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Topology 45 (2006) 691–705
TOPOLOGY
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# Embedding infinite cyclic covers of knot spaces into 3-space

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Received 5 December 2004

Dedicated to the memory of Professor Shiing-shen Chern

**Abstract**

We say a knot $k$ in the 3-sphere $\mathbb{S}^3$ has *Property IE* if the infinite cyclic cover of the knot exterior embeds into $\mathbb{S}^3$. Clearly all fibred knots have Property $IE$.

There are infinitely many non-fibred knots with Property $IE$ and infinitely many non-fibred knots without property $IE$. Both kinds of examples are established here for the first time. Indeed we show that if a genus 1 non-fibred knot has Property $IE$, then its Alexander polynomial $\Delta_k(t)$ must be either 1 or $2t^2 - 5t + 2$, and we give two infinite families of non-fibred genus 1 knots with Property $IE$ and having $\Delta_k(t) = 1$ and $2t^2 - 5t + 2$ respectively.

Hence among genus 1 non-fibred knots, no alternating knot has Property $IE$, and there is only one knot with Property $IE$ up to ten crossings.

We also give an obstruction to embedding infinite cyclic covers of a compact 3-manifold into any compact 3-manifold.


*MSC:* 57M10; 57M25; 57N30

*Keywords:* Embedding; Non-fibre knots; Infinite cyclic coverings; Alexander polynomial

## 1. Introduction

In this paper all surfaces and 3-manifolds are orientable, and all surfaces in 3-manifolds are proper, embedded and two-sided. Suppose $S$ (resp. $P$) is a surface (resp. 3-manifold) in a 3-manifold $M$, we use

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0040-9383/$ - see front matter 
doi:10.1016/j.top.2006.01.005

$M \setminus S$ (resp. $M \setminus P$) to denote the manifold obtained by cutting $M$ along $S$ (resp. removing int $P$, the interior of $P$, from $M$).

Suppose $S$ is a connected non-separating surface in $M$. Then $X = M \setminus S$ has two copies of $S$ in $\partial X$, denoted by $S^+ \sqcup S^-$. Taking countably many copies of $X$: $\{X_i\}_{i=-\infty}^{+\infty}$, and identifying $S_{i-1}^+$ with $S_i^-$ for all $i$, we get an infinite cyclic cover of $M$, denoted by $\widetilde{M}_S$.

Let $k$ be a knot in $\mathbb{S}^3$, $E(k)$ be the exterior of $k$, $S$ be a Seifert surface of $k$. Then $E(k)$ has a unique infinite cyclic cover, simply denoted by $\widetilde{E}(k)$. If $k$ is a fibred knot with fibre $S$, then $\widetilde{E}(k)$ is homeomorphic to $S \times \mathbb{R}$ which clearly embeds into $\mathbb{S}^3$. This paper will address the following

**Question 1.** *Suppose $k$ is a non-fibred knot, when does $\widetilde{E}(k)$ embed into $\mathbb{S}^3$?*

The third named author was introduced to Question 1 during conversations with Professor Robert D. Edwards in the spring of 1984, and Edwards attributed Question 1 to Professor J. Stallings.

It is natural to ask the following more general and flexible

**Question 2.** *When does an infinite cyclic cover of a compact 3-manifold embed into a compact 3-manifold?*

**Definition 1.1.** We say a knot $k$ in $\mathbb{S}^3$ has Property $IE$, if the infinite cyclic cover $\widetilde{E}(k)$ embeds into $\mathbb{S}^3$. We say a knot $k$ in $\mathbb{S}^3$ has Property $DIE$, if $(\widetilde{E}(k), \tau) \subset (\mathbb{S}^3, f)$, that is, the deck transformation $\tau$ of $\widetilde{E}(k)$ embeds into a dynamical system $f$ on $\mathbb{S}^3$. (We say a dynamical system $g$ on a space $P$ embeds into a dynamical system $f$ on a space $Y$, denoted by $(P, g) \subset (Y, f)$, if there is an embedding $P \subset Y$ such that $f|P = g$.)

The organization of this paper is as below.

Sections 2 and 3 are the main parts of the paper. All knots involved in Sections 2 and 3 are of genus 1 and non-fibred. It is well known that the only genus 1 fibred knots are $3_1$ and $4_1$ in the knot table.

In Section 2, we give a partial positive answer to Questions 1 and 2. In Section 2.1, beginning with a discrete dynamical system $f$ on $\mathbb{S}^3$ (or a compact 3-manifold $Y$), we construct a compact 3-manifold $M$ (closed or with torus boundary) such that $(\widetilde{M}_S, \tau) \subset (\mathbb{S}^3, f)$ or $\subset (Y, f)$, where $\tau$ is the deck transformation on the infinite cyclic cover $\widetilde{M}_S$. In Section 2.2 we prove that the simplest non-trivial example provided by construction in Section 2.1 is $E(9_{46})$, the exterior of the 46-th knot of nine crossings in the knot table, see [11] or [3], therefore providing the first known positive example to Question 1. A subtle point in the verification is to choose a right projection of $9_{46}$, which significantly simplifies the process. But a key point is to choose $9_{46}$ among all knots in $\mathbb{S}^3$ to compare with. In Section 2.3, we give a sufficient condition for the 3-manifolds constructed in Section 2.1 to be complements of knots in $\mathbb{S}^3$, and then we prove that there are infinitely many non-fibred genus 1 knots having Property $DIE$ by invoking Thurston and Soma's results on Gromov volume of 3-manifolds.

In Section 3, we give a partial negative answer to Question 1. By invoking Freedman–Freedman's version of the Kneser–Haken finiteness theorem and results of Gabai (and Novikov) on foliation and on surgery, we prove that if a genus 1 non-fibred knot $k$ has Property $IE$, then $E(k)$ is constructed as in Section 2.1, and hence $k$ has Property $DIE$. It follows that the Alexander polynomial of such knots must be 1 or $2t^2 - 5t + 2$, and the Alexander invariant is also restricted. So "most" genus 1 non-fibred knots do not have Property $IE$. In particular, among all non-fibred genus 1 knots, no alternating knots have Property $IE$, and up to crossing numbers $\leq 10$ only $9_{46}$ has Property $IE$. On the other hand, two infinite families of genus 1 non-fibred knots with Property $IE$ constructed in Section 2.3 have $\Delta_k(t) = 1$ and $\Delta_k(t) = 2t^2 - 5t + 2$ respectively.

Section 4 is a remark about Property $IE$ on connected sums, which provides knots of any given genus $g$ (non-prime when $g > 1$); some of them have Property $IE$ and some do not.

Section 5 gives a homological obstruction to embedding infinite cyclic covers of a compact 3-manifold into any compact 3-manifold (Theorem 5.1), therefore giving a partial negative answer to Question 2.

**Comments.**

1. If we replace the term "unknotted solid torus" by "unknotted handlebody of genus $g$ for any $g > 1$", constructions in Section 2.1 can be used to study Property $DIE$ of knots with higher genera, although the arguments become more complicated. The knots having Property $DIE$ provide interesting dynamics in $\mathbb{S}^3$.

2. Theorem 5.1 as well as the constructions in Section 2.1 still holds for closed $n$-manifold and connected non-separating bicollared properly embedded codimension 1 submanifold $S$ in $M$.

