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Generation of sutured manifolds

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Abstract Given a compact, oriented, connected surface F, we show that the set of connected sutured manifolds (M,γ) with $R_\pm(\gamma)\cong F$ is generated by the product sutured manifold $(F,\partial F)\times[0,1]$ through surgery triads. This result has applications in Floer theories of 3-manifolds. The special case where $F=D^2$ or $F=S^2$ has been a folklore theorem, which has already been used by experts before.

Keywords sutured manifolds, Dehn surgery, surgery triads, generalized Heegaard splittings, Floer homology

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1 Introduction

The surgery triad is an important concept in various Floer theories of 3-manifolds. A surgery triad is often associated with exact triangles of Floer homologies, which have been extremely useful in many applications to low-dimensional topology.

One strategy for proving certain isomorphism theorems in Floer theories of 3-manifolds is to use surgery exact triangles and functoriality to reduce the proof of the general case to checking the isomorphism for manifolds in a generating set (see [4] for an exposition[1)]). In order to apply this strategy, we often need certain finite generation results of 3-manifolds. To explain in detail, we need a few definitions.

Definition 1.1. A *surgery triad* is a triple (Y,Y_0,Y_1) of 3-manifolds, such that there is a framed knot $K\subset Y$ such that Y_i is the i-surgery on K, $i=0,1$.

Definition 1.2. A set $\mathcal{S}$ of 3-manifolds is *closed under surgery triads*, if whenever two manifolds in a surgery triad belong to $\mathcal{S}$, the third manifold in the triad must also belong to $\mathcal{S}$.

Definition 1.3. A set $\mathcal{S}$ of 3-manifolds is *generated by a subset* $\mathcal{G}$ through surgery triads, if $\mathcal{S}$ is the minimal set containing $\mathcal{G}$ which is closed under surgery triads. In this case, we also say that the elements in $\mathcal{S}$ are *generated by* $\mathcal{G}$ through surgery triads.

In order to apply the aforementioned strategy to study closed, oriented, connected 3-manifolds, we need the following folklore theorem.

1) The author first learned this strategy from Peter Ozsváth, who had learned it from Tom Mrowka. The first application of this strategy was contained in an unpublished manuscript with the title “Connected sums in monopole Floer homology” written by Bloom, Mrowka and Ozsváth.

Theorem 1.4. *The set of closed, oriented, connected* 3*-manifolds is generated by* $\{S^3\}$ *through surgery triads.*

Baldwin and Bloom[2)] generalized the above theorem to get a finite generation result for bordered 3-manifolds with one boundary component (see [1] for further work on this topic).

The goal of this note is to prove a generation result for sutured manifolds. We first recall the definition of sutured manifolds.

Definition 1.5. A *sutured manifold* (M,γ) is a compact oriented 3-manifold M together with a (possibly empty) set $\gamma \subset \partial M$ of pairwise disjoint annuli.

Every component of $R(\gamma) = \partial M - \mathrm{int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be the union of those components of $R(\gamma)$ whose normal vectors point out of (or into) M. The orientations on $R(\gamma)$ must be coherent with respect to the core of γ, and hence every component of γ lies between a component of $R_+(\gamma)$ and a component of $R_-(\gamma)$.

Remark 1.6. In Gabai's original definition of sutured manifolds [2], γ may also have torus components. We will not consider this case in the current paper.

Example 1.7. Let F be a compact oriented surface,

$$P = F \times [0,1], \quad \delta = (\partial F) \times [0,1], \quad R_-(\delta) = F \times \{0\}, \quad R_+(\delta) = F \times \{1\}.$$

Then (P,δ) is a sutured manifold. In this case, we say that (P,δ) is a *product sutured manifold*.

Let F be a compact oriented connected surface with p boundary components, where $p \geqslant 0$. Let $\mathcal{X} = \mathcal{X}(F)$ be the set of connected sutured manifolds (M,γ) with $R_\pm(\gamma) \cong F$. Let $(P,\delta) = (F,\partial F) \times [0,1]$ be the product sutured manifold in $\mathcal{X}$.

