Embedding infinite cyclic covers of knot spaces into 3-space

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Abstract

We say a knot $k$ in the 3-sphere $S^3$ has \textit{Property IE} if the infinite cyclic cover of the knot exterior embeds into $S^3$. Clearly all fibred knots have \textit{Property IE}.

There are infinitely many non-fibred knots with \textit{Property IE} and infinitely many non-fibred knots without \textit{Property IE}. Both kinds of examples are established here for the first time. Indeed we show that if a genus 1 non-fibred knot has \textit{Property IE}, then its Alexander polynomial $\Delta_k(t)$ must be either $1$ or $2t^2 - 5t + 2$, and we give two infinite families of non-fibred genus 1 knots with \textit{Property IE} and having $\Delta_k(t) = 1$ and $2t^2 - 5t + 2$ respectively.

Hence among genus 1 non-fibred knots, no alternating knot has \textit{Property IE}, and there is only one knot with \textit{Property IE} up to ten crossings.

We also give an obstruction to embedding infinite cyclic covers of a compact 3-manifold into any compact 3-manifold.

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1. Introduction

In this paper all surfaces and 3-manifolds are orientable, and all surfaces in 3-manifolds are proper, embedded and two-sided. Suppose $S$ (resp. $P$) is a surface (resp. 3-manifold) in a 3-manifold $M$, we use
Suppose $k$ is a non-fibred knot, when does we give a partial positive answer to Question 1. By invoking Freedman–Freedman’s, we give a partial negative answer to Questions 1

Section 2.2 Question 1

To be complements of knots in $\mathbb{S}^3$. When does an infinite cyclic cover of a compact 3-manifold embed into a compact 3-manifold?

Definition 1.1. We say a knot $k$ in $\mathbb{S}^3$ has Property $\text{IE}$, if the infinite cyclic cover $\tilde{E}(k)$ embeds into $\mathbb{S}^3$. We say a knot $k$ in $\mathbb{S}^3$ has Property $\text{DIE}$, if $(\tilde{E}(k), \tau) \subset (\mathbb{S}^3, f)$, that is, the deck transformation $\tau$ of $\tilde{E}(k)$ embeds into a dynamical system $f$ on $\mathbb{S}^3$. (We say a dynamical system $g$ on a space $P$ embeds into a dynamical system $f$ on a space $Y$, denoted by $(P, g) \subset (Y, f)$, if there is an embedding $P \subset Y$ such that $f|P = g$.)

The organization of this paper is as below. Sections 2 and 3 are the main parts of the paper. All knots involved in Sections 2 and 3 are of genus 1 and non-fibred. It is well known that the only genus 1 fibred knots are $3_1$ and $4_1$ in the knot table.

In Section 2, we give a partial positive answer to Questions 1 and 2. In Section 2.1, beginning with a discrete dynamical system $f$ on $\mathbb{S}^3$ (or a compact 3-manifold $Y$), we construct a compact 3-manifold $M$ (closed or with torus boundary) such that $(\tilde{M}_S, \tau) \subset (\mathbb{S}^3, f)$ or $\subset (Y, f)$, where $\tau$ is the deck transformation on the infinite cyclic cover $\tilde{M}_S$. In Section 2.2 we prove that the simplest non-trivial example provided by construction in Section 2.1 is $E(9_{46})$, the exterior of the 46-th knot of nine crossings in the knot table, see [11] or [3], therefore providing the first known positive example to Question 1. A subtle point in the verification is to choose a right projection of $9_{46}$, which significantly simplifies the process. But a key point is to choose $9_{46}$ among all knots in $\mathbb{S}^3$ to compare with. In Section 2.3, we give a sufficient condition for the 3-manifolds constructed in Section 2.1 to be complements of knots in $\mathbb{S}^3$, and then we prove that there are infinitely many non-fibred genus 1 knots having Property $\text{DIE}$ by invoking Thurston and Soma’s results on Gromov volume of 3-manifolds.

In Section 3, we give a partial negative answer to Question 1. By invoking Freedman–Freedman’s version of the Kneser–Haken finiteness theorem and results of Gabai (and Novikov) on foliation and on surgery, we prove that if a genus 1 non-fibred knot $k$ has Property $\text{IE}$, then $E(k)$ is constructed as in Section 2.1, and hence $k$ has Property $\text{DIE}$. It follows that the Alexander polynomial of such knots must be $1$ or $2t^2 - 5t + 2$, and the Alexander invariant is also restricted. So “most” genus 1 non-fibred knots do not have Property $\text{IE}$. In particular, among all non-fibred genus 1 knots, no alternating knots have Property $\text{IE}$, and up to crossing numbers $\leq 10$ only $9_{46}$ has Property $\text{IE}$. On the other hand, two infinite
families of genus 1 non-fibred knots with Property $IE$ constructed in Section 2.3 have $\Delta_k(t) = 1$ and $\Delta_k(t) = 2t^2 - 5t + 2$ respectively.

Section 4 is a remark about Property $IE$ on connected sums, which provides knots of any given genus $g$ (non-prime when $g > 1$); some of them have Property $IE$ and some do not.

Section 5 gives a homological obstruction to embedding infinite cyclic covers of a compact 3-manifold into any compact 3-manifold (Theorem 5.1), therefore giving a partial negative answer to Question 2.

Comments.
1. If we replace the term “unknotted solid torus” by “unknotted handlebody of genus $g$ for any $g > 1$”, constructions in Section 2.1 can be used to study Property $DIE$ of knots with higher genera, although the arguments become more complicated. The knots having Property $DIE$ provide interesting dynamics in $S^3$.

2. Theorem 5.1 as well as the constructions in Section 2.1 still holds for closed $n$-manifold and connected non-separating bicollared properly embedded codimension 1 submanifold $S$ in $M$.

3. For knot $k$ in $S^3$, the homological obstruction in Theorem 5.1 vanishes for $E(k)$ (read Remark 2). We wonder if Question 2 has a positive answer when we restrict to $E(k)$ for knots $k$ in $S^3$.


