Knot Floer homology detects fibred knots

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Dedicated to Professor Boju Jiang on the occasion of his 70th birthday

Abstract. Ozsváth and Szabó conjectured that knot Floer homology detects fibred knots in S^3. We will prove this conjecture for null-homologous knots in arbitrary closed 3-manifolds. Namely, if K is a knot in a closed 3-manifold Y, $Y - K$ is irreducible, and $\widehat{HFK}(Y, K)$ is monic, then K is fibred. The proof relies on previous works due to Gabai, Ozsváth–Szabó, Ghiggini and the author. A corollary is that if a knot in S^3 admits a lens space surgery, then the knot is fibred.

1 Introduction

Knot Floer homology was introduced independently by Ozsváth–Szabó [13] and by Rasmussen [19]. For any null-homologous knot $K \subset Y^3$ with Seifert surface F, one can associate to it some abelian groups $\widehat{HFK}(Y, K, [F], i)$ for $i \in \mathbb{Z}$. The knot Floer homology

$$\widehat{HFK}(Y, K) \cong \bigoplus_{i \in \mathbb{Z}} \widehat{HFK}(Y, K, [F], i)$$

is a finitely generated abelian group.

A lot of topological information of the knot are contained in knot Floer homology, in particular in the topmost filtration level. For example, Ozsváth and Szabó proved that the topmost filtration level of $\widehat{HFK}$ for a knot in S^3 is exactly the genus of the knot (see [14]).

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When K is a fibred knot, it is shown that the topmost group of $\widehat{HFK}(Y, K)$ is a single $\mathbb{Z}$ [16]. Ozsváth and Szabó conjectured that the converse is also true for knots in S^3 [18].

In this paper, we are going to prove this conjecture. Our main theorem is:

Theorem 1.1 *Suppose K is a null-homologous knot in a closed, oriented, connected 3-manifold Y, $Y - K$ is irreducible, and F is a genus-g Seifert surface of K. If $\widehat{HFK}(Y, K, [F], g) \cong \mathbb{Z}$, then K is fibred, and F is a fibre of the fibration.*

An oriented link L in Y is called a *fibred link*, if $Y - L$ fibres over the circle, and L is the oriented boundary of the fibre. We have the following corollary of Theorem 1.1:

Corollary 1.2 *Suppose Y is a closed, oriented, connected 3-manifold, L is a null-homologous oriented link in Y, $Y - L$ is irreducible, and F is a Seifert surface of L. If $\widehat{HFK}\left(Y, L, \frac{|L| - \chi(F)}{2}\right) \cong \mathbb{Z}$, then L is a fibred link, and F is a fibre of the fibration.*

The proof of this corollary will be given in Sect. 7.

A rational homology sphere Y is called an *L-space*, if the rank of $\widehat{HF}(Y)$ is equal to $|\mathrm{H}_1(Y; \mathbb{Z})|$. Many 3-manifolds are *L*-spaces, for example, the manifolds which admit spherical structures are *L*-spaces. An immediate corollary of Theorem 1.1 is:

Corollary 1.3 *If a knot $K \subset S^3$ admits an L-space surgery, then K is a fibred knot. In particular, any knot that admits a lens space surgery is fibred.*

Proof As a corollary of [17, Proposition 9.5], if a rational surgery on K yields an L-space, then K also admits an L-space surgery with integer coefficient. Using [15, Theorem 1.2], we conclude that $\widehat{HFK}(K, g) \cong \mathbb{Z}$. Thus the desired result follows from Theorem 1.1.   □

Corollary 1.4 *Suppose Y is an L-space, $K \subset Y$ is a null-homologous knot with genus $g > 1$. If the 0-surgery on K is a surface bundle over S^1, then K is fibred.*

Proof With the above conditions, one can prove that

$$HF^+(Y_0(K), [g - 1]) \cong \widehat{HFK}(Y, K, g).$$

In fact, the proof is exactly the same as the proof of [13, Corollary 4.5], so we will not give the details. The reader should note that, since Y is an L-space, $HF^+(Y)$ is isomorphic to the direct sum of some copies of $\mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$, hence the map ψ as in the proof of [13, Corollary 4.5] is surjective. Thus the argument there can be used.

Now since $Y_0(K)$ fibres over the circle, we have $HF^+(Y_0(K), [g-1])$ $\cong \mathbb{Z}$, so $\widehat{HFK}(Y, K, g) \cong \mathbb{Z}$. It is easy to show that $Y - K$ is irreducible, hence K is fibred by Theorem 1.1.   □

Remark 1.5 The homology class $[F]$ defines a homomorphism

$$f : \pi_1(Y - K) \to \mathbb{Z}$$

by counting the intersection numbers of $[F]$ with loops. The famous Stallings' Fibration Theorem [20] says that K is a fibred knot with fibre in the homology class of $[F]$ if and only if ker f is finitely generated. Hence Theorem 1.1 indicates a mysterious relationship between Heegaard Floer homology and the fundamental group.

Theorem 1.1 was previously examined in various sporadic cases, and some theoretical evidences were given in [12], but the first real progress was made by Ghiggini in [5], where a strategy to approach this conjecture was proposed, and the special case of genus-one knots in S^3 was proved. Ghiggini's strategy plays an essential role in the present paper, we will apply this strategy by using a method introduced by Gabai [4]. Another key ingredient of this paper is the study of decomposition formulas for knot Floer homology, which is based on [12].

The paper is organized as follows. In Sect. 2, we give some backgrounds on sutured manifolds. We will introduce a sutured manifold invariant which naturally comes from knot Floer homology. We also present a construction of certain Heegaard diagrams. In Sect. 3, we prove a homological version of the main theorem. Section 4 is devoted to prove the horizontal decomposition formula for knot Floer homology. In Sect. 5, we will prove a major technical theorem: the decomposition formula for knot Floer homology, in the case of decomposing along a separating product annulus. In Sect. 6, we use Gabai's method to study Ghiggini's strategy. As a result, we get a clearer picture of the sutured manifold structure of the knot complement, namely, Theorem 6.2. Section 7 contains the proof of the main theorem, we use the decomposition formulas we proved (especially Theorem 5.1) to reduce the problem to the case that we already know.

2 Preliminaries

2.1 Sutured manifold decomposition The theory of sutured manifold decomposition was introduced by Gabai in [2]. We will briefly review the basic definitions.

Definition 2.1 A *sutured manifold* (M, γ) is a compact oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. The core of each component of $A(\gamma)$ is a *suture*, and the set of sutures is denoted by $s(\gamma)$.

Every component of $R(\gamma) = \partial M - \text{int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be the union of those components of $R(\gamma)$ whose normal vectors point out of (or into) M. The orientations on $R(\gamma)$ must be coherent with respect to $s(\gamma)$, hence every component of $A(\gamma)$ lies between a component of $R_+(\gamma)$ and a component of $R_-(\gamma)$.

Definition 2.2 [11, Definition 2.2] A *balanced sutured manifold* is a sutured manifold (M, γ) satisfying

(1) M has no closed components.
(2) Every component of ∂M intersects γ nontrivially.
(3) $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$.

Definition 2.3 Let S be a compact oriented surface with connected components $S_1, \ldots, S_n$. We define

$$x(S) = \sum_i \max\{0, -\chi(S_i)\}.$$

Let M be a compact oriented 3-manifold, A be a compact codimension-0 submanifold of ∂M. Let $h \in \mathrm{H}_2(M, A)$. The *Thurston norm* $x(h)$ of h is defined to be the minimal value of $x(S)$, where S runs over all the properly embedded surfaces in M with $\partial S \subset A$ and $[S] = h$.

Definition 2.4 A sutured manifold (M, γ) is *taut*, if M is irreducible, and $R(\gamma)$ is Thurston norm minimizing in $\mathrm{H}_2(M, \gamma)$.

Definition 2.5 Let (M, γ) be a sutured manifold, and S a properly embedded surface in M, such that no component of ∂S bounds a disk in $R(\gamma)$ and no component of S is a disk with boundary in $R(\gamma)$. Suppose that for every component λ of $S \cap \gamma$, one of (1)–(3) holds:

(1) λ is a properly embedded non-separating arc in γ.
(2) λ is a simple closed curve in an annular component A of γ in the same homology class as $A \cap s(\gamma)$.
(3) λ is a homotopically nontrivial curve in a toral component T of γ, and if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $\mathrm{H}_1(T)$.

Then S is called a *decomposition surface*, and S defines a *sutured manifold decomposition*

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma'),$$

where $M' = M - \text{int}(\text{Nd}(S))$ and

$$\gamma' = (\gamma \cap M') \cup \text{Nd}(S'_+ \cap R_-(\gamma)) \cup \text{Nd}(S'_- \cap R_+(\gamma)),$$
$$R_+(\gamma') = ((R_+(\gamma) \cap M') \cup S'_+) - \text{int}(\gamma'),$$
$$R_-(\gamma') = ((R_-(\gamma) \cap M') \cup S'_-) - \text{int}(\gamma'),$$

where S'_+ (S'_-) is that component of $\partial \text{Nd}(S) \cap M'$ whose normal vector points out of (into) M'.

Definition 2.6 A decomposition surface is called a *product disk*, if it is a disk which intersects $s(\gamma)$ in exactly two points. A decomposition surface is called a *product annulus*, if it is an annulus with one boundary component in $R_+(\gamma)$, and the other boundary component in $R_-(\gamma)$.

Definition 2.7 A decomposition surface S in a balanced sutured manifold is called a *horizontal surface*, if S has no closed component, $|\partial S| = |s(\gamma)|$, $[S] = [R_+(\gamma)] \in H_2(M, \gamma)$, and $\chi(S) = \chi(R_+)$.

Definition 2.8 A balanced sutured manifold (M, γ) is *vertically prime*, if any horizontal surface $S \subset M$ is parallel to either $R_-(\gamma)$ or $R_+(\gamma)$.