The main theorem in our paper is as follows.

Theorem 1.8. *The set* $\mathcal{X}$ *is generated by* $\{(P,\delta)\}$.

Theorem 1.4 follows from the special case of Theorem 1.8 when $p = 0$ or $p = 1$ and $g = 0$. As far as we know, this is the first publicly available proof of Theorem 1.4.

Remark 1.9. When $p > 0$, the above theorem should presumably follow from the aforementioned unpublished work of Baldwin and Bloom on the finite generation of bordered manifolds, if one carefully examines the generators of bordered manifolds.

Notation. Suppose that N is a submanifold of M. Let $M \setminus\!\!\setminus N$ be the complement of an open tubular neighborhood of N in M.

2 Preliminaries

In this section, we collect some basic concepts and results about generalized Heegaard splittings, adapted to the category of sutured manifolds.

Definition 2.1. Let $(M,\gamma) \in \mathcal{X}$. A *generalized sutured Heegaard splitting* (*GSHS*) of (M,γ) is a sequence of mutually disjoint surfaces

$$\Xi = (\Sigma_1, F_1, \Sigma_2, F_2, \ldots, \Sigma_{n-1}, F_{n-1}, \Sigma_n) \tag{2.1}$$

properly embedded in M, with the following property: (M,γ) is the union of $2n$ sutured manifolds (U_i,μ_i) and (V_i,ν_i), $i = 1,\ldots,n$, such that the interiors of these manifolds are mutually disjoint, and

$$R_-(\mu_i) = F_{i-1}, \quad R_+(\mu_i) = \Sigma_i = R_-(\nu_i), \quad R_+(\nu_i) = F_i,$$

where $F_0 = R_-(\gamma)$ and $F_n = R_+(\gamma)$. Moreover, U_i is obtained by adding 1-handles to $F_{i-1} \times [0,1]$ with feet on $\mathrm{int}(F_{i-1} \times \{1\})$, and V_i is obtained by adding 2-handles to $\Sigma_i \times [0,1]$ with attaching curves on $\mathrm{int}(\Sigma_i \times \{1\})$. The manifolds U_i and V_i are called *sutured compression bodies*.

2) This work was contained in Baldwin and Bloom's unpublished manuscript with the title "The monopole category and invariants of bordered 3-manifolds".

When $n = 1$, the above definition is essentially the "Heegaard splitting for sutured manifolds" defined by Goda [3].

Definition 2.2. A GSHS Ξ as in (2.1) is *connected* if every F_i is connected.

Every manifold in $\mathcal{X}$ has a connected GSHS. In fact, such a manifold always has a GSHS with $n = 1$ [3]. In this case, we only need to check that F_0 and F_1 are connected. By the definition of $\mathcal{X}$, both F_0 and F_1 are homeomorphic to F, so this GSHS is connected.

Definition 2.3. Given a compact, oriented surface S, define

$$c(S) = \sum_{S_j \text{ is a component of } S} \max\{1 - \chi(S_j), 0\}.$$

Definition 2.4. Let $(M, \gamma) \in \mathcal{X}$, with a GSHS as in (2.1). For each Σ_i, *a pair of non-separating attaching curves* (*a pair of NSACs*) in Σ_i is a pair (α, β), where $\alpha \subset \Sigma_i$ is a non-separating simple closed curve bounding a disk in U_i, and $\beta \subset \Sigma_i$ is a non-separating simple closed curve bounding a disk in V_i. Define $\iota(\Sigma_i) = \min\{|\alpha \cap \beta| \mid (\alpha, \beta) \text{ is a pair of NSACs in } \Sigma_i\}$. When there does not exist any pair of NSACs, define $\iota(\Sigma_i) = \infty$. The *Heegaard complexity* $hc(\Sigma_i)$ of Σ_i is defined to be $(c(\Sigma_i), \iota(\Sigma_i))$.