3. For knot $k$ in $S^3$, the homological obstruction in Theorem 5.1 vanishes for $E(k)$ (read Remark 2). We wonder if Question 2 has a positive answer when we restrict to $E(k)$ for knots $k$ in $\mathbb{S}^3$.

4. Two Refs. [9,4] were not cited in our proofs. But [9] suggested to us the construction in Section 2.1 and [4] inspired us to prove Lemma 3.1.

## 2. Infinitely many genus 1 non-fibred knots have Property DIE

### *2.1. A construction of compact 3-manifolds having infinite cyclic covers in $\mathbb{S}^3$ or in a compact 3-manifold*

Step 1. We first consider a rather general case. Let $Y$ be a closed 3-manifold, and $P \subset Y$ be a submanifold of dimension three with connected and non-empty $\partial P$. Suppose that there is a homeomorphism

$$f : Y \to Y \qquad \text{such that } f(P) \subset \text{ int } P.$$

Let $X = P \setminus f(P)$. Then $\partial X = \partial P \cup \partial f(P)$. Let $M = X/f$ be the closed 3-manifold obtained from $X$ by identifying $\partial P$ and $\partial f(P)$ via $f$, and $S \subset M$ be the image of $\partial P$ and $\partial f(P)$ after identification. Then $S$ is a connected non-separating surface in $M$. Clearly the infinite cyclic cover $\widetilde{M}_S$ is identified with $\cup_{k=-\infty}^{+\infty} f^k(X) \subset Y$ and $f| \cup_{k=-\infty}^{+\infty} f^k(X)$ gives the deck transformation $\tau$. Hence $(\widetilde{M}_S, \tau) \subset (Y, f)$.

We say the construction above is *non-trivial*, if $X$ is not homeomorphic to $\partial P \times [0, 1]$.

Step 2. Continue from Step 1. Let $Y = \mathbb{S}^3$ and let $P$ be an unknotted solid torus $P$ in $\mathbb{S}^3$, and let $P'$ be a solid torus in int $P$, such that $P'$ is still unknotted in $\mathbb{S}^3$. Since both $P$ and $P'$ are unknotted in $\mathbb{S}^3$, there is a homeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ such that $f(P) = P'$. Then $X = P \setminus f(P)$ is an example of Step 1.

Step 3. Continue from Step 2. Let $\Gamma$ be a proper arc in $X$ with one end in $\partial P$ and the other in $\partial f(P)$. Let $N(\Gamma)$ be the regular neighborhood of $\Gamma$ in $X$. Up to isotopy we may assume $f(\partial P \cap N(\Gamma)) = \partial f(P) \cap N(\Gamma)$. Let $X^* = X \setminus N(\Gamma)$. Then $X^*$ is obtained from $X$ by digging a tunnel from $\partial P$ to $\partial f(P)$. Let $M^* = X^*/f$, $S^* = M^* \cap S$, where $M^*$ is obtained from $M$ by removing a solid torus. Clearly the infinite cyclic cover $\widetilde{M}^*_{S^*}$ is identified with $\cup_{k=-\infty}^{+\infty} f^k(X^*) \subset \mathbb{S}^3$, and $f| \cup_{k=-\infty}^{+\infty} f^k(X^*)$ gives the deck transformation $\tau$. We summarize the discussion above as

**Proposition 2.1.** *$M^*$ is a compact 3-manifold with torus boundary, and $(\widetilde{M}^*_S, \tau) \subset (\mathbb{S}^3, f)$. In particular if $M^*$ is homeomorphic to $E(k)$ for a knot $k \subset S^3$, then $k$ has Property $DIE$.*

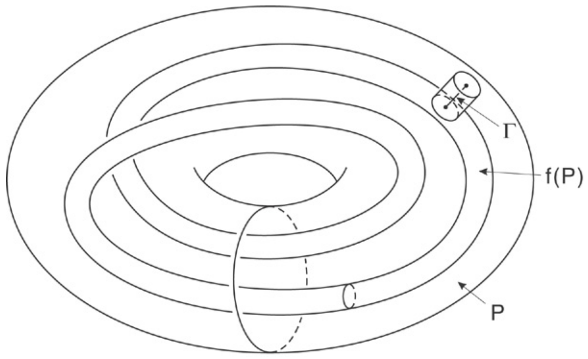


Fig. 1.

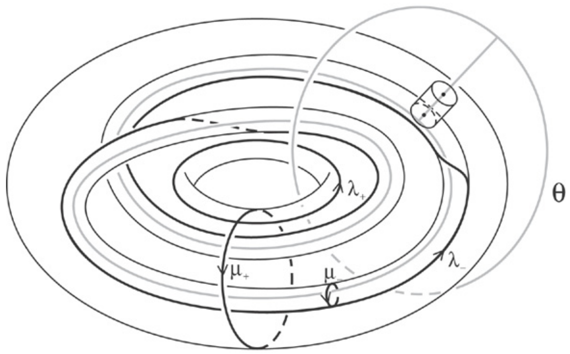


Fig. 2.

### 2.2. *The knot* $9_{46}$ *has Property DIE*

A simplest non-trivial construction in Proposition 2.1 is indicated in Fig. 1, where $P'$ is a 2-braid in $P$ and the tunnel is "unknotted". In this subsection, all notions in Step 3 of Section 2.1 refer to Fig. 1.

We will verify that $M^*$ is homeomorphic to $E = E(9_{46})$, the exterior of knot $9_{46} \subset \mathbb{S}^3$ in the knot table. Our verification consists of three steps:

Step 1. Compute $\pi_1(M^*)$ and $\pi_1(\partial M^*) \subset \pi_1(M^*)$. Cutting $M^*$ open along $S^*$, we get back to $X^*$, which is already presented in Fig. 1. Its boundary $\partial X^* = S^*_- \cup \text{annulus} \cup S^*_+$, where $S^*_-$ and $S^*_+$ are 1-punctured tori on the inner boundary $\partial f(P)$ and the outer boundary $\partial P$ respectively. The annulus is the boundary of the tunnel.

Choose meridian $\mu_+$ and longitude $\lambda_+$ on $S^*_+$ such that $\mu_+$ bounds a disc in $P$ and $\lambda_+$ bounds a disc in $\mathbb{S}^3 \setminus P$. Similarly choose meridian $\mu_-$ and longitude $\lambda_-$ on $S^*_-$ such that $\mu_-$ bounds a disc in $f(P)$ and $\lambda_-$ bounds a disc in $\mathbb{S}^3 \setminus f(P)$, where $\mu_\pm$ and $\lambda_\pm$ are as indicated in Fig. 2.

Since $f$ is a homeomorphism on $\mathbb{S}^3$ which sends the unknotted solid torus $P$ to $f(P)$, we must have $f(\lambda_+) = \lambda_-$, and $f(\mu_+) = \mu_-$. Now $M^* = X^*/f$ as in Step 3 of Section 2.1.

Note that in Fig. 2, $X^*$ is the complement of a graph $\Theta$ (shown in gray in Fig. 2) in $\mathbb{S}^3$, where $\Theta$ consists of the centerline of $f(P)$, the centerline of $\mathbb{S}^3 \setminus P$, joined by the centerline $\gamma$ of the tunnel.