2.2 Knot Floer homology and an invariant of sutured manifolds Heegaard Floer homology has been proved to have very close relationship with Thurston norm [14]. The definition of Thurston norm is purely topological (or combinatorial) [21], while the definition of Heegaard Floer homology involves symplectic geometry and analysis. The bridge that connects these two seemingly different topics is taut foliation.

The fundamental method of constructing taut foliations is sutured manifold decomposition [2]. Thus we naturally expect that, by studying the behavior of Heegaard Floer homology under sutured manifold decomposition, we can get better understanding of the relationship between Heegaard Floer homology and Thurston norm.

The first approach in such direction was taken in [12], where the "sutured Heegaard diagrams" for knots are introduced. Using them, one can study a very special case of sutured manifold decomposition: the Murasugi sum. Later on, Juhász introduced an invariant for sutured manifolds, called "sutured Floer homology", and proved some properties [11].

In this subsection, we will introduce another invariant of sutured manifolds, which naturally comes from knot Floer homology. We will prove some decomposition formulas for this invariant in Sects. 4 and 5.

Suppose $L \subset Y$ is a null-homologous oriented link, F is a Seifert surface of L. Decompose $Y - \text{int}(\text{Nd}(L))$ along F, we get a balanced sutured manifold (M, γ). The argument in [12, Proposition 3.5] shows that, if we cut open Y along F, reglue by a homeomorphism of F which is the identity on the boundary, to get a new link L' in a new manifold Y', then

$$\widehat{HFK}\left(Y, L, \frac{|\partial F| - \chi(F)}{2}\right) \cong \widehat{HFK}\left(Y', L', \frac{|\partial F| - \chi(F)}{2}\right)$$

as abelian groups. Therefore, $\widehat{HFK}\left(Y, L, \frac{|\partial F| - \chi(F)}{2}\right)$ can be viewed as an invariant for the sutured manifold (M, γ). (For simplicity, let $\mathrm{i}(F)$ denote $\frac{|\partial F| - \chi(F)}{2}$.)

More precisely, suppose (M, γ) is a balanced sutured manifold, $R_{\pm}(\gamma)$ are connected surfaces. There exists a diffeomorphism

$$\psi \colon R_+(\gamma) \to R_-(\gamma),$$

such that for each component A of γ, ψ maps one boundary component of A onto the other boundary component. We glue $R_+(\gamma)$ to $R_-(\gamma)$ by ψ, thus get a manifold with boundary consisting of tori. We fill each boundary torus by a solid torus whose meridian intersects $s(\gamma)$ exactly once. Now we get a closed 3-manifold Y. Let L be the union of the cores of the solid tori. The pair (Y, L) is denoted by $\iota(M, \gamma)$. Of course, $\iota(M, \gamma)$ depends on the way we glue R_+ to R_- and the way we fill in the solid tori. In our case, changing the filling is equivalent to changing the gluing map by Dehn twists along the components of ∂R_+. By the remark in the last paragraph, the abelian group

$$\widehat{HFK}(\iota(M, \gamma), \mathrm{i}(R_+(\gamma)))$$

is independent of the choice of the gluing, hence it is independent of the choice of $\iota(M, \gamma)$.

Proposition 2.9 *There is a well-defined invariant $\widehat{HFS}$ for balanced sutured manifolds, which is characterized by the following two properties.*

(1) If (M, γ) is a balanced sutured manifold, $R_{\pm}(\gamma)$ are connected, then

$$\widehat{HFS}(M, \gamma) \cong \widehat{HFK}(\iota(M, \gamma), \mathrm{i}(R_+(\gamma))).$$

(2) If

$$(M, \gamma) \overset{a \times I}{\leadsto} (M', \gamma'),$$

is the decomposition along a product disk then

$$\widehat{HFS}(M, \gamma) \cong \widehat{HFS}(M', \gamma').$$

Proof The inverse operation of the decomposition along a product disk, is "adding a product 1-handle" with feet at the suture. We first claim that, if (M, γ) is a balanced sutured manifold with $R_\pm(\gamma)$ connected, and (M_1, γ_1) is obtained by adding a product 1-handle to (M, γ), then

$$\widehat{HFK}(\iota(M, \gamma), \mathfrak{i}(R_+(\gamma))) \cong \widehat{HFK}(\iota(M_1, \gamma_1), \mathfrak{i}(R_+(\gamma_1))).$$

In fact, one possible choice of $\iota(M_1, \gamma_1)$ can be gotten by plumbing $\iota(M, \gamma)$ with $(S^2 \times S^1, \Pi)$. Here Π is a link in $S^2 \times S^1$ which consists of two copies of *point* $\times S^1$, but with different orientations. Now we can apply [12, Lemma 4.4] to get the claim.

Given a balanced sutured manifold (M, γ), we can add to it some product 1-handles with feet at the suture to get a new balanced sutured manifold (M_1, γ_1), such that $R_\pm(\gamma_1)$ are connected. We then define

$$\widehat{HFS}(M, \gamma) = \widehat{HFK}(\iota(M_1, \gamma_1), \mathfrak{i}(R_+(\gamma_1))).$$

Now we want to prove that $\widehat{HFS}(M, \gamma)$ is independent of the choice of (M_1, γ_1). For this purpose, let (M_2, γ_2) be another sutured manifold obtained by adding product 1-handles to (M, γ), and $R_\pm(\gamma_2)$ are connected. We can assume that the feet of the product 1-handles for (M_1, γ_1) and (M_2, γ_2) are mutually different. Let (M_3, γ_3) be the sutured manifold obtained by adding all these product 1-handles (either for M_1 or for M_2) to (M, γ). By the claim proved in the first paragraph, we have

$$\widehat{HFK}(\iota(M_3, \gamma_3), \mathfrak{i}(R_+(\gamma_3))) \cong \widehat{HFK}(\iota(M_1, \gamma_1), \mathfrak{i}(R_+(\gamma_1))),$$
$$\widehat{HFK}(\iota(M_3, \gamma_3), \mathfrak{i}(R_+(\gamma_3))) \cong \widehat{HFK}(\iota(M_2, \gamma_2), \mathfrak{i}(R_+(\gamma_2))).$$

Therefore, $\widehat{HFS}(M, \gamma)$ is well-defined.

Property (1) holds by definition, and Property (2) can be proved by the same argument as above. □

2.3 Relative Morse functions and sutured diagrams

Suppose K is a null-homologous knot in Y, F is a Seifert surface for K. In [12], the notion of "sutured Heegaard diagrams" was introduced. Such diagrams are useful to compute $\widehat{HFK}(Y, K, [F], g)$. A construction of sutured Heegaard diagrams was given in the proof of [12, Theorem 2.1].

In this subsection, we will present a slightly different construction, which is based on relative Morse functions. This construction will be useful later.

Definition 2.10 [12, Definition 2.2] A double pointed Heegaard diagram

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta_0} \cup \{\mu\}, w, z)$$

for (Y, K) is a *sutured Heegaard diagram*, if it satisfies:

(Su0) There exists a subsurface $\mathcal{P} \subset \Sigma$, bounded by two curves $\alpha_1 \in \boldsymbol{\alpha}$ and λ. g denotes the genus of $\mathcal{P}$.

(Su1) λ is disjoint from $\boldsymbol{\beta_0}$. μ does not intersect any α curves except α_1. μ intersects λ transversely in exactly one point, and intersects α_1 transversely in exactly one point. $w, z \in \lambda$ lie in a small neighborhood of $\lambda \cap \mu$, and on different sides of μ. (In practice, we often push w, z off λ into $\mathcal{P}$ or $\Sigma - \mathcal{P}$.)

(Su2) $(\boldsymbol{\alpha} - \{\alpha_1\}) \cap \mathcal{P}$ consists of $2g$ arcs, which are linearly independent in $H_1(\mathcal{P}, \partial \mathcal{P})$. Moreover, $\Sigma - \boldsymbol{\alpha} - \mathcal{P}$ is connected.

Construction 2.11 Suppose (M, γ) is the sutured manifold obtained by cutting $Y - \mathrm{int}(\mathrm{Nd}(K))$ open along F. Let $\psi \colon R_+(\gamma) \to R_-(\gamma)$ be the gluing map. Namely, if we glue $R_+(\gamma)$ to $R_-(\gamma)$ by ψ, then we get back the manifold $Y - \mathrm{int}(\mathrm{Nd}(K))$. We will construct a Heegaard diagram for the pair (Y, K). The construction consists of 4 steps.

Step 0 A relative Morse function.

Consider a self-indexed relative Morse function u on (M, γ). Namely, u satisfies:

(1) $u(M) = [0, 3]$, $u^{-1}(0) = R_-(\gamma)$, $u^{-1}(3) = R_+(\gamma)$.

(2) u has no degenerate critical points. u is the standard height function near γ. $u^{-1}\{$critical points of index $i\} = i$.

(3) u has no critical points on $R(\gamma)$.

Let $\widetilde{F} = u^{-1}\left(\frac{3}{2}\right)$. $\partial \widetilde{F}$ is denoted by $\widetilde{\lambda}$. Similarly, the boundary components of $R_\pm(\gamma)$ are denoted by $\lambda_\pm$.

Suppose u has r index-1 critical points, then the genus of $\widetilde{F}$ is $g + r$. The gradient $-\nabla u$ generates a flow ϕ_t on M. There are $2r$ points on $R_+(\gamma)$, which are connected to index-2 critical points by flowlines. We call these points "bad" points. Similarly, there are $2r$ bad points on $R_-(\gamma)$, which are connected to index-1 critical points by flowlines.

Step 1 Construct the curves.

Choose a small disk D_+ in a neighborhood of λ_+ in $R_+(\gamma)$. Choose an arc $\delta_+ \subset R_+(\gamma)$ connecting D_+ to λ_+. Flow D_+ and δ_+ by ϕ_t, their images on $\widetilde{F}$ and $R_-(\gamma)$ are $\widetilde{D}, D_-, \widetilde{\delta}, \delta_-$. (Of course, we choose D_+ and δ_+ generically, so that the flowlines starting from them do not terminate at critical points.) We can suppose the gluing map ψ maps δ_+ onto δ_-, D_+ onto D_-. Let $A_\pm = R_\pm(\gamma) - \mathrm{int}(D_\pm)$, $\widetilde{A} = \widetilde{F} - \mathrm{int}(\widetilde{D})$.