2. Infinitely many genus 1 non-fibred knots have Property $DIE$

2.1. A construction of compact 3-manifolds having infinite cyclic covers in $S^3$ or in a compact 3-manifold

Step 1. We first consider a rather general case. Let $Y$ be a closed 3-manifold, and $P \subset Y$ be a submanifold of dimension three with connected and non-empty $\partial P$. Suppose that there is a homeomorphism

$$f : Y \to Y$$

such that $f(P) \subset \text{int } P$.

Let $X = P \setminus f(P)$. Then $\partial X = \partial P \cup \partial f(P)$. Let $M = X/f$ be the closed 3-manifold obtained from $X$ by identifying $\partial P$ and $\partial f(P)$ via $f$, and $S \subset M$ be the image of $\partial P$ and $\partial f(P)$ after identification. Then $S$ is a connected non-separating surface in $M$. Clearly the infinite cyclic cover $\tilde{M}_S$ is identified with $\bigcup_{k=1}^{\infty} f^k(X) \subset Y$ and $f|\bigcup_{k=1}^{\infty} f^k(X)$ gives the deck transformation $\tau$. Hence $(\tilde{M}_S, \tau) \subset (Y, f)$.

We say the construction above is non-trivial, if $X$ is not homeomorphic to $\partial P \times [0, 1]$.

Step 2. Continue from Step 1. Let $Y = S^3$ and let $P$ be an unknotted solid torus $P$ in $S^3$, and let $P'$ be a solid torus in $P$, such that $P'$ is still unknotted in $S^3$. Since both $P$ and $P'$ are unknotted in $S^3$, there is a homeomorphism $f : S^3 \to S^3$ such that $f(P) = P'$. Then $X = P \setminus f(P)$ is an example of Step 1.

Step 3. Continue from Step 2. Let $\Gamma$ be a proper arc in $X$ with one end in $\partial P$ and the other in $\partial f(P)$. Let $N(\Gamma)$ be the regular neighborhood of $\Gamma$ in $X$. Up to isotopy we may assume $f(\partial P \cap N(\Gamma)) = \partial f(P) \cap N(\Gamma)$. Let $X^* = X \setminus N(\Gamma)$. Then $X^*$ is obtained from $X$ by digging a tunnel from $\partial P$ to $\partial f(P)$. Let $M^* = X^*/f$, $S^* = M^* \cap S$, where $M^*$ is obtained from $M$ by removing a solid torus. Clearly the infinite cyclic cover $\tilde{M}^*_S$ is identified with $\bigcup_{k=1}^{\infty} f^k(X^*) \subset S^3$, and $f|\bigcup_{k=1}^{\infty} f^k(X^*)$ gives the deck transformation $\tau$. We summarize the discussion above as

**Proposition 2.1.** $M^*$ is a compact 3-manifold with torus boundary, and $(\tilde{M}^*_S, \tau) \subset (S^3, f)$. In particular if $M^*$ is homeomorphic to $E(k)$ for a knot $k \subset S^3$, then $k$ has Property $DIE$. 
2.2. The knot 9_46 has Property DIE

A simplest non-trivial construction in Proposition 2.1 is indicated in Fig. 1, where \( P' \) is a 2-braid in \( P \) and the tunnel is “unknotted”. In this subsection, all notions in Step 3 of Section 2.1 refer to Fig. 1.

We will verify that \( M^* \) is homeomorphic to \( E = E(9_{46}) \), the exterior of knot 9_{46} \( \subset S^3 \) in the knot table. Our verification consists of three steps:

Step 1. Compute \( \pi_1(M^*) \) and \( \pi_1(\partial M^*) \subset \pi_1(M^*) \). Cutting \( M^* \) open along \( S^* \), we get back to \( X^* \), which is already presented in Fig. 1. Its boundary \( \partial X^* = S^* \cup \text{annulus} \cup S^*_+ \), where \( S^*_+ \) and \( S^*_+ \) are 1-punctured tori on the inner boundary \( \partial f(P) \) and the outer boundary \( \partial P \) respectively. The annulus is the boundary of the tunnel.

Choose meridian \( \mu_+ \) and longitude \( \lambda_+ \) on \( S^*_+ \) such that \( \mu_+ \) bounds a disc in \( P \) and \( \lambda_+ \) bounds a disc in \( S^3 \setminus P \). Similarly choose meridian \( \mu_- \) and longitude \( \lambda_- \) on \( S^*_+ \) such that \( \mu_- \) bounds a disc in \( f(P) \) and \( \lambda_- \) bounds a disc in \( S^3 \setminus f(P) \), where \( \mu_\pm \) and \( \lambda_\pm \) are as indicated in Fig. 2.

Since \( f \) is a homeomorphism on \( S^3 \) which sends the unknotted solid torus \( P \) to \( f(P) \), we must have \( f(\lambda_+) = \lambda_- \), and \( f(\mu_+) = \mu_- \). Now \( M^* = X^*/f \) as in Step 3 of Section 2.1.

Note that in Fig. 2, \( X^* \) is the complement of a graph \( \Theta \) (shown in gray in Fig. 2) in \( S^3 \), where \( \Theta \) consists of the centerline of \( f(P) \), the centerline of \( S^3 \setminus P \), joined by the centerline \( \gamma \) of the tunnel.
If we ignore the image of $X^*$ in Fig. 2, but with $\Theta$, $\lambda_{\pm}$ and $\mu_{\pm}$ remaining, then we have Fig. 3 below. Let $B^3$ be a 3-ball containing the arc $\gamma$ in $\Theta$, as indicated in Fig. 3. It is an observation that the complement of $\Theta$ is homeomorphic to the complement of two unknotted arcs in the 3-ball $S^3 \setminus B^3$. Hence $X^*$ is a handlebody of genus 2.

Two generators $a, b$ of $\pi_1(X^*)$ are indicated in Fig. 3, where we use the Wirtinger presentation [11], the base point in $X^*$ being above the page. Representing $\lambda_{\pm}$ and $\mu_{\pm}$ in terms of $a, b$, we have $\lambda_-=abab^{-1}$, $\mu_-=b$; $\lambda_+=a$, $\mu_+=baba^{-1}$. By HNN extension, we have

$$\pi_1(M^*) = \langle a, b, t| tat^{-1} = abab^{-1}, \ tbab\bar{a}^{-1}t^{-1} = b \rangle,$$

and $\pi_1(\partial M^*) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $t$ and $[\lambda_-, \mu_-] = [abab^{-1}, b]$.