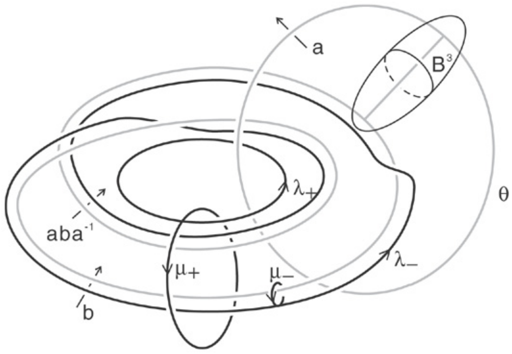


Fig. 3.

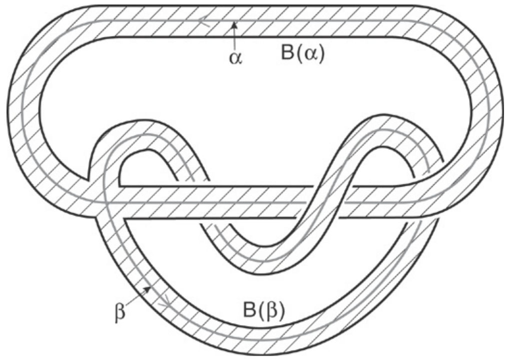


Fig. 4.

If we ignore the image of $X^*$ in Fig. 2, but with $\Theta$, $\lambda_\pm$ and $\mu_\pm$ remaining, then we have Fig. 3 below. Let $B^3$ be a 3-ball containing the arc $\gamma$ in $\Theta$, as indicated in Fig. 3. It is an observation that the complement of $\Theta$ is homeomorphic to the complement of two unknotted arcs in the 3-ball $\mathbb{S}^3 \setminus B^3$. Hence $X^*$ is a handlebody of genus 2.

Two generators $a, b$ of $\pi_1(X^*)$ are indicated in Fig. 3, where we use the Wirtinger presentation [11], the base point in $X^*$ being above the page. Representing $\lambda_\pm$ and $\mu_\pm$ in terms of $a, b$, we have $\lambda_- = abab^{-1},\ \mu_- = b$; $\lambda_+ = a$, $\mu_+ = baba^{-1}$. By HNN extension, we have

$$\pi_1(M^*) = \langle a, b, t \,|\, tat^{-1} = abab^{-1},\ tbaba^{-1}t^{-1} = b\rangle,$$

and $\pi_1(\partial M^*) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $t$ and $[\lambda_-, \mu_-] = [abab^{-1}, b]$.

Step 2. Compute $\pi_1(E)$ and $\pi_1(\partial E) \subset \pi_1(E)$. We choose the projection of $9_{46}$ provided in [11, p. 211] rather than in the knot table of [11], as Fig. 4. The Seifert surface $T$ of $9_{46}$ in Fig. 4 is the 1-punctured torus presented as a plumbing of two unknotted and untwisted bands $B(\alpha)$ and $B(\beta)$ with oriented centerlines $\alpha$ and $\beta$ respectively. $\pi_1(T)$ is generated by $\alpha$ and $\beta$.

Cutting $E$ open along $T$, we get a compact 3-manifold $Q$, which is the complement of $T$ in $\mathbb{S}^3$, therefore $Q$ is also homeomorphic to the complement of the one point union of the two circles $\alpha \cup \beta$. By a handle sliding argument (see [11, p. 95]) one can check that $Q$ is also a handlebody of genus 2. Two generators $c, d$ of $\pi_1(Q)$ are indicated in Fig. 5.

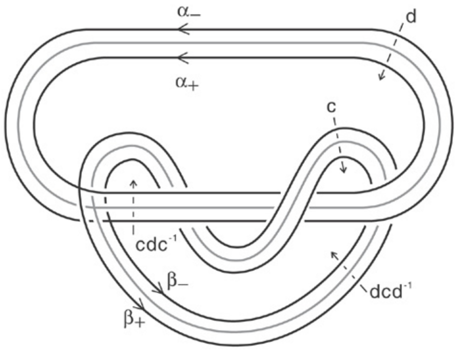


Fig. 5.

First pushing $\alpha$ and $\beta$ off $T$ towards the minus side of $T$, we get two generators $\alpha_-$, $\beta_-$ of $\pi_1(T_-)$ in $\pi_1(Q)$; and then pushing $\alpha$ and $\beta$ off $T$ towards the plus side of $T$, we get two generators $\alpha_+$, $\beta_+$ of $\pi_1(T_+)$ in $\pi_1(Q)$, all shown in Fig. 5. It can be easily computed that $\alpha_- = cdcd^{-1}$, $\beta_- = d$, $\alpha_+ = c$, $\beta_+ = dcdc^{-1}$. So

$$\pi_1(E) = \langle c, d, s | scs^{-1} = cdcd^{-1}, sdcdc^{-1}s^{-1} = d\rangle,$$

and $\pi_1(\partial E) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $s$ and $[\alpha_-, \beta_-] = [cdcd^{-1}, d]$.

Step 3. Now we have an isomorphism

$$\phi : \pi_1(M^*) \to \pi_1(E), \quad \text{such that } a \mapsto c, b \mapsto d, t \mapsto s,$$

which maps $\pi_1(\partial M^*)$ isomorphically onto $\pi_1(\partial E)$.

Both $M^*$, $E$ are $\mathbb{P}^2$-irreducible, sufficiently large manifolds, so Waldhausen's theorem [8, Theorem 13.6] (or [3, p. 308 B7] more directly) implies that $M^*$ is homeomorphic to $E$. We finished the verification.

### 2.3. *Infinitely many genus 1 knots have Property DIE*

Let $P$, $P'$, $X^*$ and $X^*/f$ be as given in Step 3 of Section 2.1.

**Proposition 2.2.** (1) *If a meridian disc $D$ of $P$ meets the core of $P'$ in exactly 2 points transversely, then $X^*/f$ is the complement of a genus 1 knot in a homotopy 3-sphere.*

(2) *Furthermore if $X^*$ is homeomorphic to a handlebody of genus 2, then $X^*/f = E(k)$ for some genus 1 knot $k \subset \mathbb{S}^3$.*

(3) *There are infinitely many genus 1 knots $k \subset \mathbb{S}^3$ such that $E(k)$ are obtained by the construction in Section 2.1.*

**Proof.** Fig. 6 indicates that there are infinitely many embeddings $P' \subset P$, such that both the conditions in Proposition 2.2(1) and (2) are satisfied. The verification of $X^*$ to be the handlebody of genus 2 is the same as we did in Fig. 3 in Section 2.2. (Note that if we choose the tunnel joining $\partial P$ and $\partial P'$ to be knotted, then the condition in (2) is not satisfied in general.)

(1) We will find a presentation for $\pi_1(M^*)$ as in Step 1 of Section 2.2 (but the process is simpler since we need less precise information about the presentation). First from Fig. 6 we get Fig. 7 (as we did from

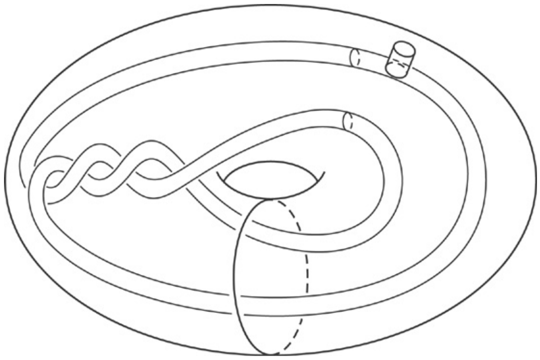

Fig. 6.