On $\widetilde{F}$, there are r simple closed curves $\widetilde{\alpha}_{2g+2}, \ldots, \widetilde{\alpha}_{2g+1+r}$, which are connected to index-1 critical points by flowlines. And there are r simple closed curves $\widetilde{\beta}_{2g+2}, \ldots, \widetilde{\beta}_{2g+1+r}$, which are connected to index-2 critical points by flowlines.

Choose $2g$ disjoint arcs $\xi_2^-, \ldots, \xi_{2g+1}^- \subset A_-$, such that their endpoints lie on λ_-, and they are linearly independent in $H_1(A_-, \partial A_-)$. We also

suppose they are disjoint from δ_- and the bad points. Let $\xi_i^+ = \psi^{-1}(\xi_i^-)$. We also flow back $\xi_2^-, \ldots, \xi_{2g+1}^-$ by ϕ_{-t} to $\widetilde{F}$, the images are denoted by $\widetilde{\xi}_2, \ldots, \widetilde{\xi}_{2g+1}$.

Choose $2g$ disjoint arcs $\eta_2^+, \ldots, \eta_{2g+1}^+ \subset A_+$, such that their endpoints lie on ∂D_+, and they are linearly independent in $H_1(A_+, \partial A_+)$. We also suppose they are disjoint from δ_+ and the bad points. Flow them by ϕ_t to $\widetilde{F}$, the images are denoted by $\widetilde{\eta}_2, \ldots, \widetilde{\eta}_{2g+1}$.

Step 2 Construct a diagram.

Let

$$\Sigma = A_+ \cup \widetilde{A} \cup \{2 \text{ annuli}\},$$

where one annulus is $\phi_{[0,3/2]}(\partial D_+)$, the other is $\phi_{[0,3/2]}(\widetilde{\lambda})$. Let

$$\alpha_i = \xi_i^+ \cup \widetilde{\xi}_i \cup \{2 \text{ arcs}\},$$

where the 2 arcs are vertical arcs connecting ξ_i^+ to $\widetilde{\xi}_i$ on an annulus, $i = 2, \ldots, 2g + 1$. Similarly, let

$$\beta_i = \eta_i^+ \cup \widetilde{\eta}_i \cup \{2 \text{ arcs}\}.$$

Let $\alpha_1 = \partial \widetilde{D}, \lambda = \widetilde{\lambda}$,

$$\mu = \delta_+ \cup \widetilde{\delta} \cup \{2 \text{ arcs}\}.$$

Let

$$\boldsymbol{\alpha} = \{\alpha_2, \ldots, \alpha_{2g+1}\} \cup \{\widetilde{\alpha}_{2g+2}, \ldots, \widetilde{\alpha}_{2g+1+r}\} \cup \{\alpha_1\},$$
$$\boldsymbol{\beta} = \{\beta_2, \ldots, \beta_{2g+1}\} \cup \{\widetilde{\beta}_{2g+2}, \ldots, \widetilde{\beta}_{2g+1+r}\} \cup \{\mu\}.$$

Pick two base points w, z near $\lambda \cap \mu$, but on different sides of μ.

Step 3 Check that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ is a Heegaard diagram for (Y, K).

This step is quite routine, we leave the reader to check the following

(A) Σ separates Y into two genus-$(2g+1+r)$ handlebodies U_1, U_2, where

$$U_1 = u^{-1}\left[0, \frac{3}{2}\right] \cup \phi_{[0,\frac{3}{2}]}(D_+), \quad U_2 = u^{-1}\left[\frac{3}{2}, 3\right] - \phi_{[0,\frac{3}{2}]}(D_+).$$

Every curve in $\boldsymbol{\alpha}$ bounds a disk in U_1, every curve in $\boldsymbol{\beta}$ bounds a disk in U_2.

(B) $\Sigma - \boldsymbol{\alpha}$ is connected, $\Sigma - \boldsymbol{\beta}$ is connected.

(C) The two base points w, z give the knot K in Y.

Then

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$$

is a Heegaard diagram for (Y, K). It is not hard to see that this is a sutured Heegaard diagram.   □

The idea of considering a relative Morse function on the sutured manifold appeared in [6]. This idea was communicated to the author by Zoltán Szabó, and was used by András Juhász to define a sutured manifold invariant [11].

Lemma 2.12 *We can wind the $\widetilde{\alpha}$-curves in $\widetilde{A} - \widetilde{\delta}$, and wind the ξ^+-curves in $A_+ - \delta_+$, to get a weakly admissible Heegaard diagram.*

Proof We claim that,

$$E = \widetilde{A} - \bigcup_{i=2}^{2g+1} \widetilde{\xi}_i - \bigcup_{j=2g+2}^{2g+1+r} \widetilde{\alpha}_j - \widetilde{\delta}$$

is connected. In fact, since A_- is connected, and ξ_j^-'s are linearly independent in $\mathrm{H}_1(A_-, \partial A_-)$, it is easy to see that $E_- = A_- - \bigcup_{i=2}^{2g+1} \xi_i^- - \delta_-$ is connected. E is obtained by removing $2r$ disks from E_-, so E is also connected.

Since E is connected, we can find simple closed curves

$$\theta_{2g+2}, \ldots, \theta_{2g+1+r} \subset \widetilde{A} - \bigcup \widetilde{\xi}_i - \widetilde{\delta},$$

which are *geometrically dual* to $\widetilde{\alpha}_{2g+2}, \ldots, \widetilde{\alpha}_{2g+1+r}$. Namely, θ_j is disjoint from $\widetilde{\alpha}_i$ when $j \neq i$, and θ_i intersects $\widetilde{\alpha}_i$ transversely at exactly one point.

We can also find closed curves in $A_+ - \delta_+$, which are geometrically dual to the ξ^+-curves. Now our desired result follows from the argument in [12, Proposition 3.3].   □

3 A homological version of the main theorem

Proposition 3.1 *Suppose $K \subset Y$ is a null-homologous knot with Seifert surface F, (M, γ) is the sutured manifold obtained by cutting $Y - \mathrm{int}(\mathrm{Nd}(K))$ open along F.*

If $\widehat{HFK}(Y, K, [F], g) \cong \mathbb{Z}$, then M is a homology product, namely,

$$\mathrm{H}_*(M, R_-(\gamma); \mathbb{Z}) \cong \mathrm{H}_*(M, R_+(\gamma); \mathbb{Z}) \cong 0.$$

Fix a field $\mathbb{F}$, let $0 \in \mathbb{F}$ be the zero element.

Lemma 3.2 *If $\widehat{HFK}(Y, K, [F], g) \cong \mathbb{Z}$, then $\mathrm{H}_2(M; \mathbb{F}) = 0$.*

Proof We use the sutured diagram constructed in Construction 2.11. For the generators of $\widehat{HFK}(Y, K, [F], -g)$, the intersection point $\mu \cap \alpha_1$ is always chosen. The generators are supported in A [13, Theorem 5.1], hence the determinant of the $(2g + r) \times (2g + r)$ matrix

$$V = \begin{pmatrix} \widetilde{\xi}_i \cdot \widetilde{\eta}_j & \widetilde{\xi}_i \cdot \widetilde{\beta}_l \\ \widetilde{\alpha}_k \cdot \widetilde{\eta}_j & \widetilde{\alpha}_k \cdot \widetilde{\beta}_l \end{pmatrix} \quad \begin{aligned} & 2 \le i, j \le 2g + 1 \\ & 2g + 2 \le k, l \le 2g + 1 + r \end{aligned}$$

computes the Euler characteristic of $\widehat{HFK}(Y, K, [F], g) \cong \mathbb{Z}$. Hence

$$\det V = \pm 1.$$

Now if $H_2(M; \mathbb{F}) \ne 0$, then there is a nontrivial $\mathbb{F}$-linear combination

$$\gamma = \sum a_k \widetilde{\alpha}_k + \sum b_l \widetilde{\beta}_l,$$

which is 0 in $H_1(\widetilde{F}; \mathbb{F})$. So $\gamma \cdot \widetilde{\alpha}_k = 0$, $\gamma \cdot \widetilde{\xi}_i = 0$.

Since $\widetilde{\alpha}$-curves are linearly independent in $H_1(\widetilde{\Sigma}; \mathbb{F})$, the coefficients b_l's are not all 0. Without loss of generality, we can assume $b_{2g+2} \ne 0$.

Since the $\widetilde{\alpha}$-curves and $\widetilde{\xi}$-curves are mutually disjoint, we have

$$\widetilde{\alpha}_k \cdot \sum b_l \widetilde{\beta}_l = 0, \quad \widetilde{\xi}_i \cdot \sum b_l \widetilde{\beta}_l = 0.$$

So by elementary column operations, we can change V into a matrix with a zero column, while the determinant of this new matrix is $b_{2g+2} \det V \ne 0$. This gives a contradiction. $\qquad \square$

Lemma 3.3 *If $\widehat{HFK}(Y, K, [F], g) \cong \mathbb{Z}$, then the map*

$$i_* : H_1(R_-(\gamma), \partial R_-(\gamma); \mathbb{F}) \to H_1(M, \gamma; \mathbb{F})$$

is injective.