Step 2. Compute $\pi_1(E)$ and $\pi_1(\partial E) \subset \pi_1(E)$. We choose the projection of $9_{46}$ provided in [11, p. 211] rather than in the knot table of [11], as Fig. 4. The Seifert surface $T$ of $9_{46}$ in Fig. 4 is the 1-punctured torus presented as a plumbing of two unknotted and untwisted bands $B(\alpha)$ and $B(\beta)$ with oriented centerlines $\alpha$ and $\beta$ respectively. $\pi_1(T)$ is generated by $\alpha$ and $\beta$.

Cutting $E$ open along $T$, we get a compact 3-manifold $Q$, which is the complement of $T$ in $S^3$, therefore $Q$ is also homeomorphic to the complement of the one point union of the two circles $\alpha \cup \beta$. By a handle sliding argument (see [11, p. 95]) one can check that $Q$ is also a handlebody of genus 2. Two generators $c, d$ of $\pi_1(Q)$ are indicated in Fig. 5.
First pushing $\alpha$ and $\beta$ off $T$ towards the minus side of $T$, we get two generators $\alpha_-, \beta_-$ of $\pi_1(T_-)$ in $\pi_1(Q)$; and then pushing $\alpha$ and $\beta$ off $T$ towards the plus side of $T$, we get two generators $\alpha_+, \beta_+$ of $\pi_1(T_+)$ in $\pi_1(Q)$, all shown in Fig. 5. It can be easily computed that $\alpha_- = cdc^{-1}, \beta_- = d, \alpha_+ = c, \beta_+ = dc^{-1}$. So 

$$\pi_1(E) = \langle c, d, s | scs^{-1} = cdc^{-1}, sdc^{-1}d^{-1}s^{-1} = d \rangle,$$

and $\pi_1(\partial E) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $s$ and $[\alpha_-, \beta_-] = [cdc^{-1}, d]$.

Step 3. Now we have an isomorphism

$$\phi : \pi_1(M^*) \to \pi_1(E),$$

such that $a \mapsto c, b \mapsto d, t \mapsto s$,

which maps $\pi_1(\partial M^*)$ isomorphically onto $\pi_1(\partial E)$.

Both $M^*, E$ are $\mathbb{P}^2$-irreducible, sufficiently large manifolds, so Waldhausen’s theorem [8, Theorem 13.6] (or [3, p. 308 B7] more directly) implies that $M^*$ is homeomorphic to $E$. We finished the verification.

2.3. Infinitely many genus 1 knots have Property DIE

Let $P, P', X^*$ and $X^*/f$ be as given in Step 3 of Section 2.1.

**Proposition 2.2.** (1) If a meridian disc $D$ of $P$ meets the core of $P'$ in exactly 2 points transversely, then $X^*/f$ is the complement of a genus 1 knot in a homotopy 3-sphere.

(2) Furthermore if $X^*$ is homeomorphic to a handlebody of genus 2, then $X^*/f = E(k)$ for some genus 1 knot $k \subset S^3$.

(3) There are infinitely many genus 1 knots $k \subset S^3$ such that $E(k)$ are obtained by the construction in Section 2.1.

**Proof.** Fig. 6 indicates that there are infinitely many embeddings $P' \subset P$, such that both the conditions in Proposition 2.2(1) and (2) are satisfied. The verification of $X^*$ to be the handlebody of genus 2 is the same as we did in Fig. 3 in Section 2.2. (Note that if we choose the tunnel joining $\partial P$ and $\partial P'$ to be knotted, then the condition in (2) is not satisfied in general.)

(1) We will find a presentation for $\pi_1(M^*)$ as in Step 1 of Section 2.2 (but the process is simpler since we need less precise information about the presentation). First from Fig. 6 we get Fig. 7 (as we did from
Fig. 1 to Fig. 3 in Step 1 of Section 2.2) where $a, b, b'$ are elements in $G = \pi_1(X^*)$. Then as in Step 1 of Section 2.2 we can compute $\pi_1(X^*/f)$ via HNN extension as

$$\pi_1(X^*/f) = \langle G, t | tat^{-1} = c, tbb't^{-1} = b \rangle,$$

where $c$ is the element in $G$ representing $\lambda_-$, and $t$ is represented by a loop $\gamma$ in $\partial(X^*/f) = T^2$. Note $\mu_+ = bb'$ because the meridian disc intersects $P'$ twice.

A Dehn filling along $\gamma$ will kill $t$ and provide a new manifold $M_1 = (X^*/f)(\gamma)$ with

$$\pi_1(M_1) = \langle G | a = c, bb' = b \rangle = \langle G | a = c, b' = 1 \rangle.$$

If we add a 2-handle to $X^*$ along the loop representing $b'$, the new manifold is obviously a solid torus. So $\langle G | b' = 1 \rangle \cong \mathbb{Z}$. Thus $\pi_1(M_1)$ is a quotient group of $\mathbb{Z}$. A computation in homology will show that $H_1(M_1; \mathbb{Z}) = 0$, hence $\pi_1(M_1) = 1$. Thus $X^*/f$ is the complement of a knot $k$ in the homotopy 3-sphere $M_1$.

(2) Furthermore suppose $X^*$ is homeomorphic to a handlebody of genus 2. Note $X^*/f = X^* \cup_f N(\partial P \setminus N(\Gamma))$, and $M_1 = (X^*/f)(\gamma)$ can be viewed as a quotient of $X^* \cup_f N(\partial P \setminus N(\Gamma))$ by identifying the annulus $\partial(X^*/f) \cap N(\partial P \setminus N(\Gamma))$ with the annulus $\partial(X^*/f) \cap X^*$. Hence $M_1$ has a Heegaard splitting $X^* \cup_h N(\partial P \setminus N(\Gamma))$ of genus 2, where $h$ is determined by $f$ and $\gamma$. By Theorem 1 of [1] $M_1$ is a 2-fold cyclic covering of $S^3$, branched over a 3-bridge link. It follows that $M_1$ is
homeomorphic to \( S^3 \) by Thurston’s orbifold theorem (see [2]), and hence \( X^*/f = E(k) \) for a knot \( k \) in \( S^3 \).