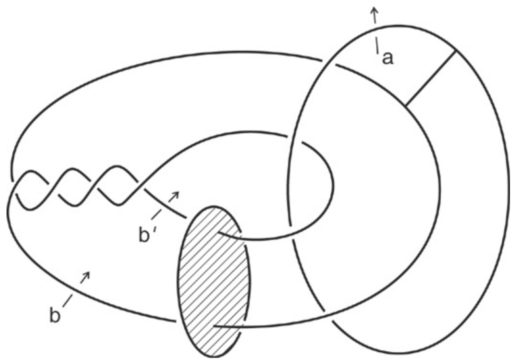


Fig. 7.

Fig. 1 to Fig. 3 in Step 1 of Section 2.2) where $a, b, b'$ are elements in $G = \pi_1(X^*)$. Then as in Step 1 of Section 2.2 we can compute $\pi_1(X^*/f)$ via HNN extension as

$$\pi_1(X^*/f) = \langle G, t | tat^{-1} = c, tbb't^{-1} = b\rangle,$$

where $c$ is the element in $G$ representing $\lambda_-$, and $t$ is represented by a loop $\gamma$ in $\partial(X^*/f) = T^2$. Note $\mu_+ = bb'$ because the meridian disc intersects $P'$ twice.

A Dehn filling along $\gamma$ will kill $t$ and provide a new manifold $M_1 = (X^*/f)(\gamma)$ with

$$\pi_1(M_1) = \langle G | a = c, bb' = b\rangle = \langle G | a = c, b' = 1\rangle.$$

If we add a 2-handle to $X^*$ along the loop representing $b'$, the new manifold is obviously a solid torus. So $\langle G \mid b' = 1\rangle \cong \mathbb{Z}$. Thus $\pi_1(M_1)$ is a quotient group of $\mathbb{Z}$. A computation in homology will show that $H_1(M_1; \mathbb{Z}) = 0$, hence $\pi_1(M_1) = 1$. Thus $X^*/f$ is the complement of a knot $k$ in the homotopy 3-sphere $M_1$.

(2) Furthermore suppose $X^*$ is homeomorphic to a handlebody of genus 2. Note $X^*/f = X^* \cup_f N(\partial P \setminus N(\Gamma))$, and $M_1 = (X^*/f)(\gamma)$ can be viewed as a quotient of $X^* \cup_f N(\partial P \setminus N(\Gamma))$ by identifying the annulus $\partial(X^*/f) \cap N(\partial P \setminus N(\Gamma))$ with the annulus $\partial(X^*/f) \cap X^*$. Hence $M_1$ has a Heegaard splitting $X^* \cup_h N(\partial P \setminus N(\Gamma))$ of genus 2, where $h$ is determined by $f$ and $\gamma$. By Theorem 1 of [1] $M_1$ is a 2-fold cyclic covering of $S^3$, branched over a 3-bridge link. It follows that $M_1$ is

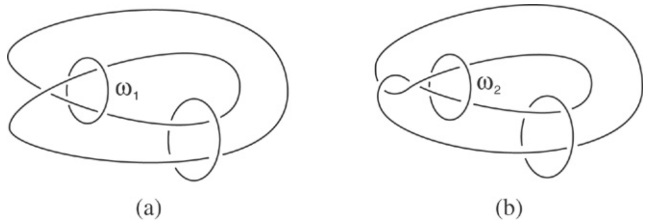


Fig. 8.

homeomorphic to $\mathbb{S}^3$ by Thurston's orbifold theorem (see [2]), and hence $X^*/f = E(k)$ for a knot $k$ in $\mathbb{S}^3$.

(3) We refine our notations related to Fig. 6. Denote $P'$, $X^*$ and $f$ by $P'_n$, $X^*_n$ and $f_n$, if the crossing number of the core of $P' \subset P$ in Fig. 6 is $n$, $n \in \mathbb{Z}$. Then we have $X^*_n/f^*_n = E(k_n)$ for some knot $k_n \subset \mathbb{S}^3$ according to (2). If there are only finitely many different homeomorphism types for $E(k_n)$, then there are only finitely many $E(k_n, 0)$, the zero surgery manifold on $k_n$. It follows that the Gromov volumes $\{V(E(k_n, 0))\}$ take only finitely many values. Note that $E(k_n, 0)$ is homeomorphic to $(P \setminus P'_n)/f_n$, and $E(k_n, 0) \setminus S_n = P \setminus P'_n$. Since $\partial P$ is incompressible in $P \setminus P'_n$, $V(E(k_n, 0)) = V(P \setminus P'_n)$ by a theorem of Soma [12, Theorem 1], it follows that $\{V(P \setminus P'_n)\}$ take only finitely many values.

Consider two 3-component links $L_1$ and $L_2$ with marked components $\omega_1$ and $\omega_2$ respectively, indicated in Fig. 8, (a) and (b). Note $\omega_i$ is unknotted in $\mathbb{S}^3$; the standard arguments (see [11, Chap. 9]) show that

$$P \setminus P'_{2n+1} = E(L_1)(\omega_1, 1/n), \qquad P \setminus P'_{2n+2} = E(L_2)(\omega_2, 1/n),$$

where $E(L_i)(\omega_i, 1/n)$ is the $1/n$-Dehn filling along $\omega_i$. It is also known that both $L_1$ and $L_2$ are hyperbolic links. (This fact can be checked by SnapPea [14].) According to Thurston's theory about Gromov volume on 3-manifolds (see [13, Chapters 5 and 6]), we have

(i) $V(E(L_i)(\omega_i, 1/n)) < V(E(L_i))$,

(ii) $\lim_{n\to\infty} V(E(L_i)(\omega_i, 1/n)) = V(E(L_i))$.

It follows that $\{V(P \setminus P'_n)\}$ take infinitely many values, a contradiction. □

**Remark 1.** All knots constructed in Proposition 2.2 bound the genus 1 surface. All knots $k_n$ in Proposition 2.2(3) are non-fibred, see the end of Section 3, and also $k_1 = 9_{46}$.

## 3. "Most" genus 1 knots do not have Property *IE*

Let $k$ be a non-fibred knot of genus 1 in $\mathbb{S}^3$. Recall the notations $E(k)$, $S$, $X = E(k) \setminus S$, $\widetilde{E}(k)$ defined at the beginning of the paper. Suppose also $S$ is of genus 1.

Let $S_n$ $(n = \ldots, -2, -1, 0, 1, 2, \ldots)$ denote the copies of $S$ in $\widetilde{E}(k)$. For integers $m < n$, let $X_{[m,n]}$ denote the sub-manifold of $\widetilde{E}(k)$ between $S_m$ and $S_n$, and $A_{[m,n]}$ denote the annulus bounded by $\partial S_m \sqcup \partial S_n$ on $\partial X_{[m,n]}$. Assume $\widetilde{E}(k)$ is already embedded in $\mathbb{S}^3$, and $Y_{]m,n[} = \mathbb{S}^3 \setminus X_{[m,n]}$. We always use $X_n$ to denote $X_{[n,n+1]}$ for simplicity. The readers should be aware that the subscript $n$ here has a different meaning from the $n$ in the last section.