Proof $H_1(R_-(\gamma), \partial R_-(\gamma))$ is generated by the ξ^--curves. If i_* is not injective, then there exists a nontrivial linear combination

$$\zeta = \sum c_i \widetilde{\xi}_i,$$

which is homologous to a linear combination

$$\gamma = \sum a_k \widetilde{\alpha}_k + \sum b_l \widetilde{\beta}_l$$

in $H_1(\widetilde{F}, \partial \widetilde{F}; \mathbb{F})$. We have $(\zeta - \gamma) \cdot \widetilde{\beta}_l = 0$. The $\widetilde{\eta}$-curves are non-proper arcs in $\widetilde{F}$, but we can connect the two endpoints of $\widetilde{\eta}_j$ by an arc in $\widetilde{D}$ to get a closed curve in $\widetilde{F}$. $\widetilde{\xi}_i, \widetilde{\alpha}_k, \widetilde{\beta}_l$ do not intersect $\widetilde{D}$, so $(\zeta - \gamma) \cdot \widetilde{\eta}_j = 0$. Since $\widetilde{\beta}$-curves and $\widetilde{\eta}$-curves are mutually disjoint, we have

$$\widetilde{\eta}_j \cdot \left(\zeta - \sum a_k \widetilde{\alpha}_k\right) = 0, \quad \widetilde{\beta}_l \cdot \left(\zeta - \sum a_k \widetilde{\alpha}_k\right) = 0.$$

Now we can get a contradiction as in the proof of Lemma 3.2. $\qquad \square$

Proof of Proposition 3.1 Let $R_{\pm} = R_{\pm}(\gamma)$. By Lemma 3.3, we have the exact sequence

$$0 \to H_2(R_-, \partial R_-; \mathbb{F}) \to H_2(M, \partial R_-; \mathbb{F}) \to H_2(M, R_-; \mathbb{F}) \to 0.$$

Compare this exact sequence with

$$H_2(M; \mathbb{F}) \to H_2(M, \gamma; \mathbb{F}) \to H_1(\gamma; \mathbb{F}) \to \cdots.$$

Note that

$$H_2(M, \gamma) = H_2(M, \partial R_-), \quad H_2(R_-, \partial R_-) \cong H_1(\gamma) \cong \mathbb{Z},$$

and $H_2(M; \mathbb{F}) = 0$ by Lemma 3.2, so

$$H_2(M, R_-; \mathbb{F}) = 0.$$

By Poincaré duality, we have

$$H_1(M, \partial M; \mathbb{F}) \cong H^2(M; \mathbb{F}) \cong H_2(M; \mathbb{F}) \cong 0.$$

So we have the exact sequence

$$0 \to H_2(M, \partial M; \mathbb{F}) \to H_1(\partial M, R_-; \mathbb{F}) \to H_1(M, R_-; \mathbb{F}) \to 0.$$

Compare this exact sequence with

$$0 \to H_1(R_-; \mathbb{F}) \to H_1(M; \mathbb{F}) \to H_1(M, R_-; \mathbb{F}) \to 0.$$

Note that

$$H_1(\partial M, R_-; \mathbb{F}) \cong H_1(R_-; \mathbb{F}) \cong \mathbb{F}^{2g},$$
$$H_1(M; \mathbb{F}) \cong H^1(M; \mathbb{F}) \cong H_2(M, \partial M; \mathbb{F}),$$

we should have

$$H_1(M, R_-; \mathbb{F}) = 0.$$

Hence we have proved that $H_*(M, R_-; \mathbb{F}) = 0$ for any field $\mathbb{F}$. So

$$H_*(M, R_-; \mathbb{Z}) = 0.$$

Similarly, we have $H_*(M, R_+; \mathbb{Z}) = 0.$ $\square$

4 Horizontal decomposition

Theorem 4.1 *Let $K' \subset Y'$, $K'' \subset Y''$ be two null-homologous knots. Suppose F', F'' are two genus-g Seifert surfaces for K', K'', respectively. We construct a new manifold Y and a knot $K \subset Y$ as follows. Cut open Y', Y'' along F', F'', we get sutured manifolds (M', γ'), (M'', γ''). Now glue $R_+(\gamma')$ to $R_-(\gamma'')$, glue $R_+(\gamma'')$ to $R_-(\gamma')$, by two diffeomorphisms. We get a manifold Z with torus boundary. There is a simple closed curve $\mu \subset \partial Z$, which is the union of the two cut-open meridians of K', K''. We do Dehn filling along μ to get the manifold Y, the knot K is the core of the filled-in solid torus.*

Our conclusion is

$$\widehat{HFK}(Y, K, [F'], g) \cong \widehat{HFK}(Y', K', [F'], g) \otimes \widehat{HFK}(Y'', K'', [F''], g),$$

as linear spaces over any field $\mathbb{F}$.

Remark 4.2 We did not specify the gluings, since they will not affect our result, thanks to [12, Proposition 3.5].

Remark 4.3 We clarify some convention we are going to use throughout this paper. A holomorphic disk in the symmetric product is seen as an immersed subsurface of the Heegaard surface Σ. Suppose Q is a subsurface of Σ, D_1, ..., D_n are the closures of the components of $Q - \bigcup \alpha_i - \bigcup \beta_j$, choose a point z_k in the interior of D_k for each k. If Φ is a holomorphic disk, then $\Phi \cap Q$ denotes the immersed surface $\sum_k n_{z_k}(\Phi) D_k$.

Proof of Theorem 4.1 The proof uses the techniques from [12]. We construct a sutured Heegaard diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', w', z')$ for (Y', K'), as in the proof of [12, Theorem 2.1]. The reader may refer to Fig. 1 there for a partial picture.

As a result, Σ' is the union of two compact surfaces A', B', where A' is a genus g surface with two boundary components α'_{1A}, λ'_A, and B' is a genus $g + r'$ surface with two boundary components α'_{1B}, λ'_B. A' and B' are glued together, so that α'_{1A} and α'_{1B} become one curve α'_1, λ'_A and λ'_B become one curve λ'.

We have

$$\boldsymbol{\alpha}' = \{\alpha'_1, \alpha'_2, \dots, \alpha'_{2g+1}, \alpha'_{2g+2}, \dots, \alpha'_{2g+1+r'}\},$$
$$\boldsymbol{\beta}' = \{\mu', \beta'_2, \beta'_3, \dots, \beta'_{2g+1+r'}\}.$$

Here α'_i is the union of two arcs $\xi'_i \subset A'$, $\overline{\xi'_i} \subset B'$, for $i = 2, \dots, 2g + 1$. α'_j lie in B', for $j = 2g + 2, \dots, 2g + 1 + r'$. μ' is the union of two arcs $\delta' \subset A'$, $\overline{\delta'} \subset B'$. μ' intersects α'_1 transversely in one point, and is disjoint from all other α'-curves. β'_i's are disjoint from λ'.

Similarly, we construct a sutured diagram $(\Sigma'', \boldsymbol{\alpha}'', \boldsymbol{\beta}'', w'', z'')$. Σ'' is the union of A'', B''. And the corresponding curves are denoted by α''_i, β''_i, ...

Now we glue A', B', A'', B'' together, so that α'_{1A} and α'_{1B} become one curve γ'_1, λ'_B and λ''_A become one curve λ', α''_{1A} and α''_{1B} become one curve γ''_1, λ''_B and λ'_A become one curve λ''. $\overline{\xi'_i}$ and ξ''_i are glued together to be a curve γ'_i, $\overline{\xi''_i}$ and ξ'_i are glued together to be a curve γ''_i, $i = 2, \ldots, 2g + 1$. $\gamma'_j = \alpha'_j$ when $j = 2g+2, \ldots, 2g+1+r'$, $\gamma''_k = \alpha''_k$ when $k = 2g+2, \ldots, 2g+1+r''$. β'_i and β''_i are as before. δ', $\overline{\delta'}$, δ'', $\overline{\delta''}$ are glued together to a closed curve μ. We also pick two basepoints w, z near $\lambda' \cap \mu$, but on different sides of μ.

Let

$$\Sigma = A' \cup B' \cup A'' \cup B'',$$
$$\gamma = \{\gamma'_1, \gamma'_2, \ldots, \gamma'_{2g+1+r'}, \gamma''_2, \ldots, \gamma''_{2g+1+r''}\},$$
$$\beta = \{\mu, \beta'_2, \ldots, \beta'_{2g+1+r'}, \beta''_2, \ldots, \beta''_{2g+1+r''}\}.$$

Then $(\Sigma, \gamma, \beta, w, z)$ is a Heegaard diagram for (Y, K).

As in the proof of [12, Proposition 3.3], we can wind $\xi'_2, \ldots, \xi'_{2g+1}$ in $A' - \delta'$, $\gamma'_{2g+2}, \ldots, \gamma'_{2g+1+r'}$ in $B' - \overline{\delta'}$, $\xi''_2, \ldots, \xi''_{2g+1}$ in $A'' - \delta''$, $\gamma''_{2g+2}, \ldots, \gamma''_{2g+1+r''}$ in $B'' - \overline{\delta''}$, so that the diagrams

$$(\Sigma, \gamma, \beta, w, z), (\Sigma', \alpha', \beta', w', z'), (\Sigma'', \alpha'', \beta'', w'', z'')$$

become admissible, and any nonnegative relative periodic domain in A' or A'' (for these diagrams) is supported away from λ'', λ'.

Claim If $\mathbf{x}$ is a generator of $\widehat{CFK}(Y, K, -g)$, then $\mathbf{x}$ is supported outside $\text{int}(A') \cup \text{int}(A'')$.

Let $Y_0(K)$ be the manifold obtained from Y by 0-surgery on K, $\underline{\mathfrak{s}}_{w,z}(\mathbf{y}) \in \text{Spin}^c(Y_0)$ be the Spin^c structure associated to an intersection point $\mathbf{y}$, $\widehat{F'}$ be the surface in Y_0 obtained by capping off the boundary of F'.

We want to compute $\langle c_1(\underline{\mathfrak{s}}_{w,z}(\mathbf{y})), [\widehat{F'}]\rangle$.

$(\Sigma, \gamma, (\beta \backslash \{\mu\}) \cup \{\lambda''\}, w'')$ is a Heegaard diagram for Y_0, we wind λ'' once along $\delta' \cup \overline{\delta'}$ to create two new intersection points with γ'_1. The variant of λ'' after winding is denoted by λ^*. Let $\mathbf{y}^*$ be an intersection point close to $\mathbf{y}$ in this new diagram. A standard computation of $\langle c_1(\mathfrak{s}(\mathbf{y}^*)), [\widehat{F'}]\rangle$ shows that, an intersection point $\mathbf{x}$ is a generator of $\widehat{CFK}(Y, K, -g)$, if and only if $\mathbf{x}$ is supported outside $\text{int}(A')$. Now if $\mathbf{x}$ is supported outside $\text{int}(A')$, then the $\beta'_2, \ldots, \beta'_{2g+1+r'}$ components of $\mathbf{x}$ have to lie in B'. Hence they are also the $\gamma'_2, \ldots, \gamma'_{2g+1+r'}$ components of $\mathbf{x}$. So $\mathbf{x}$ has no component in $\text{int}(A'')$. This finishes the proof of the claim.