Definition 2.5. Given a GSHS Ξ as in (2.1), suppose that τ is a permutation of $1, 2, \dots, n$, such that $hc(\Sigma_{\tau(1)}) \geqslant hc(\Sigma_{\tau(2)}) \geqslant \cdots \geqslant hc(\Sigma_{\tau(n)})$, where the order is the lexicographic order. Let $HC(\Xi) = (hc(\Sigma_{\tau(1)}), hc(\Sigma_{\tau(2)}), \dots, hc(\Sigma_{\tau(n)}))$ be the *Heegaard complexity* of Ξ.

For $(M, \gamma) \in \mathcal{X}$, define its *Heegaard complexity*

$$HC(M, \gamma) = \min\{HC(\Xi) \mid \Xi \text{ is a connected GSHS of } (M, \gamma)\},$$

where the order is the lexicographic order.

Example 2.6. The product sutured manifold (P, δ) has the trivial Heegaard splitting Ω with neither 1-handles nor 2-handles. We have $HC(\Omega) = ((c(F), \infty))$. Suppose that $(M, \gamma) \in \mathcal{X}$, and Ξ as in (2.1) is a GSHS of (M, γ). Then we always have $c(\Sigma_1) \geqslant c(F)$, and the equality holds if and only if U_1 is a product sutured manifold, in which case we have $\iota(\Sigma_1) = \infty$. As a result, we always have $HC(\Xi) \geqslant HC(\Omega)$.

If $HC(\Xi) = HC(\Omega)$, then $n = 1$ and $c(\Sigma_1) = c(F)$, which means that both U_1 and V_1 are product sutured manifolds. Hence $\Xi = \Omega$.

It follows that $HC(P, \delta) = HC(\Omega) \leqslant HC(M, \gamma)$, and the equality holds if and only if (M, γ) is the product sutured manifold.

Remark 2.7. Generalized Heegaard splittings for compact oriented 3-manifolds are defined by Scharlemann and Thompson [6]. Our Heegaard complexity of a GSHS is a refinement of the *width* defined in [6] in the sutured category. Note that the generalized Heegaard splittings in Scharlemann and Thompson's definition of the width need not to be connected.

Lemma 2.8. *Suppose that (M, γ) has a GSHS Ξ as in* (2.1)*, and $K \subset \Sigma_i$ is a knot with the surface framing. Let Σ_i^- be a parallel copy of Σ_i below Σ_i, and let Σ_i^+ be a parallel copy of Σ_i above Σ_i. Let $K^\pm \subset \Sigma_i^\pm$ be the copies of K in $\Sigma_i^\pm$. Let M' be the sutured manifold obtained by* 0*-surgery on K, and let $F' \subset M'$ be the surface obtained from $\Sigma_i \setminus\setminus K$ by capping off the two boundary components parallel to K with disks. Then M' has a GSHS*

$$\Xi' = (\Sigma_1, F_1, \dots, F_{i-1}, \Sigma_i^-, F', \Sigma_i^+, F_i, \dots, \Sigma_{n-1}, F_{n-1}, \Sigma_n). \tag{2.2}$$

The sutured compression body between F_{i-1} and Σ_i^- is isotopic to U_i in $M \setminus\setminus K$, and the sutured compression body between Σ_i^- and F' is obtained by adding a 2*-handle to $\Sigma_i^- \times [0,1]$, with attaching curve K^-. Similarly, the sutured compression body between F' and Σ_i^+ is obtained by adding a* 1*-handle to $F' \times [0,1]$, with belt curve K^+, and the sutured compression body between Σ_i^+ and F_i is isotopic to V_i in $M \setminus\setminus K$.*

Moreover, if Ξ is connected and K is non-separating in Σ_i, then Ξ' is also connected.