(3) We refine our notations related to Fig. 6. Denote \( P', X^* \) and \( f \) by \( P'_n, X^*_n \) and \( f_n \), if the crossing number of the core of \( P' \subset P \) in Fig. 6 is \( n, n \in \mathbb{Z} \). Then we have \( X^*_n/f_n^* = E(k_n) \) for some knot \( k_n \subset S^3 \) according to (2). If there are only finitely many different homeomorphism types for \( E(k_n) \), then there are only finitely many \( E(k_n, 0) \), the zero surgery manifold on \( k_n \). It follows that the Gromov volumes \( \{V(E(k_n, 0))\} \) take only finitely many values. Note that \( E(k_n, 0) \) is homeomorphic to \( (P \setminus P'_n)/f_n \), and \( E(k_n, 0) \setminus S_n = P \setminus P'_n \). Since \( \partial P \) is incompressible in \( P \setminus P'_n \), \( V(E(k_n, 0)) = V(P \setminus P'_n) \) by a theorem of Soma [12, Theorem 1], it follows that \( \{V(P \setminus P'_n)\} \) take only finitely many values.

Consider two 3-component links \( L_1 \) and \( L_2 \) with marked components \( \omega_1 \) and \( \omega_2 \) respectively, indicated in Fig. 8, (a) and (b). Note \( \omega_i \) is unknotted in \( S^3 \); the standard arguments (see [11, Chap. 9]) show that

\[
P \setminus P'_{2n+1} = E(L_1)(\omega_1, 1/n), \quad P \setminus P'_{2n+2} = E(L_2)(\omega_2, 1/n),
\]

where \( E(L_i)(\omega_i, 1/n) \) is the \( 1/n \)-Dehn filling along \( \omega_i \). It is also known that both \( L_1 \) and \( L_2 \) are hyperbolic links. (This fact can be checked by SnapPea [14].) According to Thurston’s theory about Gromov volume on 3-manifolds (see [13, Chapters 5 and 6]), we have

(i) \( V(E(L_i)(\omega_i, 1/n)) < V(E(L_i)) \),

(ii) \( \lim_{n \to \infty} V(E(L_i)(\omega_i, 1/n)) = V(E(L_i)) \).

It follows that \( \{V(P \setminus P'_n)\} \) take infinitely many values, a contradiction. \( \square \)

**Remark 1.** All knots constructed in Proposition 2.2 bound the genus 1 surface. All knots \( k_n \) in Proposition 2.2(3) are non-fibred, see the end of Section 3, and also \( k_1 = 9_{46} \).

### 3. “Most” genus 1 knots do not have Property IE

Let \( k \) be a non-fibred knot of genus 1 in \( S^3 \). Recall the notations \( E(k), S, X = E(k) \setminus S, \tilde{E}(k) \) defined at the beginning of the paper. Suppose also \( S \) is of genus 1.

Let \( S_n \) \((n = \ldots , -2, -1, 0, 1, 2, \ldots )\) denote the copies of \( S \) in \( \tilde{E}(k) \). For integers \( m < n \), let \( X_{[m,n]} \) denote the sub-manifold of \( \tilde{E}(k) \) between \( S_m \) and \( S_n \), and \( A_{[m,n]} \) denote the annulus bounded by \( \partial S_m \sqcup \partial S_n \) on \( \partial X_{[m,n]} \). Assume \( \tilde{E}(k) \) is already embedded in \( S^3 \), and \( Y_{[m,n]} = S^3 \setminus X_{[m,n]} \). We always use \( X_n \) to denote \( X_{[n,n+1]} \) for simplicity. The readers should be aware that the subscript \( n \) here has a different meaning from the \( n \) in the last section.

**Lemma 3.1.** For any integer \( N > 0 \), \( \partial S_0 \) bounds a disc \( D \) in \( Y_{[-N,N]} \).
Lemma 3.2

Consider the separating surfaces $S_n^* = A_{[N,n]} \cup S_n$ ($n = N + 1, N + 2, \ldots$) in $Y_{[-N,N]}$. They are mutually non-parallel, since $k$ is non-fibred. Since each $S_n^*$ has the first Betti number 2, Freedman–Freedman’s version of the Kneser–Haken finiteness theorem [5] implies that they must be compressible in $Y_{[-N,N]}$ when $n$ is sufficiently large. Suppose $D$ is a compressing disc of $S_n^*$. If $\partial D$ is parallel to $\partial S_n^*$ on $S_n^*$, then the lemma is proved, since $\partial S_n^*$ is parallel to $\partial Y_{[-N,N]}$. If $\partial D$ is not parallel to $\partial S_n^*$, surger $S_n^*$ along $\partial D$, we still get a disc $D'$ in $Y_{[-N,N]}$, with $\partial D' = \partial S_n^*$, since $S_n^*$ is a 1-punctured torus. 

Now fix an $N$ sufficiently large, we can thicken $D \cup A_{[-N,N]}$ in $Y_{[-N,N]}$ to get a 2-handle $D \times I$, which is attached to $X_{[-N,N]}$ along the annulus $A_{[-N,N]}$. Let $D_{-N}, \ldots, D_N$ be a collection of $D \times \{t\}$’s in the 2-handle, so that $\partial D_i = \partial S_i, i = -N, \ldots, N$. From now on, all subscripts in this section are bounded by $N$, as is understood.

Let $\hat{S}_i$ denote the torus $S_i \cup D_i$. Let $\hat{X}_i$ be the manifold bounded by $\hat{S}_i$ and $\hat{S}_{i+1}$ in $S^3$, and more generally, $\hat{X}_{[m,n]}$ be the manifold bounded by $\hat{S}_m$ and $\hat{S}_n$ in $S^3$.