Using the previous claim, one sees that

$$\widehat{CFK}(Y, K, -g) \cong \widehat{CFK}(Y', K', -g) \otimes \widehat{CFK}(Y'', K'', -g)$$

as abelian groups. Suppose Φ is a holomorphic disk for $\widehat{CFK}(Y, K, -g)$, by the previous claim all the corners of Φ are supported outside $A' \cup A''$,

so $\Phi \cap (A' \cup A'')$ is a nonnegative relative periodic domain in $A' \cup A''$, our previous conclusion before the claim shows that Φ is supported away from λ'', λ'. Moreover, Φ is supported away from $\gamma_1' \cap \mu$, since Φ should avoid w, z, which lie on different sides of μ. By the same reason, if Φ', Φ'' are holomorphic disks for $\widehat{CFK}(Y', K', -g)$ and $\widehat{CFK}(Y'', K'', -g)$, respectively, then they are supported away from $\alpha_1' \cap \mu'$ and $\alpha_1'' \cap \mu''$, respectively. Hence Φ is the disjoint union of two holomorphic disks for $\widehat{CFK}(Y', K', -g)$ and $\widehat{CFK}(Y'', K'', -g)$, respectively. Now our desired result is obvious. □

As a corollary, we have

Corollary 4.4 *Let $K \subset Y$ be a null-homologous knot with a genus g Seifert surface F, Y_m be the m-fold cyclic branched cover of Y over K, with respect to F, and K_m is the image of K in Y_m. Then*

$$\widehat{HFK}(Y_m, K_m, [F], g; \mathbb{F}) \cong \widehat{HFK}(Y, K, [F], g; \mathbb{F})^{\otimes m}$$

as linear spaces over any field $\mathbb{F}$. □

Knot Floer homology of knots in cyclic branched covers has been studied by Grigsby [7], with emphasis on 2-bridge knots in S^3.

Theorem 4.1 can be re-stated in the language of $\widehat{HFS}(M, \gamma)$ as follows.

Theorem 4.5 *Suppose (M, γ) is a balanced sutured manifold, and $S \subset M$ is a horizontal surface. Decompose (M, γ) along S, we get two balanced sutured manifolds (M_1, γ_1), (M_2, γ_2). Then*

$$\widehat{HFS}(M, \gamma) \cong \widehat{HFS}(M_1, \gamma_1) \otimes \widehat{HFS}(M_2, \gamma_2)$$

as linear spaces over any field $\mathbb{F}$.

5 Product decomposition

In this section, we will study sutured manifold decomposition along product annuli. We are not able to obtain a formula for non-separating product annuli, but the formula for separating product annuli is already enough for many applications.

Theorem 5.1 *Suppose (M, γ) is a balanced sutured manifold, $R_{\pm}(\gamma)$ are connected. $\mathcal{A} \subset M$ is a separating product annulus, and $\mathcal{A}$ separates M into two balanced sutured manifolds (M_1, γ_1), (M_2, γ_2).*
Then we have

$$\widehat{HFS}(M, \gamma) \cong \widehat{HFS}(M_1, \gamma_1) \otimes \widehat{HFS}(M_2, \gamma_2)$$

as vector spaces over any field $\mathbb{F}$.

In the first two subsections, we will consider the case that γ has only one component, which lies in M_2, and $M_1 = R_1 \times [0, 1]$, where R_1 is a compact genus-1 surface with one boundary component.

5.1 A Heegaard diagram related to (M, γ)

Construction 5.2 Let $\psi \colon R_+(\gamma) \to R_-(\gamma)$ be a homeomorphism, such that $\psi(R_+(\gamma_i)) = R_-(\gamma_i)$, $i = 1, 2$, and $\psi|R_+(\gamma_1)$ maps $x \times 1$ to $x \times 0$ for any $x \in R_1$. If we glue $R_+(\gamma)$ to $R_-(\gamma)$ by ψ, then we get a manifold with boundary consisting of a torus. This manifold can be viewed as the complement of a knot K in a manifold Y. We will construct a Heegaard diagram for the pair (Y, K). The construction is similar to Construction 2.11.

Step 0 A relative Morse function.

Consider a self-indexed relative Morse function u on (M_2, γ_2). Let $\widetilde{F}_2 = u^{-1}(\frac{3}{2})$. $\widetilde{F}_2$ has two boundary components. We denote the one that lies in the separating annulus $\mathcal{A}$ by $\widetilde{a}$. The other boundary component is denoted by $\widetilde{\lambda}$. Similarly, the boundary components of $R_\pm(\gamma_2)$ are denoted by $a_\pm, \lambda_\pm$.

Suppose the genus of $R_+(\gamma_2)$ is $g - 1$, and u has r index-1 critical points, then the genus of $\widetilde{F}_2$ is $g + r - 1$. The gradient $-\nabla u$ generates a flow ϕ_t on M_2.

*Step 1* Construct curves for (M_2, γ_2).

Choose

$$\widetilde{D}, \widetilde{\delta} \subset \widetilde{F}_2, \quad D_\pm, \delta_\pm \subset R_\pm(\gamma_2)$$

as in Construction 2.11. Let $B_\pm = R_\pm(\gamma_2) - \text{int}(D_\pm)$, $\widetilde{B} = \widetilde{F}_2 - \text{int}(\widetilde{D})$.

On $\widetilde{F}_2$, there are simple closed curves $\widetilde{\alpha}_{2g+2}, \ldots, \widetilde{\alpha}_{2g+1+r}, \widetilde{\beta}_{2g+2}, \ldots, \widetilde{\beta}_{2g+1+r}$, which correspond to the critical points of u.

The $\widetilde{\alpha}$-curves do not separate $\widetilde{F}_2$, so there is an arc $\widetilde{\sigma} \subset \widetilde{B}$ connecting $\widetilde{\lambda}$ to $\widetilde{a}$, and $\widetilde{\sigma}$ is disjoint from $\widetilde{\delta}$ and $\widetilde{\alpha}$-curves. Similarly, there is an arc $\widetilde{\tau} \subset \widetilde{F}_2$ connecting $\partial \widetilde{D}$ to $\widetilde{a}$, and $\widetilde{\tau}$ is disjoint from $\widetilde{\delta}$ and $\widetilde{\beta}$-curves. Moreover, by stabilization as shown in Fig. 1, we can assume $\widetilde{\sigma} \cap \widetilde{\tau} = \varnothing$, and $\widetilde{\sigma}$ intersects exactly one $\widetilde{\beta}$-curve transversely once. Suppose this curve is $\widetilde{\beta}_{2g+2}$. Let $\sigma_- \subset B_-$ be the image of $\widetilde{\sigma}$ under the flow ϕ_t, and $\sigma_+ = \psi^{-1}(\sigma_-)$. Let $\tau_+ \subset B_+$ be the image of $\widetilde{\tau}$ under the flow ϕ_{-t}.
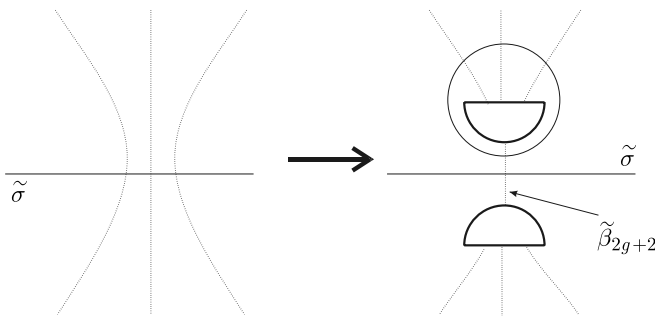
Fig. 1: A stabilization will eliminate the extra intersection points of $\widetilde{\sigma}$ with $\widetilde{\tau}$ and $\widetilde{\beta}$-curves. On the right hand side the two semicircular holes are glued together

Choose $2g-2$ disjoint arcs $\xi_4^-, \ldots, \xi_{2g+1}^- \subset B_-$, such that their endpoints lie on λ_-, and they are linearly independent in $H_1(B_-, \partial B_-)$. We also suppose they are disjoint from δ_-, σ_- and the bad points. Let $\xi_i^+ = \psi^{-1}(\xi_i^-)$. We also flow back $\xi_4^-, \ldots, \xi_{2g+1}^-$ by ϕ_{-t} to $\widetilde{F}_2$, the images are denoted by $\widetilde{\xi}_4, \ldots, \widetilde{\xi}_{2g+1}$.

Choose $2g-2$ disjoint arcs $\eta_4^+, \ldots, \eta_{2g+1}^+ \subset B_+$, such that their endpoints lie on ∂D_+, and they are linearly independent in $H_1(B_+, \partial B_+)$. We also suppose they are disjoint from δ_+, τ_+ and the bad points. Flow them by ϕ_t to $\widetilde{F}$, the images are denoted by $\widetilde{\eta}_4, \ldots, \widetilde{\eta}_{2g+1}$.

We can slide $\widetilde{\eta}$-curves over $\widetilde{\beta}_{2g+2}$ to eliminate the possible intersection points between $\widetilde{\eta}$-curves and $\widetilde{\sigma}$.

By stabilization, we can assume $\widetilde{\tau}$ does not intersect $\widetilde{\xi}$-curves, and it intersects exactly one $\widetilde{\alpha}$-curve transversely once. This curve is denoted by $\widetilde{\alpha}_{2g+2}$.