Proof. The two new disks in F' split the new solid torus in M' into two $D^2 \times [0,1]$'s, which may be viewed as 1-handles attached to two different sides of F'. The belt curves of these two 1-handles are $K^\pm$.

The 1-handle attached to the bottom side of F' can be viewed as a 2-handle attached to the top side of Σ_i^-. So we get a GSHS as in the statement of this lemma.

If Ξ is connected and K is non-separating in Σ_i, then F' is also connected, so Ξ' is connected. $\square$

The following lemma is well known (see, for example, [5, Lemma I.4, Chapter 9]).

Lemma 2.9. *Suppose that (M,γ) has a GSHS as in (2.1), and $K\subset\Sigma_i$ is a knot with the surface framing. We push K slightly into U_i and do $(+1)$-surgery (resp. (-1)-surgery) on it, the resulting manifold is denoted by M''. Then M'' has a GSHS*

$$\Xi''=(\Sigma_1,F_1,\dots,F_{i-1},\Sigma_i'',F_i,\dots,\Sigma_{n-1},F_{n-1},\Sigma_n), \tag{2.3}$$

where Σ_i'' is isotopic to Σ_i in $M\setminus\setminus K$. The only difference between Ξ and Ξ'' is that the curves on Σ_i'' which bound disks in U_i are exactly the images of the corresponding curves in Σ_i under the left-handed (resp. right-handed) Dehn twist along K.

3 Proof of the main theorem

In this section, we prove Theorem 1.8. We induct on $HC(M,\gamma)$, and the inductive step is the following proposition.

Proposition 3.1. *Let $(M,\gamma)\in\mathcal{X}$ be a non-product sutured manifold. Then (M,γ) is generated by*

$$\mathcal{X}^{<M}=\{(N,\zeta)\in\mathcal{X}\mid HC(N,\zeta)<HC(M,\gamma)\}.$$

From now on, let $(M,\gamma)\in\mathcal{X}$ be a non-product sutured manifold, and let Ξ as in (2.1) be a connected GSHS of (M,γ) with $HC(\Xi)=HC(M,\gamma)$. Let $i=\tau(1)$, where τ is as in Definition 2.5, and let (α,β) be a pair of NSACs in Σ_i with $|\alpha\cap\beta|=\iota(\Sigma_i)$ if $\iota(\Sigma_i)<\infty$. We orient α and β arbitrarily.

Lemma 3.2. *With the above notations, we have $\iota(\Sigma_i)\neq 1$ and $\iota(\Sigma_i)\neq\infty$.*

Proof. If $|\alpha\cap\beta|=1=\iota(\Sigma_i)$, we can cancel the 1-handle corresponding to α with the 2-handle corresponding to β, and thus decrease $c(\Sigma_i)$ and hence $HC(\Xi)$. The new GSHS has the same collection of F_i's, so it is still connected. This contradicts the choice of Ξ.

If $\iota(\Sigma_i)=\infty$, since Ξ is connected, either there are no 1-handles in U_i, or there are no 2-handles in V_i, i.e., one of U_i,V_i must be a product sutured manifold.

Without loss of generality, we may assume that U_i is a product sutured manifold. Then the 2-handles in V_i may be viewed as attached to Σ_{i-1}. We then get a new connected GSHS of (M,γ):

$$\Xi'=(\Sigma_1,F_1,\dots,F_{i-2},\Sigma_{i-1}',F_i,\Sigma_{i+1},F_{i+1},\dots,\Sigma_n),$$

where Σ_{i-1}' is just Σ_{i-1}, but now there are additional attaching curves bounding disks in V_{i-1}', the sutured compression body between Σ_{i-1}' and F_i. So we have

$$c(\Sigma_{i-1}')=c(\Sigma_{i-1}),\quad \iota(\Sigma_{i-1}')\leqslant\iota(\Sigma_{i-1}),$$

which implies $hc(\Sigma_{i-1}')\leqslant hc(\Sigma_{i-1})$.