**Lemma 3.1.** *For any integer $N > 0$, $\partial S_0$ bounds a disc $D$ in $Y_{]-N,N[}$.*

**Proof.** Consider the separating surfaces $S_n^* = A_{[N,n]} \cup S_n$ $(n = N+1, N+2, \ldots)$ in $Y_{]-N,N[}$. They are mutually non-parallel, since $k$ is non-fibred. Since each $S_n^*$ has the first Betti number 2, Freedman–Freedman's version of the Kneser–Haken finiteness theorem [5] implies that they must be compressible in $Y_{]-N,N[}$ when $n$ is sufficiently large. Suppose $D$ is a compressing disc of $S_n^*$. If $\partial D$ is parallel to $\partial S_n^*$ on $S_n^*$, then the lemma is proved, since $\partial S_n^*$ is parallel to $\partial S_0$ on $\partial Y_{]-N,N[}$. If $\partial D$ is not parallel to $\partial S_n^*$, surger $S_n^*$ along $\partial D$, we still get a disc $D'$ in $Y_{]-N,N[}$, with $\partial D' = \partial S_n^*$, since $S_n^*$ is a 1-punctured torus. □

Now fix an $N$ sufficiently large, we can thicken $D \cup A_{[-N,N]}$ in $Y_{]-N,N[}$ to get a 2-handle $D \times I$, which is attached to $X_{[-N,N]}$ along the annulus $A_{[-N,N]}$. Let $D_{-N}, \ldots, D_N$ be a collection of $D \times \{t\}$'s in the 2-handle, so that $\partial D_i = \partial S_i,\ i = -N, \ldots, N$. From now on, all subscripts in this section are bounded by $N$, as is understood.

Let $\widehat{S}_i$ denote the torus $S_i \cup D_i$. Let $\widehat{X}_i$ be the manifold bounded by $\widehat{S}_i$ and $\widehat{S}_{i+1}$ in $\mathbb{S}^3$, and more generally, $\widehat{X}_{[m,n]}$ be the manifold bounded by $\widehat{S}_m$ and $\widehat{S}_n$ in $\mathbb{S}^3$.

**Lemma 3.2.** *$\widehat{X}_i$ is irreducible and $\partial$-irreducible. Moreover $\widehat{X}_i$ is not a product.*

**Proof.** Since $S$ is a minimal genus Seifert surface of $k$, $E(k)$ admits a taut foliation $\mathcal{F}$ such that $S$ is a leaf of $\mathcal{F}$ and $\mathcal{F}|\partial E(k)$ is foliated by circles by [6, Theorem 3.1]. Then $\mathcal{F}$ can be extended to a taut foliation $\widehat{\mathcal{F}}$ on $E(k, 0)$, the zero surgery manifold on $k$, such that $\widehat{S}$ is a leaf of $\widehat{\mathcal{F}}$, where $\widehat{S}$ is obtained by capping disc on $S$. Moreover since $E(k)$ is not fibred, $E(k, 0)$ is not fibred by [6, Corollary 8.19], in particular $E(k, 0) \neq S^2 \times S^1$. By Novikov's theorem [10], each leaf of the taut foliation $\widehat{\mathcal{F}}$ is $\pi_1$-injective in $E(k, 0)$ and $\pi_2(E(k, 0)) = 0$. Then $E(k, 0)$ is irreducible by the sphere theorem [8, Chap. 3], and furthermore $\widehat{S}$ is incompressible. It follows that $E(k, 0) \setminus \widehat{S}$ is irreducible, $\partial$-irreducible, and is not a product.

Since each $\widehat{X}_i$ is homeomorphic to $E(k, 0) \setminus \widehat{S}$, Lemma 3.2 is proved. □

Each $\widehat{S}_i$ separates $\mathbb{S}^3$ into 2 components. We say the component containing $\widehat{X}_i$ *lies on the plus side of* $\widehat{S}_i$, the component containing $\widehat{X}_{i-1}$ lies on the *minus* side of $\widehat{S}_i$. $\widehat{S}_i$ bounds a solid torus on the plus side or the minus side, since every torus in $\mathbb{S}^3$ bounds a solid torus. In fact, we can prove the stronger

**Proposition 3.3.** *Each $\widehat{S}_i$ bounds solid tori on both sides.*

**Proof.** Without loss of generality, we can assume $\widehat{S}_0$ bounds a solid torus $P_0$ on the minus side. Our argument proceeds in the following steps.

**Step 1.** For each $n < 0$, $\widehat{S}_n$ bounds a solid torus $P_n$ on the minus side.

Otherwise, assume some $\widehat{S}_n$ does not bound a solid torus on the minus side, then $\widehat{S}_n$ bounds a solid torus $P_+$ on the positive side. Hence $\widehat{S}_n$ cuts $P_0$ into 2 parts: $\widehat{X}_{[n,0]}$ and $P_0 \setminus \widehat{X}_{[n,0]} = \mathbb{S}^3 \setminus P_+$. By Lemma 3.2, $\widehat{S}_n$ is incompressible in $\widehat{X}_{[n,0]}$; $\widehat{S}_n$ is also incompressible in $\mathbb{S}^3 \setminus P_+$ since $P_+$ is knotted. So $P_0 = (\mathbb{S}^3 \setminus P_+) \cup_{\widehat{S}_n} \widehat{X}_{[n,0]}$ cannot have $\pi_1 = \mathbb{Z}$.

By Step 1, we have a nested sequence of solid tori

$$\cdots \subset P_{n-1} \subset P_n \subset P_{n+1} \subset \cdots \subset P_0.$$

We assume that these tori adapt the orientation of $\mathbb{S}^3$. Let $\mu_n,\ \lambda_n \subset \widehat{S}_n$ be an oriented meridian–longitude system of $P_n$, $n < 0$, so that

(1) the algebraic intersection number of $\mu_n$ and $\lambda_n$ is 1,

(2) the linking number of $\lambda_n$ and $\mu_{n+1}$, which is defined as the winding number of $P_n$ in $P_{n+1}$, is $\geq 0$.

**Step 2.** Suppose $P_n$ has winding number $w_n$ in $P_{n+1}$, $n < 0$. Then all $w_n$ are equal, denoted by $w$.

Clearly $P_n \setminus P_{n-1}$ is homeomorphic to the complement in $\mathbb{S}^3$ of a 2-component link with linking number $w_{n-1}$, so $H_1(P_n \setminus P_{n-1}; \mathbb{Z})$ has a basis $\lambda_n, \mu_{n-1}$, and $H_1(P_n \setminus P_{n-1}, \widehat{S}_n; \mathbb{Z})$ is isomorphic to $Z_{w_{n-1}}$.

Note that the deck translation $\tau : \widetilde{E}(k) \to \widetilde{E}(k)$, which sends $X_i$ to $X_{i+1}$, induces a homeomorphism $\widehat{\tau} : \widehat{X}_{n-1} = P_n \setminus P_{n-1} \to \widehat{X}_n = P_{n+1} \setminus P_n$ with $\widehat{\tau}_n(\widehat{S}_n) = \widehat{S}_{n+1}$ for each $n < 0$. It follows that $w_n = w_{n-1}$.