A picture of $\widetilde{F}_2$ is shown in Fig. 2.
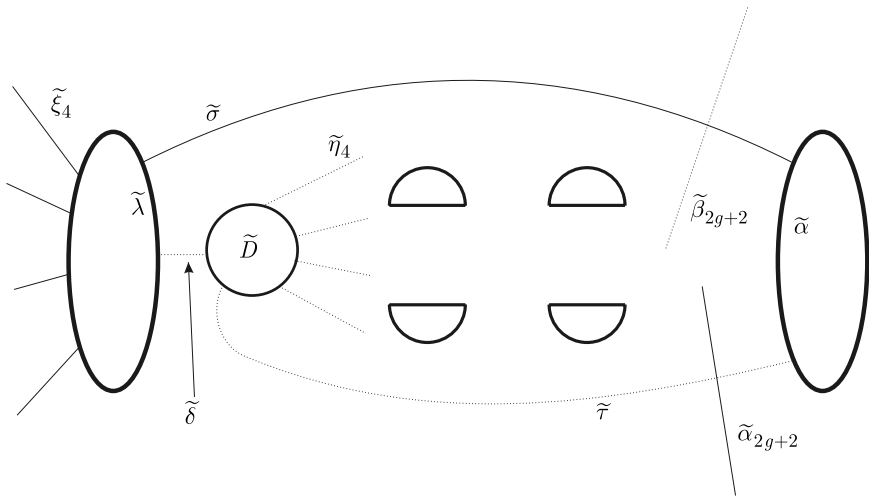
Fig. 2: A schematic picture of $\widetilde{F}_2$. The pairs of semicircular holes are glued together

Step 2 Find geometric duals of $\widetilde{\alpha}$-curves.

As in Lemma 2.12, we can prove that

$$E = \widetilde{B} - \bigcup_{i=4}^{2g+1} \widetilde{\xi}_i - \bigcup_{j=2g+2}^{2g+1+r} \widetilde{\alpha}_j - \widetilde{\delta} - \widetilde{\sigma}$$

is connected. Then there are simple closed curves

$$\theta_{2g+2}, \ldots, \theta_{2g+1+r} \subset \widetilde{B} - \bigcup \widetilde{\xi}_i - \widetilde{\sigma} - \widetilde{\delta},$$

which are geometrically dual to $\widetilde{\alpha}_{2g+2}, \ldots, \widetilde{\alpha}_{2g+1+r}$.

We can slide $\widetilde{\theta}_{2g+3}, \ldots, \widetilde{\theta}_{2g+1+r}$ over $\widetilde{\alpha}_{2g+2}$ to eliminate the possible intersection points between $\widetilde{\theta}_{2g+3}, \ldots, \widetilde{\theta}_{2g+1+r}$ and $\widetilde{\tau}$.

Step 3 Construct curves for (M_1, γ_1).

As in Fig. 3, we choose 4 properly embedded arcs $\xi_2, \xi_3, \eta_2, \eta_3$ on R_1. ξ_2 intersects η_2 at one point, and ξ_3 intersects η_3 at one point. There are no more intersection points between these arcs.

Let

$$\widetilde{A} = \widetilde{B} \cup_{\widetilde{a}=(\partial R_1)\times\frac{1}{2}} \left(R_1 \times \frac{1}{2}\right),$$

$$A_+ = B_+ \cup_{a_+=(\partial R_1)\times 1} (R_1 \times 1).$$

Take 4 parallel copies of $\widetilde{\sigma}$, glue them to $\xi_2 \times \frac{1}{2}, \xi_3 \times \frac{1}{2}$, we get two arcs $\widetilde{\xi}_2, \widetilde{\xi}_3 \subset \widetilde{A}$. Glue 4 parallel copies of $\widetilde{\tau}$ with $\eta_2 \times \frac{1}{2}, \eta_3 \times \frac{1}{2}$, we get two arcs $\widetilde{\eta}_2, \widetilde{\eta}_3 \subset \widetilde{A}$. Similarly, we can construct $\xi_2^+, \xi_3^+, \eta_2^+, \eta_3^+ \subset A_+$.
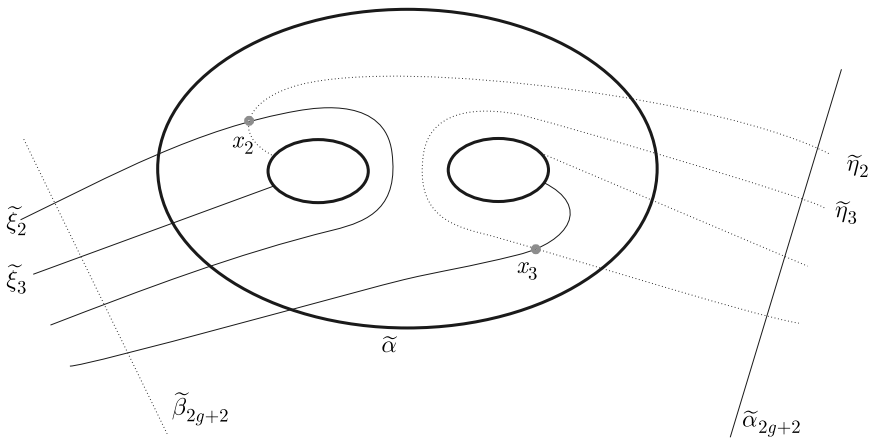
Fig. 3: Local picture of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ near $R_1 \times \frac{1}{2}$. The surface R_1 is obtained by gluing the boundaries of two holes in the disk by a reflection

Step 4 Construct a Heegaard diagram.

Let

$$\Sigma = A_+ \cup \widetilde{A} \cup \{2 \text{ annuli}\},$$

where one annulus is $\phi_{[0,3/2]}(\partial D_+)$, the other is $\phi_{[0,3/2]}(\widetilde{\lambda})$. Construct α_i, β_i, $(i = 2, \ldots, 2g+1)$ α_1, λ, μ as in Step 2 of Construction 2.11.

Let

$$\boldsymbol{\alpha} = \{\alpha_2, \ldots, \alpha_{2g+1}\} \cup \{\widetilde{\alpha}_{2g+2}, \ldots, \widetilde{\alpha}_{2g+1+r}\} \cup \{\alpha_1\},$$

$$\boldsymbol{\beta} = \{\beta_2, \ldots, \beta_{2g+1}\} \cup \{\widetilde{\beta}_{2g+2}, \ldots, \widetilde{\beta}_{2g+1+r}\} \cup \{\mu\}.$$

Pick two base points w, z near $\lambda \cap \mu$, but on different sides of μ. As in Construction 2.11,

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$$

is a Heegaard diagram for (Y, K).   □

It is easy to check that the Heegaard diagram constructed above is a sutured Heegaard diagram. In order to prove our desired result, we still need to change the diagram by handleslides.

Construction 5.3 In the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ constructed in Construction 5.2, the curve $\widetilde{\beta}_{2g+2}$ has 4 intersection points with $\widetilde{\xi}_2$ and $\widetilde{\xi}_3$. Since $\widetilde{\xi}_2$ intersects β_2 exactly once, we can slide $\widetilde{\beta}_{2g+2}$ over β_2 twice, to eliminate the intersection points between $\widetilde{\beta}_{2g+2}$ and $\widetilde{\xi}_2$. Note that in the two handleslides, the orientations of β_2 are different, so the new curve is homologous to $\widetilde{\beta}_{2g+2}$. Similarly, we can slide $\widetilde{\beta}_{2g+2}$ over β_3 twice, to eliminate the intersection points between $\widetilde{\beta}_{2g+2}$ and $\widetilde{\xi}_3$. The local picture is shown in Fig. 4. The new $\widetilde{\beta}$-curve is denoted by β°_{2g+2}.

Moreover, we can isotope β°_{2g+2} to eliminate its intersection points with $\widetilde{\alpha}_{2g+2}$ and α_1, then isotope it so that it lies outside $R_1 \times \frac{1}{2}$, as shown in Figs. 4 and 5. The new $\widetilde{\beta}$-curve is denoted by β^*_{2g+2}. β^*_{2g+2} is homologous to $\widetilde{\beta}_{2g+2}$ in $H_1(\Sigma)$, thus $\beta^*_{2g+2} \cap A_+$ is null-homologous in $H_1(A_+, \partial A_+)$.
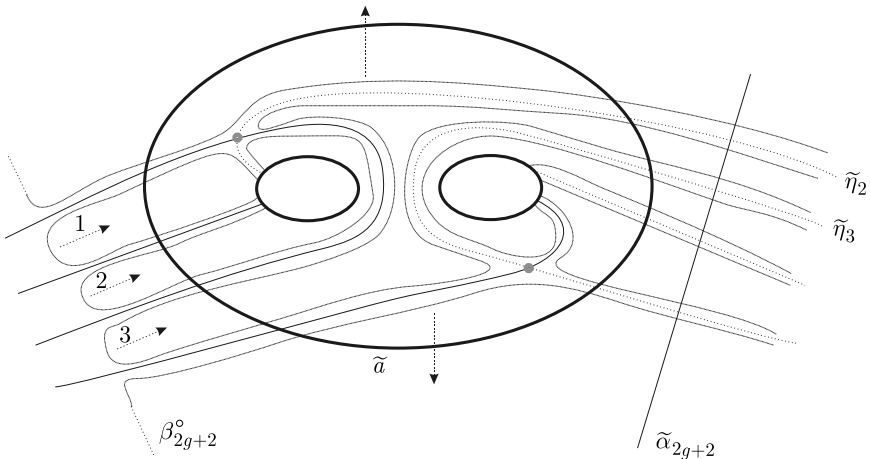
Fig. 4: Local picture of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^\circ, w, z)$ near $R_1 \times \frac{1}{2}$. The arrows indicate the directions of a further isotopy

The new Heegaard diagram after handlesliding is denoted by $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^*, w, z)$. This diagram is still a sutured Heegaard diagram.   □

Fig. 5: Local picture of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^*, w, z)$

Lemma 5.4 *After winding transverse to $\widetilde{\alpha}$-curves in $\widetilde{B}$, and transverse to ξ^+-curves in A_+, we can get a weakly admissible Heegaard diagram for (Y, K). Moreover, in this diagram, any holomorphic disk for $\widehat{CFK}(Y, K, -g)$ is supported in $\widetilde{B}$.*