Moreover, in $HC(\Xi')$, there is no component corresponding to the component $hc(\Sigma_i)$ in $HC(\Xi)$. So we have $HC(\Xi')<HC(\Xi)$, which is a contradiction to the choice of Ξ. $\square$

Lemma 3.3. *Suppose that there exist a non-separating simple closed curve $K\subset\Sigma_i$, and a curve α_1 which is the image of α under a (left-handed or right-handed) Dehn twist along K, satisfying the following two conditions:*

- *either*

$$\max\{|K\cap\alpha|,|K\cap\beta|\}<|\alpha\cap\beta|, \tag{3.1}$$

or

$$|K\cap\alpha|=|K\cap\beta|=1; \tag{3.2}$$

- α_1 *can be isotoped so that either*

$$|\alpha_1 \cap \beta| < |\alpha \cap \beta| \tag{3.3}$$

or

$$|\alpha_1 \cap \beta| = 1. \tag{3.4}$$

Then the conclusion of Proposition 3.1 *holds for* (M, γ).

Proof. Let M' be the 0-surgery on K, and let M'' be the $(+1)$ or (-1)-surgery on K, with the sign chosen so that the corresponding Dehn twist (as in Lemma 2.9) applying to α yields α_1.

By Lemma 2.8, M' has a GSHS Ξ' as in (2.2). Since K is non-separating in Σ, F' is still connected, so Ξ' is connected. On Σ_i^-, we have a pair of NSACs (α, K). On Σ_i^+, we have a pair of NSACs (K, β). If (3.1) holds, then $\max\{\iota(\Sigma_i^-), \iota(\Sigma_i^+)\} < \iota(\Sigma_i)$. If (3.2) holds, then we can cancel the only 2-handle between Σ_i^- and F' with the 1-handle corresponding to α, and cancel the only 1-handle between F' and Σ_i^+ with the 2-handle corresponding to β. This gives us a new connected GSHS

$$(\Sigma_1, F_1, \dots, F_{i-1}, \Sigma_i^*, F_i, \dots, \Sigma_{n-1}, F_{n-1}, \Sigma_n)$$

with $c(\Sigma_i^*) = c(F') < c(\Sigma_i)$. In either case, we have $HC(M') < HC(M)$.

By Lemma 2.9, M'' has a connected GSHS Ξ'' as in (2.3), and a pair of NSACs (α_1, β) on Σ_i''. If (3.3) holds, $\iota(\Sigma_i'') < \iota(\Sigma_i)$. If (3.4) holds, we can cancel the corresponding 1-handle/2-handle pair to decrease $c(\Sigma_i'')$. In either case, we have $HC(M'') < HC(M)$.

Now our conclusion holds since (M, M', M'') is a surgery triad. □

Lemma 3.4. *If* $\alpha \cap \beta = \emptyset$, *then the conclusion of Proposition* 3.1 *holds for* (M, γ).

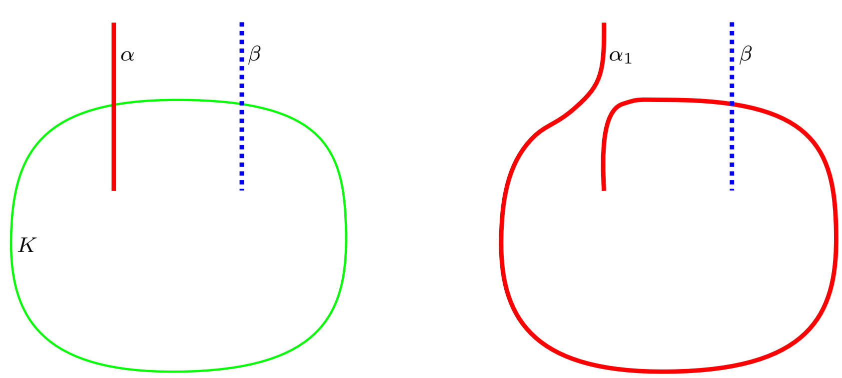
Proof. Since α is non-separating, $\Sigma_i \setminus\!\!\setminus \alpha$ is connected. Since β is non-separating in Σ_i, $\Sigma_i \setminus\!\!\setminus (\alpha \cup \beta)$ has 1 or 2 components, and none of the components has only one boundary component. In either case, we can find a simple closed curve K, which intersects each of α and β exactly once.