**Step 3.** We claim that $\widehat{\tau}$ sends $\mu_{n-1}$ to $\mu_n$ for $n \leq -1$. There are 2 cases:

Case 1. $w = 0, 1$. Now $P_n$ cannot be a braid in $P_{n+1}$, otherwise $w = 1$ and $\widehat{X}_n$ is a product $T^2 \times I$, contrary to Lemma 3.2. Then the results in [7] imply that only trivial surgery on $P_n$ yields a solid torus. Since the Dehn surgery on the knot $P_n$ in the solid torus $P_{n+1}$ along $\widehat{\tau}(\mu_{n-1})$ again yields a solid torus, $\widehat{\tau}(\mu_{n-1}) = \mu_n$ for $n \leq -1$.

Case 2. $w \geq 2$. Fix $n < 0$. Now $\lambda_i, \mu_i$ is a basis of $H_1(\widehat{S}_i; Z)$, $i \leq 0$.

$$\widehat{\tau}_*(\lambda_n) = p\lambda_{n+1} + q\mu_{n+1}, \qquad \widehat{\tau}_*(\mu_n) = r\lambda_{n+1} + s\mu_{n+1}, \qquad ps - qr = 1.$$

For each integer $m > 0$, since $\mu_n$ is a $w^m$ multiple in $H_1(\widehat{X}_{[n-m,n]}; \mathbb{Z})$, $\widehat{\tau}_*(\mu_n)$ is also a $w^m$ multiple in $H_1(\widehat{X}_{[n-m+1,n+1]}; \mathbb{Z})$. Since $\mu_{n+1}$ is already a $w^m$ multiple in $H_1(\widehat{X}_{[n-m+1,n+1]}; \mathbb{Z})$, $r\lambda_{n+1}$ is also a $w^m$ multiple.

Since $\{\lambda_{n+1}, \mu_{n-m+1}\}$ is a basis of $H_1(\widehat{X}_{[n-m+1,n+1]})$ for $m > 0$, $r$ should be a $w^m$ multiple. Since $r$ is a given integer, letting $m$ be sufficiently large, we must have $r = 0$. Then $p = s = \pm 1$, i.e., $\widehat{\tau}_*(\mu_n) = \pm\mu_{n+1}$, and the conclusion holds.

**Step 4.** When $n > 0$, $\widehat{S}_n$ bounds a solid torus on the minus side.

There is a properly embedded planar surface $G$ in $\widehat{X}_{-2}$, $G \cap \widehat{S}_{-1} = \mu_{-1}$, $G \cap \widehat{S}_{-2}$ consists of parallel copies of $\mu_{-2}$. By Step 3, $\widehat{\tau}(G)$ is a planar surface in $\widehat{X}_{-1}$, $\widehat{\tau}(G) \cap \widehat{S}_{-1}$ consists of parallel copies of $\mu_{-1}$. $\widehat{\tau}(G) \cap \widehat{S}_0$ bounds a disc on the minus side of $\widehat{S}_0$, since each copy of $\mu_{-1}$ bounds a disc in $P_{-1}$. So $\widehat{\tau}(\mu_{-1}) = \widehat{\tau}(G) \cap \widehat{S}_0 = \mu_0$. Let $\mu_n = \widehat{\tau}^n(\mu_0)$ for $n > 0$, the same argument as above shows that $\mu_n$ bounds a disc on the minus side of $\widehat{S}_n$, by induction.

**Step 5.** All $\widehat{S}_n$ bounds solid tori on both sides, $n \in \mathbb{N}$.

By Lemma 3.2, $\widehat{S}_n$ and $\widehat{S}_m$ are not parallel for $m \neq n$. By Haken's finiteness theorem, $\widehat{S}_n$ is compressible in $\mathbb{S}^3 \setminus P_0$ when $n$ is sufficiently large. The compressing disc cannot lie on the minus side, since $\widehat{X}_{[0,n]}$ is $\partial$-irreducible by Lemma 3.2. So $\widehat{S}_n$ bounds a solid torus on the plus side when $n$ is sufficiently large. Now proceed from Step 1 to Step 4, but reverse the direction, to get our conclusion. □

**Theorem 3.4.** *Suppose $k$ is a non-fibred knot of genus* 1 *in $\mathbb{S}^3$. If $k$ has Property $IE$, then $k$ has Property $DIE$. Indeed, $E(k)$ can be obtained by the construction in Section* 2.1.

*Moreover, the winding number $w$ involved is either* 0 *or* 2 . *Correspondingly, the Alexander invariant of $k$ is either* 0 *or* $\mathbb{Z}[t, t^{-1}]/(2t-1) \oplus \mathbb{Z}[t, t^{-1}]/(t-2)$, *and the Alexander polynomial of $k$ is either* 1 *or* $2t^2 - 5t + 2$.

**Proof.** Suppose $\widetilde{E}(k)$ is embedded into $\mathbb{S}^3$. We keep the notation in the proof of Proposition 3.3. First, extend $\tau|X_0 : (X_0, S_0) \to (X_1, S_1)$ to a homeomorphism $\widehat{\tau}_1 : (\widehat{X}_0, \widehat{S}_0) \to (\widehat{X}_1, \widehat{S}_1)$ as in the proof of Proposition 3.3.

According to Proposition 3.3, each $\widehat{S}_n$ bounds a solid torus $P_n^-$ on the minus side, and a solid torus $P_n^+$ on the plus side. Suppose $\mu_n^-, \mu_n^+ \subset S_n \subset \widehat{S}_n$ are meridians of $P_n^-$, $P_n^+$ respectively. By Step 3 (and its counterpart in Step 5) of Proposition 3.3,

$$\widehat{\tau}(\mu_n^-) = \mu_{n+1}^-, \qquad \widehat{\tau}(\mu_n^+) = \mu_{n+1}^+. \tag{3.1}$$

Hence we can further extend $\widehat{\tau}_1$ to $\widehat{\tau}_2 : P_0^+ \to P_1^+$, and finally we extend $\widehat{\tau}_2$ to $f : \mathbb{S}^3 \to \mathbb{S}^3$ since both $P_0^+$ and $P_1^+$ are unknotted. Now we can reconstruct $E(k)$ from $f$ as in Section 2.1, so $k$ has Property $DIE$. We have finished the proof of the first part of Theorem 3.4.