Proof In Step 2 of Construction 5.2, we find circles $\theta_{2g+2}, \ldots, \theta_{2g+1+r}$, which are geometrically dual to $\widetilde{\alpha}_{2g+2}, \ldots, \widetilde{\alpha}_{2g+1+r}$. Moreover, $\theta_{2g+3}, \ldots,$ θ_{2g+1+r} are disjoint with $\widetilde{\xi}$-curves, $\widetilde{\delta}$, $\widetilde{\sigma}$ and $\widetilde{\tau}$. As in the proof of [12, Proposition 3.3], we can wind $\widetilde{\alpha}_{2g+3}, \ldots, \widetilde{\alpha}_{2g+1+r}$ along $\theta_{2g+3}, \ldots, \theta_{2g+1+r}$, and wind ξ^+-curves in A_+, to get a new Heegaard diagram. In this diagram, if $\mathcal{P}$ is a nonnegative periodic domain, then $\partial \mathcal{P}$ does not contain $\widetilde{\alpha}_{2g+3}, \ldots,$ $\widetilde{\alpha}_{2g+1+r}$ and ξ^+-curves. Obviously, ∂P does not contain μ. Note that α_1 is the only attaching curve in the Heegaard diagram which intersects μ, and $\partial P \cdot \mu = 0 \neq \alpha_1 \cdot \mu$, so ∂P does not contain α_1.

$\partial(\mathcal{P} \cap A_+) - \partial A_+$ is a linear combination of $\beta_{2g+2}^* \cap A_+$ and η_i^+'s. Note that $\eta_2^+, \ldots, \eta_{2g+1}^+$ are linearly independent in $\mathrm{H}_1(A_+, \partial A_+)$, and $\beta_{2g+2}^* \cap A_+$ is null-homologous in $\mathrm{H}_1(A_+, \partial A_+)$, so $\partial(\mathcal{P} \cap A_+) - \partial A_+$ is a multiple of $\beta_{2g+2}^* \cap A_+$.

Hence $\partial \mathcal{P}$ is a linear combination of $\widetilde{\alpha}_{2g+2}$ and $\widetilde{\beta}$-curves. Here β_{2g+2}^* is also viewed as a $\widetilde{\beta}$-curve, since it is homologous to $\widetilde{\beta}_{2g+2}$. In Step 1 of Construction 5.2, $\widetilde{\alpha}_{2g+2}$ is obtained by stabilization. Thus there is a $\widetilde{\beta}$-curve, which intersects $\widetilde{\alpha}_{2g+2}$ exactly once. $\widetilde{\beta}$-curves are mutually disjoint, so $\widetilde{\alpha}_{2g+2}$ is not homologous to the linear combination of $\widetilde{\beta}$-curves. This shows that the new diagram is weakly admissible. This diagram is still denoted by $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^*, w, z)$.

The generators of $\widehat{CFK}(Y, K, -g)$ are supported outside $\mathrm{int}(A_+)$.

α_2 and α_3 intersect only two β-curves outside A_+. Therefore the two intersection points $x_2 = \widetilde{\xi}_2 \cap \widetilde{\eta}_2$ and $x_3 = \widetilde{\xi}_3 \cap \widetilde{\eta}_3$ must be chosen.

Suppose Φ is a holomorphic disk for $\widehat{CFK}(Y, K, -g)$. $\Phi \cap A_+$ is a relative periodic domain [12, Definition 3.1] in A_+. As before, Φ does not contain any ξ_+-curve after winding, so Φ is supported away from λ. Now $\partial(\Phi \cap A_+) - \partial A_+$ is a linear combination of $\beta_{2g+2}^* \cap A_+$ and η_i^+'s. As in the second paragraph of this proof, we have that

$$\partial(\Phi \cap A_+) - \partial A_+ = m(\beta_{2g+2}^* \cap A_+).$$

Φ is supported away from μ, μ intersects α_1 at exactly one point, so the contribution of α_1 to $\partial \Phi$ is 0. $\beta_{2g+2}^* \cap A_+$ separates λ from $\eta_2^+ \cap \alpha_1$. The local multiplicities of Φ in a small neighborhood of x_2, x_3 are 0, and Φ does not intersect λ. As in Fig. 6, we conclude that the local multiplicities of Φ in two corners at x are 0, and the local multiplicities of Φ in two corners at y are 0.
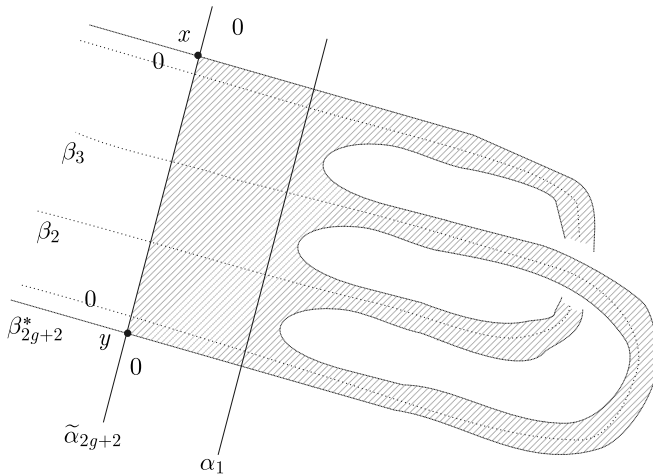
Fig. 6: A component of the possible holomorphic disk. The two bands lie in $R_1 \times 1$

If $m > 0$, then Φ (which is possibly reducible) has a component Φ_0, which has two vertices x, y and two edges lying in $\widetilde{\alpha}_{2g+2}$ and β_{2g+2}^*. See Fig. 6 for the picture of Φ_0. The genus of Φ_0 is 1, and its Maslov index is -1. So Φ has no contribution to the boundary map in $\widehat{CFK}(Y, K, -g)$.

Hence if Φ is a holomorphic disk for $\widehat{CFK}(Y, K, -g)$, then $m = 0$, so Φ is supported in $\widetilde{B}$. $\qquad\square$

5.2 A Heegaard diagram related to (M_2, γ_2) We also need to construct a Heegaard diagram to compute $\widehat{HFS}(M_2, \gamma_2)$. As in Sect. 2.2, we can add a product 1-handle to M_2 with feet at different components of γ_2 to get a new sutured manifold (M_3, γ_3). Then we glue $R_+(\gamma_3)$ with $R_-(\gamma_3)$ to get the complement of a knot K_3 in a manifold Y_3. Then

$$\widehat{HFS}(M_2, \gamma_2) = \widehat{HFK}(Y_3, K_3, g).$$

Our next task is to construct a Heegaard diagram for (Y_3, K_3).
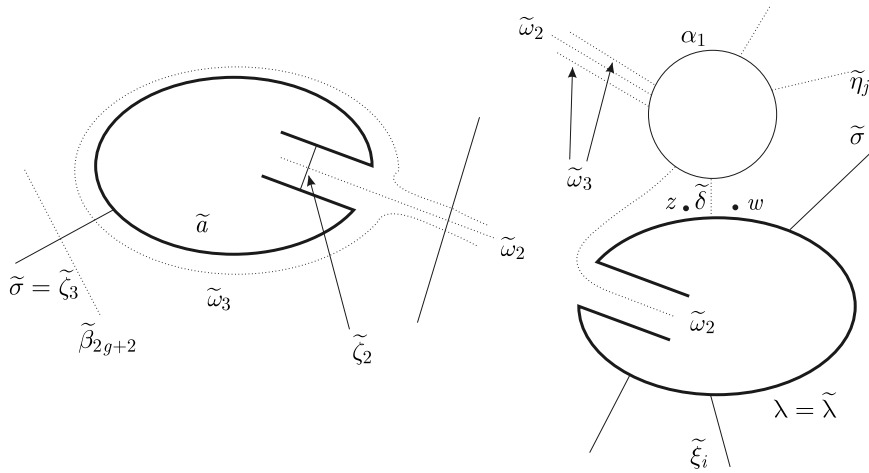
Fig. 7: Local picture of $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', w, z)$, near $\widetilde{a}$ and near $\widetilde{\lambda}$. There is a band connecting $\widetilde{a}$ to $\widetilde{\lambda}$

Construction 5.5 Notations as in Construction 5.2. As in Fig. 7, let $\widetilde{C}$ be the surface obtained by gluing a rectangle $I \times [0, 1]$ to $\widetilde{B}$, such that

$$(I \times [0, 1]) \cap \widetilde{B} = I \times \{0, 1\}, (I \times 0) \subset \widetilde{\lambda}, (I \times 1) \subset \widetilde{a}.$$

Moreover, $\widetilde{\delta} \cap \widetilde{\lambda}$ is not contained in $I \times 0$, but it is close to $I \times 0$. $\widetilde{\tau} \cap \widetilde{a} \in \text{int}(I \times 1)$, and $\widetilde{\sigma} \cap \widetilde{a} \notin I \times 1$. Let $\widetilde{\zeta_2} = I \times \frac{1}{2}$, $\widetilde{\zeta_3} = \widetilde{\sigma}$. Let $\widetilde{\omega_2}$ be the union of $\widetilde{\tau}$, $p \times [0, 1]$ ($p \in \text{int}(I)$) and a parallel copy of $\widetilde{\delta}$. Let T be the neighborhood of $\widetilde{\tau} \cup \widetilde{a}$ in $\widetilde{B}$. $\widetilde{\omega_3} = T \cap (\widetilde{B} - \text{int}(T))$.

Similarly, construct a surface C_+, and curves $\zeta_2^+, \zeta_3^+, \omega_2^+, \omega_3^+$ on it.