**Lemma 3.2.** $\hat{X}_i$ is irreducible and $\partial$-irreducible. Moreover $\hat{X}_i$ is not a product.

**Proof.** Since $S$ is a minimal genus Seifert surface of $k$, $E(k)$ admits a taut foliation $\mathcal{F}$ such that $S$ is a leaf of $\mathcal{F}$ and $\mathcal{F} \cap E(k)$ is foliated by circles by [6, Theorem 3.1]. Then $\mathcal{F}$ can be extended to a taut foliation $\hat{\mathcal{F}}$ on $E(k, 0)$, the zero surgery manifold on $k$, such that $\hat{S}$ is a leaf of $\hat{\mathcal{F}}$, where $\hat{S}$ is obtained by capping disc on $S$. Moreover since $E(k)$ is not fibred, $E(k, 0)$ is not fibred by [6, Corollary 8.19], in particular $E(k, 0) \neq S^2 \times S^1$. By Novikov’s theorem [10], each leaf of the taut foliation $\hat{\mathcal{F}}$ is $\pi_1$-injective in $E(k, 0)$ and $\pi_2(E(k, 0)) = 0$. Then $E(k, 0)$ is irreducible by the sphere theorem [8, Chap. 3], and furthermore $\hat{S}$ is incompressible. It follows that $E(k, 0) \setminus \hat{S}$ is irreducible, $\partial$-irreducible, and is not a product.

Since each $\hat{X}_i$ is homeomorphic to $E(k, 0) \setminus \hat{S}$, Lemma 3.2 is proved. 

Each $\hat{S}_i$ separates $S^3$ into 2 components. We say the component containing $\hat{X}_i$ lies on the plus side of $\hat{S}_i$, the component containing $\hat{X}_{i-1}$ lies on the minus side of $\hat{S}_i$. $\hat{S}_i$ bounds a solid torus on the plus side or the minus side, since every torus in $S^3$ bounds a solid torus. In fact, we can prove the stronger

**Proposition 3.3.** Each $\hat{S}_i$ bounds solid tori on both sides.

**Proof.** Without loss of generality, we can assume $\hat{S}_0$ bounds a solid torus $P_0$ on the minus side. Our argument proceeds in the following steps.

**Step 1.** For each $n < 0$, $\hat{S}_n$ bounds a solid torus $P_n$ on the minus side.

Otherwise, assume some $\hat{S}_n$ does not bound a solid torus on the minus side, then $\hat{S}_n$ bounds a solid torus $P_+$ on the positive side. Hence $\hat{S}_n$ cuts $P_0$ into 2 parts: $\hat{X}_{[n,0]}$ and $P_0 \setminus \hat{X}_{[n,0]} = S^3 \setminus P_+$. By Lemma 3.2, $\hat{S}_n$ is incompressible in $\hat{X}_{[n,0]}$, $\hat{S}_n$ is also incompressible in $S^3 \setminus P_+$ since $P_+$ is knotted. So $P_0 = (S^3 \setminus P_+) \cup \hat{S}_n \hat{X}_{[n,0]}$ cannot have $\pi_1 = \mathbb{Z}$.

By Step 1, we have a nested sequence of solid tori

$$\cdots \subset P_{n-1} \subset P_n \subset P_{n+1} \subset \cdots \subset P_0.$$  

We assume that these tori adapt the orientation of $S^3$. Let $\mu_n, \lambda_n \subset \hat{S}_n$ be an oriented meridian–longitude system of $P_n, n < 0$, so that

1. the algebraic intersection number of $\mu_n$ and $\lambda_n$ is 1,
2. the linking number of $\lambda_n$ and $\mu_{n+1}$, which is defined as the winding number of $P_n$ in $P_{n+1}$, is $\geq 0$. 


699
Lemma 3.2

Suppose \( k \) is a non-fibred knot of genus \( g \). So \( H_1(P_n) \) has a basis \( \lambda, \mu \), and \( H_1(P_n) \) is isomorphic to \( \mathbb{Z}^2 \).

Note that the deck translation \( \tau : \tilde{E}(k) \to \tilde{E}(k) \), which sends \( X_i \) to \( X_{i+1} \), induces a homeomorphism \( \hat{\tau} : \tilde{X}_{n-1} = P_n \setminus P_{n-1} \to \tilde{X}_n = P_{n+1} \setminus P_n \) for each \( n \). It follows that \( w_n = w_{n-1} \).

Step 3. We claim that \( \hat{\tau} \) sends \( \mu_{n-1} \) to \( \mu_n \) for \( n \leq -1 \). There are two cases:

Case 1. \( w = 0 \). Now \( P_n \) cannot be a braid in \( P_{n+1} \), otherwise \( w = 1 \) and \( \tilde{X}_n \) is a product \( T^2 \times I \), contrary to Lemma 3.2. Then the results in [7] imply that only trivial surgery on \( P_n \) yields a solid torus.

Since the Dehn surgery on the knot \( P_n \) in the solid torus \( P_{n+1} \) along \( \hat{\tau}(\mu_{n-1}) \) again yields a solid torus, \( \hat{\tau}(\mu_{n-1}) = \mu_n \) for \( n \leq -1 \).

Case 2. \( w \geq 2 \). Fix \( n < 0 \). Now \( \lambda_i, \mu_i \) is a basis of \( H_1(\tilde{S}_i; \mathbb{Z}), \) \( i \leq 0 \).

\[
\hat{\tau}_*(\lambda) = p\lambda + q\mu, \quad \hat{\tau}_*(\mu) = r\lambda + s\mu + 1, \quad ps - qr = 1.
\]

For each integer \( m > 0 \), since \( \mu_n \) is a \( w^m \) multiple in \( H_1(\tilde{X}_{[m-n, n]}; \mathbb{Z}) \), \( \hat{\tau}_*(\mu_n) \) is also a \( w^m \) multiple in \( H_1(\tilde{X}_{[n-m+1, n+1]}; \mathbb{Z}) \). Since \( \mu_{n+1} \) is already a \( w^m \) multiple in \( H_1(\tilde{X}_{[n-m+1, n+1]}; \mathbb{Z}) \), \( r\lambda_{n+1} \) is also a \( w^m \) multiple.