By Step 2 (and its counterpart in Step 5) of Proposition 3.3, the winding number of $P_n^-$ in $P_{n+1}^-$ is a constant $w^-$, and the winding number of $P_{n+1}^+$ in $P_n^+$ is a constant $w^+$. It is easy to see that both $w^-$ and $w^+$ are the linking number between $\mu_{n+1}^-$ and $\mu_n^+$ (see the paragraph after Step 1 in the proof of Proposition 3.3), and we have $w^- = w^+ = w$. Since $\widehat{\tau}| : \widehat{S}_n \to \widehat{S}_{n+1}$ is orientation preserving, by (3.1) we have

$$\widehat{\tau}_*^{-1}([\mu_n^+]) = \pm w[\mu_n^+], \qquad \widehat{\tau}_*([\mu_n^-]) = \pm w[\mu_n^-]. \tag{3.2}$$

Note that $X_n \hookrightarrow \widehat{X}_n$ induces an isomorphism on 1-dimensional homology. Then by (3.2) the Alexander invariant of $k$ has presentation [11, Chap. 7]

$$H_1(\widetilde{E}(k); \mathbb{Z}[t, t^{-1}]) = \langle \mu_n^+, \mu_n^-, t \mid t^{-1}([\mu_n^+]) = \pm w[\mu_n^+], t([\mu_n^-]) = \pm w[\mu_n^-]\rangle,$$

and the Alexander matrix of $k$ is

$$\begin{pmatrix} wt \mp 1 & 0 \\ 0 & t \mp w \end{pmatrix}.$$

Since $\Delta_k(1) = \pm 1$, $w$ can only be 0 or 2, and the corresponding Alexander polynomials are 1 or $2t^2 - 5t + 2$ respectively, and the Alexander invariant of $k$ are either 0 or $\mathbb{Z}[t, t^{-1}]/(2t-1) \oplus \mathbb{Z}[t, t^{-1}]/(t-2)$. We have finished the proof of Theorem 3.4. □

**Corollary 3.5.** *Among all genus* 1 *non-fibred knots in* $\mathbb{S}^3$*,*
(1) *up to ten crossings,* $9_{46}$ *is the only one that has Property* $IE$*,*
(2) *no alternating knot has Property* $IE$*.*

**Proof.** (1) For knots with ≤10 crossings, no non-fibred knot has Alexander polynomial 1, and only $6_1$ and $9_{46}$ have Alexander polynomial $2t^2 - 5t + 2$, see the tables in [3] and in [11]. But their Alexander invariants are not isomorphic (see [11, p. 211]), so $6_1$ does not have Property $IE$. Then by Section 2.2 (1) follows.

(2) If a genus 1 non-fibred knot $k$ has Property $IE$, then $\Delta_k(-1) = 1$ or 9. Now suppose $k$ is alternating, by a theorem of R.H. Crowell (see [3, Proposition 13.30]) $\Delta_k(-1)$ is not smaller than the crossing number of $k$, and $9_{46}$ is not alternating. Hence (2) follows from (1). □

Recall the two infinite families of knots $k_{2n}$ and $k_{2n+1}$ with Property $IE$, as well as the notion $P'_n$, defined in the proof of Proposition 2.2(3). Since the winding number of $P'_{2n}$ is 0 and the winding number of $P'_{2n+1}$ is 2, according to the calculation in the proof of Theorem 3.4 we have $\Delta_{k_{2n}}(t) = 1$ and $\Delta_{k_{2n+1}}(t) = 2t^2 - 5t + 2$.

**Corollary 3.6.** *Among non-fibred genus 1 knots, both the subsets defined by* $\Delta_k(t) = 1$ *and by* $\Delta_k(t) = 2t^2 - 5t + 2$ *have infinitely many elements with Property* $IE$*.* □

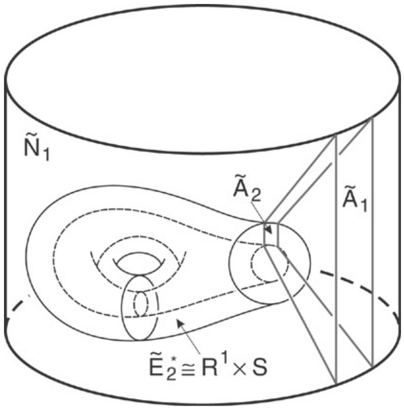


Fig. 9.

## 4. A remark on connected sums

**Lemma 4.1.** *Suppose $k_1$ and $k_2$ are two knots in $\mathbb{S}^3$.*

(1) *If $k_1\#k_2$ has Property $IE$, then both $k_1$ and $k_2$ have Property $IE$.*

(2) *If $k_1$ has Property $IE$ and $k_2$ is fibred, then $k_1\#k_2$ has Property $IE$.*

Note that there are fibred knots of any genus (just consider the connected sum of genus 1 fibred knots), and that $k_1\#k_2$ is fibred if and only if both $k_1$ and $k_2$ are fibred (it follows from the definitions of connected sum, fibred knot, and Stallings' fibration Theorem [8, Theorem 11.1]). Then by the main results in Sections 2 and 3 and Lemma 4.1 we have the following

**Corollary 4.2.** *Among non-fibred knots of genus $g$ for any given integer $g > 0$, both the subsets defined by having Property $IE$ and not having Property $IE$ have infinitely many elements.* □

**Proof of Lemma 4.1.** Denote $E(k_i)$ by $E_i$. Let $N_i = N(\mu_i)$ be the regular neighborhood of the meridian $\mu_i \subset \partial E_i$ in $E_i$. Let $E_i^* = E_i \setminus N_i$, and $A_i = E_i^* \cap N_i$. Then $E_i^*$ is homeomorphic to $E_i$ and $A_i$ is an annulus. By definition of the connected sum, we have $E(k_1\#k_2) = E_1^* \cup_h E_2^*$, where $h$ is a homeomorphism identifying $A_1$ and $A_2$.

Let $p_i : \widetilde{E}_i \to E_i$ be the infinite cyclic covering, and let $\widetilde{E}_i^*, \widetilde{N}_i, \widetilde{A}_i$ be the preimage of $E_i^*, N_i, A_i$ under $p_i$. Clearly the restriction of

$$p_i : (\widetilde{E}_i, \widetilde{E}_i^*, \widetilde{N}_i, \widetilde{A}_i) \to (E_i, E_i^*, N_i, A_i)$$

is the infinite cyclic covering on each of the four corresponding pairs. Moreover $\widetilde{E}_i^*, \widetilde{N}_i, \widetilde{A}_i$ are homeomorphic to $\widetilde{E}_i, R^1 \times D^2, R^1 \times I$ respectively and $\widetilde{E}(k_1\#k_2) = \widetilde{E}_1^* \cup_{\widetilde{h}} \widetilde{E}_2^*$, where $\widetilde{h}$ is a homeomorphism identifying $\widetilde{A}_1$ with $\widetilde{A}_2$. Hence (1) follows.

We are going to prove (2). Now $\widetilde{E}_2^* = R^1 \times S$ for a once punctured surface $S$ with $\widetilde{A}_2 = R^1 \times I$ properly embedded in $R^1 \times \partial S$.

