 5.4 In [1], Cromwell proved that an alternative link is fibred if and only if its Alexander–Conway polynomial is monic, hence if and only if $\widehat{HFK}(L, \mathrm{Fil}_{\max}) \cong \mathbb{Z}$. Cromwell actually proved his result for a (possibly but unlikely larger) class of links called "homogeneous links".

It is interesting to ask if the converse of Proposition 2.2 is true for classical links.

References

[1] **P R Cromwell**, *Homogeneous links*, J. London Math. Soc. (2) 39 (1989) 535–552 MR1002465

[2] **Y Eliashberg**, *A few remarks about symplectic filling*, Geom. Topol. 8 (2004) 277–293 MR2023279

[3] **Y M Eliashberg**, **W P Thurston**, *Confoliations*, University Lecture Series 13, American Mathematical Society, Providence, RI (1998) MR1483314

[4] **J B Etnyre**, *On symplectic fillings*, Algebr. Geom. Topol. 4 (2004) 73–80 MR2023278

[5] **D Gabai**, *private communication*

[6] **D Gabai**, *The Murasugi sum is a natural geometric operation*, from: "Low-dimensional topology (San Francisco, CA 1981)", Contemp. Math. 20, Amer. Math. Soc., Providence, RI (1983) 131–143 MR718138

[7] **D Gabai**, *Foliations and the topology of 3-manifolds. III*, J. Differential Geom. 26 (1987) 479–536 MR910018

[8] **L H Kauffman**, *Formal knot theory*, Mathematical Notes 30, Princeton University Press, Princeton, NJ (1983) MR712133 Reprinted with corrections by Dover Publications (2006)

[9] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology II* arXiv:math.QA/0505056

[10] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and integer surgeries* arXiv:math.GT/0410300

[11] **P Ozsváth**, **Z Szabó**, *Heegaard Floer homology and alternating knots*, Geom. Topol. 7 (2003) 225-254 MR1988285

[12] **P Ozsváth**, **Z Szabó**, *Heegaard diagrams and holomorphic disks*, from: ''Different faces of geometry'', Int. Math. Ser. (N. Y.), Kluwer/Plenum, New York (2004) 301-348 MR2102999

[13] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and genus bounds*, Geom. Topol. 8 (2004) 311-334 MR2023281

[14] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58-116 MR2065507

[15] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) 159 (2004) 1159-1245 MR2113020

[16] **P Ozsváth**, **Z Szabó**, *Holomorphic triangle invariants and the topology of symplectic four-manifolds*, Duke Math. J. 121 (2004) 1-34 MR2031164

[17] **P Ozsváth**, **Z Szabó**, *Heegaard Floer homology and contact structures*, Duke Math. J. 129 (2005) 39-61 MR2153455

[18] **J Rasmussen**, *Floer homology and knot complements*, PhD thesis, Harvard University (2003) arXiv:math.QA/0306378

[19] **D Rolfsen**, *Knots and links*, Mathematics Lecture Series 7, Publish or Perish Inc., Houston, TX (1990) MR1277811 Corrected reprint of the 1976 original

[20] **J R Stallings**, *Constructions of fibred knots and links*, from: ''Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2'', Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I. (1978) 55-60 MR520522

[21] **W P Thurston**, *A norm for the homology of 3-manifolds*, Mem. Amer. Math. Soc. 59 (1986) i-vi and 99-130 MR823443

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