Let α_1 be the image of α under the left-handed Dehn twist along K, as shown in Figure 1. Then $|\alpha_1 \cap \beta| = 1$.

Now our conclusion holds by Lemma 3.3. □

Lemma 3.5. *If* $\alpha \cap \beta$ *has exactly two points, which have opposite signs, then the conclusion of Proposition* 3.1 *holds for* (M, γ).

Proof. Let $x \in \alpha \cap \beta$. Consider the local picture near x as in Figure 2. Since $|\alpha \cap \beta| = 2$, every component of $\Sigma \setminus (\alpha \cup \beta)$ must contain at least one of the four corners at x. The corners are labelled so that Corners 1 and 3 are opposite, and Corners 2 and 4 are opposite. We claim that there exists a pair of opposite corners at x, which are contained in the same component.



(a) The original diagram (b) The new diagram

Figure 1 (Color online) The local picture near K

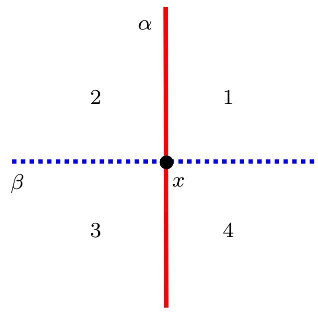


Figure 2 (Color online) The local picture near x. The four angles are labeled with 1–4

Assume that the claim is not true. Since α is non-separating, there exists a path in $\Sigma_i \setminus \alpha$ connecting the two sides of α. Note that whenever the path crosses β, we either move between the component containing Corner 1 and the component containing Corner 4, or move between the component containing Corner 2 and the component containing Corner 3. It follows that one of Corners 1 and 4 is in the same component as one of Corners 2 and 3. By our assumption, these two angles cannot be opposite corners. Without loss of generality, we may assume that Corners 1 and 2 are in the same component. Similarly, since β is non-separating, one of Corners 1 and 2 is in the same component as one of Corners 3 and 4. Since Corners 1 and 2 are already in the same component, this forces a pair of opposite corners to be in the same component.

Using the claim, we can find a simple closed curve K which intersects each of α and β exactly once, and

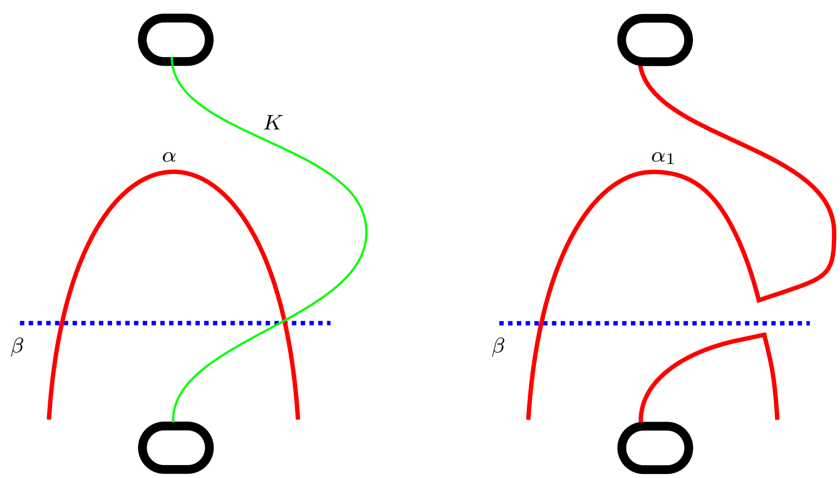
$$K \cap \alpha = K \cap \beta = \{x\}.$$