Since \( \{\lambda_{n+1}, \mu_{n-m+1}\} \) is a basis of \( H_1(\tilde{X}_{[n-m+1, n+1]}; \mathbb{Z}) \) for \( m > 0 \), \( r \) should be a \( w^m \) multiple. Since \( r \) is a given integer, letting \( m \) be sufficiently large, we must have \( r = 0 \). Then \( p = s = \pm 1 \), i.e., \( \hat{\tau}_*(\mu_n) = \pm \mu_{n+1} \), and the conclusion holds.

Step 4. When \( n > 0 \), \( \hat{S}_n \) bounds a solid torus on the minus side.

There is a properly embedded planar surface \( G \) in \( \tilde{X}_{-2}, G \cap \hat{S}_{-1} = \mu_{-1}, G \cap \hat{S}_{-2} \) consists of parallel copies of \( \mu_{-2} \). By Step 3, \( \hat{\tau}(G) \) is a planar surface in \( \tilde{X}_{-1}, \hat{\tau}(G) \cap \hat{S}_{-1} \) consists of parallel copies of \( \mu_{-1} \). \( \hat{\tau}(G) \cap \hat{S}_0 \) bounds a disc on the minus side of \( \hat{S}_0 \), since each copy of \( \mu_{-1} \) bounds a disc in \( P_{-1} \). So \( \hat{\tau}(\mu_{-1}) = \hat{\tau}(G) \cap \hat{S}_0 = \mu_0 \). Let \( \mu_n = \tilde{\tau}^n(\mu_0) \) for \( n > 0 \). The same argument as above shows that \( \mu_n \) bounds a disc on the minus side of \( \hat{S}_n \), by induction.

Step 5. All \( \hat{S}_n \) bounds solid tori on both sides, \( n \in \mathbb{N} \).

By Lemma 3.2, \( \hat{S}_n \) and \( \hat{S}_m \) are not parallel for \( m \neq n \). By Haken’s finiteness theorem, \( \hat{S}_n \) is compressible in \( \mathbb{S}^3 \setminus P_0 \) when \( n \) is sufficiently large. The compressing disc cannot lie on the minus side, since \( \tilde{X}_{[0, n]} \) is \( \partial \)-irreducible by Lemma 3.2. So \( \hat{S}_n \) bounds a solid torus on the plus side when \( n \) is sufficiently large. Now proceed from Step 1 to Step 4, but reverse the direction, to get our conclusion. □

Theorem 3.4. Suppose \( k \) is a non-fibred knot of genus 1 in \( \mathbb{S}^3 \). If \( k \) has Property I E, then \( k \) has Property DI E. Indeed, \( E(k) \) can be obtained by the construction in Section 2.1.

Moreover, the winding number \( w \) involved is either 0 or 2. Correspondingly, the Alexander invariant of \( k \) is either 0 or \( \mathbb{Z}[t, t^{-1}]/(2t - 1) \oplus \mathbb{Z}[t, t^{-1}]/(t - 2) \), and the Alexander polynomial of \( k \) is either 1 or \( 2t^2 - 5t + 2 \).

Proof. Suppose \( \tilde{E}(k) \) is embedded into \( \mathbb{S}^3 \). We keep the notation in the proof of Proposition 3.3. First, extend \( \tau : (X_0, S_0) \to (X_1, S_1) \) to a homeomorphism \( \hat{\tau} : (\tilde{X}_0, S_0) \to (\tilde{X}_1, S_1) \) as in the proof of Proposition 3.3.
According to Proposition 3.3, each $\widehat{S}_n$ bounds a solid torus $P_n^-$ on the minus side, and a solid torus $P_n^+$ on the plus side. Suppose $\mu_n^-, \mu_n^+ \subset S_n \subset \widehat{S}_n$ are meridians of $P_n^-, P_n^+$ respectively. By Step 3 (and its counterpart in Step 5) of Proposition 3.3,
\[ \hat{\tau}(\mu_n^-) = \mu_{n+1}^-, \quad \hat{\tau}(\mu_n^+) = \mu_{n+1}^+. \] (3.1)

Hence we can further extend $\hat{\tau}$ to $\hat{\tau}_2 : P_0^+ \to P_1^+$, and finally we extend $\hat{\tau}_2$ to $f : S^3 \to S^3$ since both $P_0^+$ and $P_1^+$ are unknotted. Now we can reconstruct $E(k)$ from $f$ as in Section 2.1, so $k$ has Property $DIE$. We have finished the proof of the first part of Theorem 3.4.

By Step 2 (and its counterpart in Step 5) of Proposition 3.3, the winding number of $P_n^-$ in $P_n^-$ is a constant $w^-$, and the winding number of $P_n^+$ in $P_n^+$ is a constant $w^+$. It is easy to see that both $w^-$ and $w^+$ are the linking number between $\mu_n^-$ and $\mu_n^+$ (see the paragraph after Step 1 in the proof of Proposition 3.3), and we have $w^- = w^+ = w$. Since $\hat{\tau} : \widehat{S}_n \to \widehat{S}_{n+1}$ is orientation preserving, by (3.1) we have
\[ \hat{\tau}_n^{-1}(\mu_n^+) = \pm w[\mu_n^+], \quad \hat{\tau}_n(\mu_n^-) = \pm w[\mu_n^-]. \] (3.2)

Note that $X_n \leftrightarrow \widehat{X}_n$ induces an isomorphism on 1-dimensional homology. Then by (3.2) the Alexander invariant of $k$ has presentation [11, Chap. 7]
\[ H_1(\widehat{E}(k); \mathbb{Z}[t, t^{-1}]) = (\mu_n^+, \mu_n^-, t | t^{-1}(\mu_n^+) = \pm w[\mu_n^+], t(\mu_n^-) = \pm w[\mu_n^-]), \]
and the Alexander matrix of $k$ is
\[ \begin{pmatrix} wt \mp 1 & 0 \\ 0 & t \mp w \end{pmatrix}. \]

Since $\Delta_k(1) = \pm 1$, $w$ can only be 0 or 2, and the corresponding Alexander polynomials are 1 or $2t^2 - 5t + 2$ respectively, and the Alexander invariant of $k$ are either 0 or $\mathbb{Z}[t, t^{-1}]/(2t - 1) \oplus \mathbb{Z}[t, t^{-1}]/(t - 2)$. We have finished the proof of Theorem 3.4.