Since there is an embedding $e : \widetilde{E}_2^* = R^1 \times S \to \widetilde{N}_1$ such that $e$ sends $\widetilde{A}_2$ to $\widetilde{A}_1 \subset \partial \widetilde{N}_1$ homeomorphically, and $e(\widetilde{E}_2^*) \cap \widetilde{N}_1 = \widetilde{A}_1$ (see Fig. 9), $\widetilde{E}(k_1\#k_2) = \widetilde{E}_1 \cup_{\widetilde{h}} \widetilde{E}_2$ can be embedded into $E_1^* \cup \widetilde{N}_1 = \widetilde{E}_1$. Hence (2) follows. □

## 5. A partial negative answer to Question 2

In this section we use the notation in the first two paragraphs of Section 1. We will use $H_i(\cdot)$ to denote $H_i(\cdot\,;\mathbb{Q})$. Recall the following standard fact: let

$$\cdots \to A \to B \to C \to \cdots$$

be an exact sequence of vector spaces. Then

$$\dim A + \dim C \geq \dim B. \qquad (*)$$

**Theorem 5.1.** *Suppose $M$ is a compact 3-manifold, $S$ is a connected non-separating 2-sided proper surface in $M$. Let $X = M \setminus S$.*

(1) *In the case $\partial M \neq \emptyset$, if $[S \cap T] \neq 0 \in H_1(\partial M;\mathbb{Z})$ for each boundary component $T$ of $M$ and $\beta_1(X) > \beta_1(S) - \chi(\partial M)$, then $\widetilde{M}_S$ cannot be embedded into any compact 3-manifold.*

(2) *In the case $\partial M = \emptyset$, if $\beta_1(X) > \beta_1(S)$, then $\widetilde{M}_S$ cannot be embedded into any compact 3-manifold.*

**Proof.** Suppose $\partial M \neq \emptyset$, $\widetilde{M} = \cup_{k=-\infty}^{+\infty} X_k$ can be embedded into a compact 3-manifold $Y$. We may assume $\partial Y = \emptyset$. Denote $\cup_{k=1}^{m} X_k$ by $P_m$.

We first need to estimate $\beta_1(P_m)$. From $P_m = P_{m-1} \cup X_m$ and $S_m = P_{m-1} \cap X_m$, we have the Mayer–Vietoris sequence:

$$\cdots \to H_1(S_m) \to H_1(P_{m-1}) \oplus H_1(X_m) \to H_1(P_m) \to \cdots.$$

By (*), we have the inequality:

$$\beta_1(P_m) \geq \beta_1(P_{m-1}) + \beta_1(X) - \beta_1(S).$$

Hence we can easily deduce:

$$\beta_1(P_m) \geq m\beta_1(X) - (m-1)\beta_1(S). \qquad (1)$$

We need then to estimate $\beta_1(\partial P_m)$.

Cutting $\partial M$ open along $\partial S$, we get a surface $T'$. $\partial P_m$ is the union of $S_1^- \sqcup S_m^+$ and $m$ copies of $T'$. Note that the cutting and gluing of surfaces are all along circles, which have Euler characteristic 0. So

$$\begin{aligned}\chi(\partial P_m) &= \chi(S_1^- \sqcup S_m^+) + m\chi(T')\\ &= 2\chi(S) + m\chi(\partial M)\\ &= 2(1 - \beta_1(S)) + m\chi(\partial M).\end{aligned}$$

Then one can verify that

$$\beta_1(\partial P_m) = 2\beta_0(\partial P_m) - \chi(\partial P_m) = 2\beta_0(\partial P_m) + 2(\beta_1(S) - 1) - m\chi(\partial M). \qquad (2)$$

**Lemma 5.2.** *$\beta_0(\partial P_m) \leq 2\beta_0(S \cap \partial M)$ for any $m$.*

**Proof.** The bottom and the top of $P_m$ are $S_1^- \sqcup S_m^+$, which consists of $2\beta_0(S \cap \partial M)$ boundary components. If for some $m$, $\beta_0(\partial P_m) > 2\beta_0(S \cap \partial M)$, then some component $F$ of $\partial P_m$ does not meet the top and the bottom of $P_m$. It follows that $F \subset P_m \subset \widetilde{M}_S$ provides a component of $\partial \widetilde{M}_S$, therefore $p(F)$ is a component of $\partial M$, where $p : \widetilde{M}_S \to M$ is the infinite cyclic covering map. Since the deck transformation group of the covering $p : \widetilde{M}_S \to M$ is the infinite cyclic group which contains no non-trivial finite subgroup, it follows that $p : F \to p(F)$ is a homeomorphism. Now $S \cap p(F) = \cup_{i=2}^{m} p(S_i \cap F)$.

Since $S_i$ separates $P_m$, $S_i$ separates $F$. Since $F$ is closed, $S_i \cap F$ is homologically trivial in $F$. Hence $p(S_i \cap F)$ is homologically trivial in $p(F)$, and then $[S \cap p(F)] = 0$, contradicting the assumption in Theorem 5.1(1). □

By using (*) to various homology sequences, we have

$$\begin{aligned}
\beta_1(Y) &\geq \beta_1(Y, Y \setminus P_m) - \beta_0(Y \setminus P_m) && \text{by (*)}\\
&= \beta_1(P_m, \partial P_m) - \beta_0(Y \setminus P_m) && \text{by excision}\\
&\geq \beta_1(P_m, \partial P_m) - \beta_0(\partial P_m) && \text{since } \beta_0(Y \setminus P_m) \leq \beta_0(\partial P_m)\\
&\geq \beta_1(P_m) - \beta_1(\partial P_m) - \beta_0(\partial P_m) && \text{by (*)}\\
&\geq m(\beta_1(X) - \beta_1(S) + \chi(\partial M)) + C && \text{by (1), (2) and Lemma 5.2}
\end{aligned}$$

where $C = 2 - \beta_1(S) - 6\beta_0(S \cap \partial M)$ is independent of $m$.

It follows that if $\beta_1(X) > \beta_1(S) - \chi(\partial M)$, $\beta_1(Y)$ would be arbitrarily large when $m$ gets large. We reach a contradiction, since $\beta_1(Y)$ should be finite for a compact manifold $Y$. Theorem 5.1(1) is proved.

A similar and more direct argument proves Theorem 5.1(2). □

**Remark 2.** Consider the connected sum $M = P \# E(k)$, where $P$ is a homology 3-sphere with $\pi_1(P) \neq 1$ and $k$ is a knot in $\mathbb{S}^3$. Let $S \subset M$ be a Seifert surface of $E(k)$, and $X = M \setminus S$. Then $\beta_1(X) \leq \beta_1(S)$ and $\chi(\partial M) = 0$. So the inequality in Theorem 5.1(1) is not met. There is an essential 2-sphere $S^2$ in the connected sum, and $p^{-1}(S^2)$ is an infinite family of essential 2-spheres in $\widetilde{M}_S$, where $p : \widetilde{M}_S \to M$ is the infinite cyclic covering. Then $\widetilde{M}_S$ cannot stay in a compact 3-manifold.

Otherwise suppose $\widetilde{M}_S \subset Y$ for a compact 3-manifold $Y$. Let $\cup_{i=1}^{n} S_i^2$ be $n$ components in $p^{-1}(S^2)$ for any given $n$. Then clearly each component of $Y \setminus \cup_{i=1}^{n} S_i^2$ contains a copy of the 1-punctured homology 3-sphere $P^*$ with $\pi_1(P^*) \neq 1$. Since $P^*$ is not a subset of a punctured 3-sphere, no component of $Y \setminus \cup_{i=1}^{n} S_i^2$ is a punctured 3-sphere, which contradicts the Kneser finiteness theorem [8, Lemma 3.14].

## Acknowledgements

We are grateful to Dr. Hao Zheng for drawing the pictures, to Professor William Browder for a helpful conversation with the second author, to Professor Robert D. Edwards for bringing the third author to this simply stated intuitive question, and to Professor David Gabai for a comment on alternating knots. All authors are partially supported by a MOSTC grant and a MOEC grant. The second author is partially supported by the Centennial fellowship of Princeton University. Part of the paper is revised from the Master thesis of the second named author submitted to Peking University.

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