As in Figure 3(b), the curve α_1 is obtained from α by a Dehn twist along K, and $|\alpha_1 \cap \beta| = 1$. Now our conclusion follows from Lemma 3.3. □

Lemma 3.6. *If there exist two intersection points $x, y \in \alpha \cap \beta$ of the same sign, and an arc a in α connecting x to y, such that the interior of a does not intersect β, then the conclusion of Proposition* 3.1 *holds for (M, γ).*

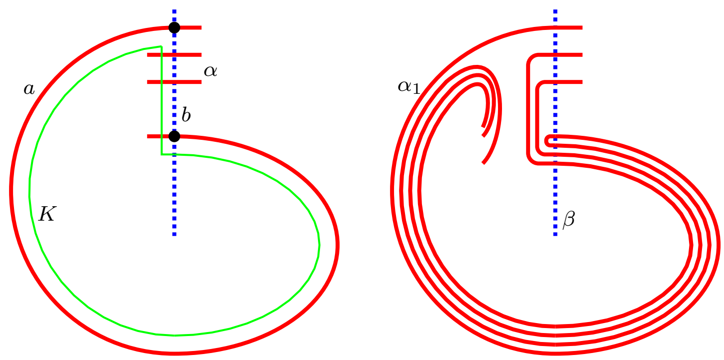
Proof. Let b be an arc in β connecting x to y. Then $a \cap b = \{x, y\}$.

Let $K \subset \Sigma_i$ be a small perturbation of $a \cup b$, such that $K \cap (a \cup b) = \emptyset$.



(a) The original diagram (b) The new diagram

Figure 3 (Color online) In each part of the picture, two ovals are glued together by a vertical reflection to form a tube in the surface



(a) The original diagram (b) The new diagram

Figure 4 (Color online) The two black dots are intersection points between α and β of the same sign. The arc b may also intersect α in its interior. In this case, α_1 is obtained from α by a left-handed Dehn twist along K

As in Figure 4(a), we clearly have

$$|K \cap \alpha| < |\alpha \cap \beta|$$

and

$$|K \cap \beta| = 1 < |\alpha \cap \beta|.$$

Since $|K \cap \beta| = 1$, K is non-separating.

As in Figure 4(b), we can choose an appropriate Dehn twist along K, such that α_1 can be isotoped so that (3.3) holds.

Now our conclusion follows from Lemma 3.3. □

By Lemmas 3.2 and 3.4–3.6, in order to prove Proposition 3.1, the only remaining case is that the signs of the intersection points in $\alpha \cap \beta$ appear as positive and negative alternatively on α, and there are more than 2 intersection points.

Let $G = \Sigma_i \setminus\setminus \beta$, with $\partial G = \beta_0 \sqcup \beta_1$. Then $\alpha \cap G$ consists of arcs $a_1, \dots, a_{2m}$ for some $m > 1$, in the cyclic order they appear on α, with

$$\partial a_j \subset \beta_k, \quad k = 0, 1, \quad j \equiv k \pmod 2.$$

Moreover, α splits β into arcs $b_1, \dots, b_{2m}$, in the cyclic order they appear on β. For each $j \in \{1, \dots, 2m\}$, pick two points z_j^0 and z_j^1, one by each side of b_j, such that z_j^k is near β_k in G, $k = 0, 1$ (see Figure 5(a) for a picture of z_j^k).