Corollary 3.5. Among all genus 1 non-fibred knots in $S^3$,
(1) up to ten crossings, $9_{46}$ is the only one that has Property $IE$,
(2) no alternating knot has Property $IE$.

Proof. (1) For knots with $\leq 10$ crossings, no non-fibred knot has Alexander polynomial 1, and only $6_1$ and $9_{46}$ have Alexander polynomial $2t^2 - 5t + 2$, see the tables in [3] and in [11]. But their Alexander invariants are not isomorphic (see [11, p. 211]), so $6_1$ does not have Property $IE$. Then by Section 2.2 (1) follows.

(2) If a genus 1 non-fibred knot $k$ has Property $IE$, then $\Delta_k(-1) = 1$ or 9. Now suppose $k$ is alternating, by a theorem of R.H. Crowell (see [3, Proposition 13.30]) $\Delta_k(-1)$ is not smaller than the crossing number of $k$, and $9_{46}$ is not alternating. Hence (2) follows from (1).

Recall the two infinite families of knots $k_{2n}$ and $k_{2n+1}$ with Property $IE$, as well as the notion $P'_n$, defined in the proof of Proposition 2.2(3). Since the winding number of $P'_n$ is 0 and the winding number of $P'_{2n+1}$ is 2, according to the calculation in the proof of Theorem 3.4 we have $\Delta_{k_{2n}}(t) = 1$ and $\Delta_{k_{2n+1}}(t) = 2t^2 - 5t + 2$.

Corollary 3.6. Among non-fibred genus 1 knots, both the subsets defined by $\Delta_k(t) = 1$ and by $\Delta_k(t) = 2t^2 - 5t + 2$ have infinitely many elements with Property $IE$. □
4. A remark on connected sums

Lemma 4.1. Suppose $k_1$ and $k_2$ are two knots in $S^3$.

1. If $k_1 \# k_2$ has Property I E, then both $k_1$ and $k_2$ have Property I E.

2. If $k_1$ has Property I E and $k_2$ is fibred, then $k_1 \# k_2$ has Property I E.

Note that there are fibred knots of any genus (just consider the connected sum of genus 1 fibred knots), and that $k_1 \# k_2$ is fibred if and only if both $k_1$ and $k_2$ are fibred (it follows from the definitions of connected sum, fibred knot, and Stallings’ fibration Theorem [8, Theorem 11.1]). Then by the main results in Sections 2 and 3 and Lemma 4.1 we have the following

Corollary 4.2. Among non-fibred knots of genus $g$ for any given integer $g > 0$, both the subsets defined by having Property I E and not having Property I E have infinitely many elements. □

Proof of Lemma 4.1. Denote $E(k_i)$ by $E_i$. Let $N_i = N(\mu_i)$ be the regular neighborhood of the meridian $\mu_i \subset \partial E_i$ in $E_i$. Let $E_i^* = E_i \setminus N_i$, and $A_i = E_i^* \cap N_i$. Then $E_i^*$ is homeomorphic to $E_i$ and $A_i$ is an annulus. By definition of the connected sum, we have $E(k_1 \# k_2) = E_1^* \cup_h E_2^*$, where $h$ is a homeomorphism identifying $A_1$ and $A_2$.

Let $p_i : \tilde{E}_i \rightarrow E_i$ be the infinite cyclic covering, and let $\tilde{E}_i^*, \tilde{N}_i, \tilde{A}_i$ be the preimage of $E_i^*, N_i, A_i$ under $p_i$. Clearly the restriction of

$$p_i : (\tilde{E}_i, \tilde{E}_i^*, \tilde{N}_i, \tilde{A}_i) \rightarrow (E_i, E_i^*, N_i, A_i)$$

is the infinite cyclic covering on each of the four corresponding pairs. Moreover $\tilde{E}_i^*, \tilde{N}_i, \tilde{A}_i$ are homeomorphic to $\tilde{E}_i, R^1 \times D^2, R^1 \times I$ respectively and $E(k_1 \# k_2) = \tilde{E}_1^* \cup_h \tilde{E}_2^*$, where $\tilde{h}$ is a homeomorphism identifying $\tilde{A}_1$ with $\tilde{A}_2$. Hence (1) follows.

We are going to prove (2). Now $\tilde{E}_2^* = R^1 \times S$ for a once punctured surface $S$ with $\tilde{A}_2 = R^1 \times I$ properly embedded in $R^1 \times \partial S$.

Since there is an embedding $e : \tilde{E}_2^* = R^1 \times S \rightarrow \tilde{N}_1$ such that $e$ sends $\tilde{A}_2$ to $\tilde{A}_1 \subset \partial \tilde{N}_1$ homeomorphically, and $e(\tilde{E}_2^*) \cap \tilde{N}_1 = \tilde{A}_1$ (see Fig. 9), $E(k_1 \# k_2) = \tilde{E}_1^* \cup_h \tilde{E}_2^*$ can be embedded into $E_1^* \cup \tilde{N}_1 = \tilde{E}_1$. Hence (2) follows. □
5. A partial negative answer to Question 2

In this section we use the notation in the first two paragraphs of Section 1. We will use $H_i(\cdot)$ to denote $H_i(\cdot; \mathbb{Q})$. Recall the following standard fact: let

\[ \cdots \to A \to B \to C \to \cdots \]

be an exact sequence of vector spaces. Then

\[ \dim A + \dim C \geq \dim B. \tag{\text{*}} \]

**Theorem 5.1.** Suppose $M$ is a compact 3-manifold, $S$ is a connected non-separating 2-sided proper surface in $M$. Let $X = M \setminus S$.

1. In the case $\partial M \neq \emptyset$, if $[S \cap T] \neq 0 \in H_1(\partial M; \mathbb{Z})$ for each boundary component $T$ of $M$ and $\beta_1(X) > \beta_1(S) - \chi(\partial M)$, then $\widetilde{M}_S$ cannot be embedded into any compact 3-manifold.

2. In the case $\partial M = \emptyset$, if $\beta_1(X) > \beta_1(S)$, then $\widetilde{M}_S$ cannot be embedded into any compact 3-manifold.

**Proof.** Suppose $\partial M \neq \emptyset$, $\tilde{M} = \bigcup_{k=-\infty}^{+\infty} X_k$ can be embedded into a compact 3-manifold $Y$. We may assume $\partial Y = \emptyset$. Denote $\bigcup_{k=1}^{m} X_k$ by $P_m$.

We first need to estimate $\beta_1(P_m)$. From $P_m = P_{m-1} \cup X_m$ and $S_m = P_{m-1} \cap X_m$, we have the Mayer–Vietoris sequence:

\[ \cdots \to H_1(S_m) \to H_1(P_{m-1}) \oplus H_1(X_m) \to H_1(P_m) \to \cdots. \]

By (\text{*}), we have the inequality:

\[ \beta_1(P_m) \geq \beta_1(P_{m-1}) + \beta_1(X) - \beta_1(S). \]

Hence we can easily deduce:

\[ \beta_1(P_m) \geq m\beta_1(X) - (m-1)\beta_1(S). \tag{1} \]

We need then to estimate $\beta_1(\partial P_m)$.

Cutting $\partial M$ open along $\partial S$, we get a surface $T'$. $\partial P_m$ is the union of $S_1^{-} \sqcup S_m^{+}$ and $m$ copies of $T'$. Note that the cutting and gluing of surfaces are all along circles, which have Euler characteristic 0. So

\[
\chi(\partial P_m) = \chi(S_1^{-} \sqcup S_m^{+}) + m\chi(T') \\
= 2\chi(S) + m\chi(\partial M) \\
= 2(1 - \beta_1(S)) + m\chi(\partial M).
\]

Then one can verify that

\[ \beta_1(\partial P_m) = 2\beta_0(\partial P_m) - \chi(\partial P_m) = 2\beta_0(\partial P_m) + 2(\beta_1(S) - 1) - m\chi(\partial M). \tag{2} \]

**Lemma 5.2.** $\beta_0(\partial P_m) \leq 2\beta_0(S \cap \partial M)$ for any $m$.

**Proof.** The bottom and the top of $P_m$ are $S_1^{-} \sqcup S_m^{+}$, which consists of $2\beta_0(S \cap \partial M)$ boundary components. If for some $m$, $\beta_0(\partial P_m) > 2\beta_0(S \cap \partial M)$, then some component $F$ of $\partial P_m$ does not meet the top and
the bottom of $P_m$. It follows that $F \subset P_m \subset \tilde{M}_S$ provides a component of $\partial \tilde{M}_S$, therefore $p(F)$ is a component of $\partial M$, where $p : \tilde{M}_S \to M$ is the infinite cyclic covering map. Since the deck transformation group of the covering $p : \tilde{M}_S \to M$ is the infinite cyclic group which contains no non-trivial finite subgroup, it follows that $p : F \to p(F)$ is a homeomorphism. Now $S \cap p(F) = \bigcup_{i=2}^{m} p(S_i \cap F)$.

Since $S_i$ separates $P_m$, $S_i$ separates $F$. Since $F$ is closed, $S_i \cap F$ is homologically trivial in $F$. Hence $p(S_i \cap F)$ is homologically trivial in $p(F)$, and then $[S \cap p(F)] = 0$, contradicting the assumption in Theorem 5.1. □

By using (*) to various homology sequences, we have

$$\beta_1(Y) \geq \beta_1(Y, Y \setminus P_m) - \beta_0(Y \setminus P_m) \quad \text{by (*)}$$

$$= \beta_1(P_m, \partial P_m) - \beta_0(Y \setminus P_m) \quad \text{by excision}$$

$$\geq \beta_1(P_m, \partial P_m) - \beta_0(\partial P_m) \quad \text{since } \beta_0(Y \setminus P_m) \leq \beta_0(\partial P_m)$$

$$\geq \beta_1(P_m) - \beta_1(\partial P_m) - \beta_0(\partial P_m) \quad \text{by (*)}$$

$$\geq m(\beta_1(X) - \beta_1(S) + \chi(\partial M)) + C \quad \text{by (1), (2) and Lemma 5.2}$$

where $C = 2 - \beta_1(S) - 6\beta_0(S \cap \partial M)$ is independent of $m$.

It follows that if $\beta_1(X) > \beta_1(S) - \chi(\partial M)$, $\beta_1(Y)$ would be arbitrarily large when $m$ gets large. We reach a contradiction, since $\beta_1(Y)$ should be finite for a compact manifold $Y$. Theorem 5.1(1) is proved.

A similar and more direct argument proves Theorem 5.1(2). □

**Remark 2.** Consider the connected sum $M = P#E(k)$, where $P$ is a homology 3-sphere with $\pi_1(P) \neq 1$ and $k$ is a knot in $\mathbb{S}^3$. Let $S \subset M$ be a Seifert surface of $E(k)$, and $X = M \setminus S$. Then $\beta_1(X) \leq \beta_1(S)$ and $\chi(M) = 0$. So the inequality in Theorem 5.1(1) is not met. There is an essential 2-sphere $S^2$ in the connected sum, and $p^{-1}(S^2)$ is an infinite family of essential 2-spheres in $\tilde{M}_S$, where $p : \tilde{M}_S \to M$ is the infinite cyclic covering. Then $\tilde{M}_S$ cannot stay in a compact 3-manifold.

Otherwise suppose $\tilde{M}_S \subset Y$ for a compact 3-manifold $Y$. Let $\bigcup_{i=1}^{n} S_i^2$ be $n$ components in $p^{-1}(S^2)$ for any given $n$. Then clearly each component of $Y \setminus \bigcup_{i=1}^{n} S_i^2$ contains a copy of the 1-punctured homology 3-sphere $P^*$ with $\pi_1(P^*) \neq 1$. Since $P^*$ is not a subset of a punctured 3-sphere, no component of $Y \setminus \bigcup_{i=1}^{n} S_i^2$ is a punctured 3-sphere, which contradicts the Kneser finiteness theorem [8, Lemma 3.14].

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