Lemma 3.7. *There exist a component A of $G \setminus\setminus (\bigcup_j a_j)$, and two different indices $j_0, j_1 \in \{1, \dots, 2m\}$, such that $z_{j_0}^0, z_{j_1}^1 \in A$.*

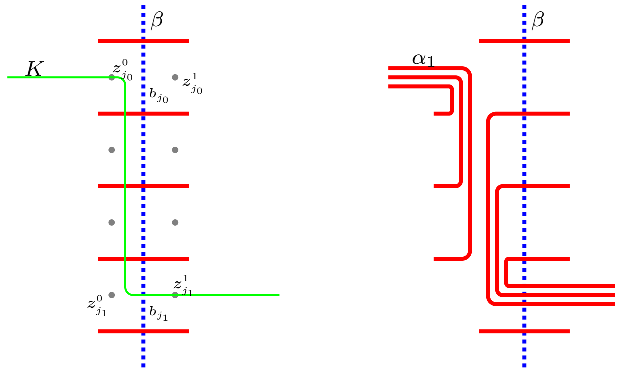
Proof. Let $G_0 = G \setminus\setminus (\bigcup_{j \text{ is even}} a_j)$. Since $a_j \cap \beta_1 = \emptyset$ when j is even, there exists a distinguished component A_0 of G_0, such that $\beta_1 \subset A_0$. Of course, $A_0 \cap \beta_0 \neq \emptyset$ since every a_j intersects β_0 when j is even. Let A be a component of $A_0 \setminus\setminus (\bigcup_{j \text{ is odd}} a_j)$ which intersects β_0. Then A also intersects β_1 since every a_j intersects β_1 when j is odd. So $A \cap \beta_0 \neq \emptyset$ and $A \cap \beta_1 \neq \emptyset$. It follows that

$$A \cap \{z_1^k, \dots, z_{2m}^k\} \neq \emptyset \quad \text{for each } k \in \{0, 1\}.$$

Assume that the conclusion of this lemma does not hold. Then there exists $j \in \{1, \dots, 2m\}$, such that

$$A \cap \{z_1^k, \dots, z_{2m}^k\} = \{z_j^k\} \quad \text{for each } k \in \{0, 1\}. \tag{3.5}$$

Let $x \in \partial b_j$. Suppose that x is the common endpoint of a_ℓ and $a_{\ell+1}$ for some ℓ. Since the other endpoint of a_ℓ is also a corner of A, it follows from (3.5) that $\partial a_\ell = \partial b_j$. The same argument shows that $\partial a_{\ell+1} = \partial b_j$. This implies that $|\alpha \cap \beta| = 2$, which is a contradiction to our assumption that $m > 1$. □



(a) The original diagram (b) The new diagram

Figure 5 (Color online) The local picture in a neighborhood of an arc in β

Now we are ready to deal with the last case of Proposition 3.1.

Lemma 3.8. *Suppose that the signs of the intersection points in $\alpha \cap \beta$ appear as positive and negative alternatively on α. Then the conclusion of Proposition* 3.1 *holds for* (M, γ).

Proof. Let $z^0_{j_0}$ and $z^1_{j_1}$ be as in Lemma 3.7. Since they are in the same component of $G \setminus\!\!\setminus (\cup_j a_j)$, we can connect them by an arc $e_1 \subset \Sigma_i$ with $e_1 \cap (\alpha \cup \beta) = \emptyset$. Since $j_0 \neq j_1$, we can connect $z^0_{j_0}$ and $z^1_{j_1}$ by an arc $e_2 \subset \Sigma_i$ satisfying

$$|e_2 \cap \beta| = 1, \quad 0 < d = |e_2 \cap \alpha| < 2m. \tag{3.6}$$

Let $K = e_1 \cup e_2$. Then (3.6) holds if we replace e_2 with K. In particular, this implies that K is non-separating (see Figure 5(a)).

As in Figure 5(b), we can apply a Dehn twist along K to α, to get a curve α_1. After an isotopy, we get $|\alpha_1 \cap \beta| = 2m - d < 2m$.

Now our conclusion follows from Lemma 3.3. □

This finishes the proof of Proposition 3.1. Combining Proposition 3.1 with Example 2.6, we get Theorem 1.8 by a straightforward induction.

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