As in Step 4 of Construction 5.2, for $i = 2, 3$, let

$$\alpha_i' = \zeta_i^+ \cup \widetilde{\zeta_i} \cup \{2 \text{ arcs}\},$$
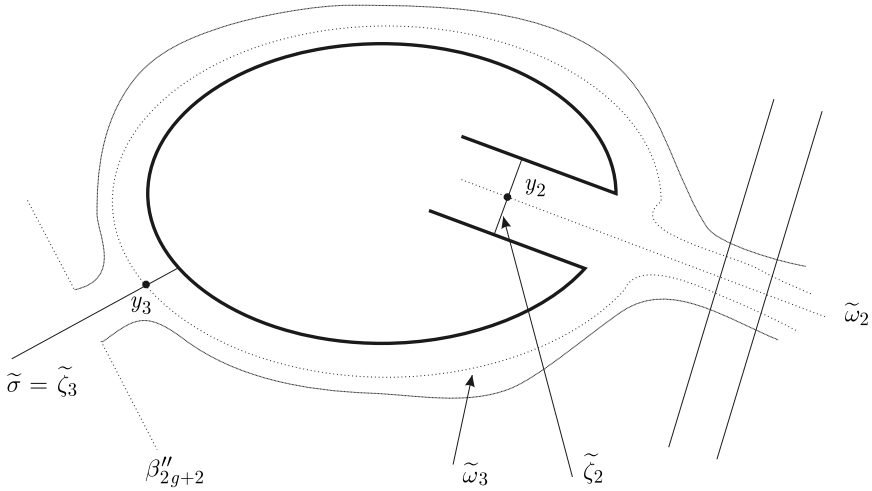$$\beta_i' = \omega_i^+ \cup \widetilde{\omega_i} \cup \{2 \text{ arcs}\}.$$

Let

$$\Sigma' = C_+ \cup \widetilde{C} \cup \{2 \text{ annuli}\},$$
$$\boldsymbol{\alpha}' = \{\alpha_2', \alpha_3', \alpha_4, \ldots, \alpha_{2g+1}\} \cup \{\widetilde{\alpha}_{2g+2}, \ldots, \widetilde{\alpha}_{2g+1+r}\} \cup \{\alpha_1\},$$
$$\boldsymbol{\beta}' = \{\beta_2', \beta_3', \beta_4, \ldots, \beta_{2g+1}\} \cup \{\widetilde{\beta}_{2g+2}, \ldots, \widetilde{\beta}_{2g+1+r}\} \cup \{\mu\}.$$

Then

$$(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', w, z)$$

is a Heegaard diagram for (Y_3, K_3).   □

Construction 5.6 In the diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', w, z)$, we can slide $\widetilde{\beta}_{2g+2}$ over β_3' once, to get a new curve β_{2g+2}''. The new diagram is denoted by $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}'', w, z)$. See Fig. 8 for the local picture.   □

Fig. 8: Local picture of $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}'', w, z)$ near $\widetilde{a}$

Lemma 5.7 *After winding transverse to $\widetilde{\alpha}$-curves in $\widetilde{B}$, and transverse to ξ^+-curves in B_+, we can get a weakly admissible Heegaard diagram for (Y_3, K_3). In this diagram, any holomorphic disk for $\widehat{CFK}(Y_3, K_3, -g)$ is supported in $\widetilde{B}$.*

Proof The proof of this lemma is similar to the proof of Lemma 5.4, and the argument here is even simpler. Again, we can wind some curves, such that the diagram becomes weakly admissible, and the holomorphic disks for $\widehat{CFK}(Y_3, K_3, -g)$ are supported away from λ'. Here $\lambda' \subset \partial\widetilde{C}$ is the connected sum of λ and $\widetilde{a}$. Suppose Φ' is a holomorphic disk for $\widehat{CFK}(Y_3, K_3, -g)$, then $\Phi' \cap C_+$ is a relative periodic domain in C_+. Since Φ' is disjoint with λ', $\partial\Phi' \cap C_+$ does not contain ξ^+-curves, and the local multiplicities of Φ' near y_2 and y_3 are zero. So $\partial(\Phi' \cap C_+) - \partial C_+$ is a linear combination of $\eta_4^+, \ldots, \eta_{2g+1}^+, \beta_{2g+2}'' \cap C_+$. These arcs are linearly independent in $H_1(C_+, \partial C_+)$, so Φ' is supported in $\widetilde{B}$. □

5.3 Proof of the product decomposition formula

Proof of Theorem 5.1 The surfaces

$$F_\pm = R_\pm(\gamma_1) \cup \mathcal{A} \cup R_\mp(\gamma_2)$$

are homologous to $R_+(\gamma)$. $F_\pm$ can be isotoped to horizontal surfaces in (M, γ).

F_+ decomposes (M, γ) into two sutured manifolds (M_1', γ_1'), (M_2', γ_2'). Here (M_1', γ_1') is the sutured manifold obtained by gluing $R_-(\gamma_2) \times I$ to (M_1, γ_1) along $\mathcal{A}$, and (M_2', γ_2') is the sutured manifold obtained by gluing $R_+(\gamma_1) \times I$ to (M_2, γ_2) along $\mathcal{A}$. Now one can apply Theorem 4.5 to conclude

that

$$\widehat{HFS}(M, \gamma) \cong \widehat{HFS}(M'_1, \gamma'_1) \otimes \widehat{HFS}(M'_2, \gamma'_2).$$

We only need to show that $\widehat{HFS}(M'_i, \gamma'_i) \cong \widehat{HFS}(M_i, \gamma_i)$ for $i = 1, 2$. Hence we can reduce our theorem to the case that one of the two sutured submanifolds M_1, M_2 is a product.

From now on, we assume M_1 is a product.

According to Definition 2.2, $\gamma \neq \emptyset$. If $M_1 \cap \gamma \neq \emptyset$, then one can decompose M along product disks to get M_2. Now we apply Proposition 2.9(2) to conclude that $\widehat{HFS}(M, \gamma) \cong \widehat{HFS}(M_2, \gamma_2)$.

Now we consider the case that $M_1 \cap \gamma = \emptyset$. By adding product 1-handles with feet at γ, we can get a sutured manifold with connected suture.

$R_+(\gamma_1)$ contains a subsurface G which is a once-punctured torus. $\partial G \times I$ splits M into two sutured manifolds $G \times I$ and (M^*, γ^*). One can then decompose M^* along product disks to get (M_2, γ_2). Hence we only need to prove the decomposition formula for the case of splitting along $\partial G \times I$. From now on, we focus on this case, namely, the case that the genus of $R_+(\gamma_1)$ is 1.

We apply the constructions in the previous two subsections to get Heegaard diagrams for (Y, K) and (Y_3, K_3). See Figs. 5 and 8 for the local pictures. For generators of $\widehat{CFK}(Y, K, -g)$, the two intersection points x_2, x_3 in Fig. 5 must be chosen; for generators of $\widehat{CFK}(Y_3, K_3, -g)$, the two intersection points y_2, y_3 in Fig. 8 must be chosen. Thus the generators of $\widehat{CFK}(Y, K, -g)$ and $\widehat{CFK}(Y_3, K_3, -g)$ are in one-to-one correspondence. By Lemma 5.4 and Lemma 5.7, the holomorphic disks for these two chain complexes are also the same. Hence

$$\widehat{HFK}(Y, K, -g) = \widehat{HFK}(Y_3, K_3, -g),$$

which means that

$$\widehat{HFS}(M, \gamma) = \widehat{HFS}(M_2, \gamma_2)$$

by definition.   □

5.4 An application to satellite knots As an application, we can compute the topmost terms in the knot Floer homology of satellite knots with nonzero winding numbers. We recall the following definition from [12].

Definition 5.8 Suppose K is a null-homologous knot in Y, F is a Seifert surface of K (not necessarily of minimal genus). V is a 3-manifold, $\partial V = T^2$, $L \subset V$ is a nontrivial knot. $G \subset V$ is a compact connected oriented surface so that L is a component of ∂G, and $\partial G - L$ (may be empty) consists of parallel essential circles on ∂V. Orientations on these circles are induced from the orientation on G, we require that these circles are parallel as oriented ones. We glue V to $Y - \text{int}(\text{Nd}(K))$, so that any component of $\partial G - L$ is null-homologous in $Y - \text{int}(\text{Nd}(K))$. The new manifold is denoted by Y^*,

and the image of L in Y^* is denoted by K^*. We then say K^* is a *satellite knot* of K, and K a *companion knot* of K^*. Let p denote the number of components of $\partial G - L$, p will be called the *winding number* of L in V.

Suppose $p > 0$, F is a minimal genus Seifert surface for K, then a minimal genus Seifert surface F^* for K^* can be obtained as follows: take p parallel copies of F, and glue them to a certain surface G in $V - \mathrm{int}(\mathrm{Nd}(L))$. We decompose $V - \mathrm{int}(\mathrm{Nd}(L))$ along G, the resulting sutured manifold is denoted by $(M(L), \gamma(L))$, where $\gamma(L)$ consists of $p + 1$ annuli, p of them lie on ∂V, denoted by $A_1, \ldots, A_p$.

Corollary 5.9 *With notations as above, suppose the genus of K is $g > 0$, and the genus of K^* is g^*, then*

$$\widehat{HFK}(Y^*, K^*, [F^*], g^*) \cong \widehat{HFK}(Y, K, [F], g) \otimes \widehat{HFS}(M(L), \gamma(L))$$

as linear spaces over any field $\mathbb{F}$.

Proof Let (M, γ) be the sutured manifold obtained by decomposing $Y^* - \mathrm{int}(\mathrm{Nd}(K^*))$ along F^*. Note that $A_1, \ldots, A_p$ are separating product annuli in M. The desired result holds by Theorem 5.1. $\qquad\square$

Matthew Hedden also got some interesting results regarding knot Floer homology of satellite knots with nonzero winding numbers [8]. Our result can be compared with his.

6 Characteristic product regions

Definition 6.1 Suppose (M, γ) is an irreducible sutured manifold, γ has no toral component, $R_-(\gamma)$, $R_+(\gamma)$ are incompressible and diffeomorphic to each other. A *product region* for M is a submanifold $\Phi \times I$ of N, such that Φ is a compact, possibly disconnected, surface, and $\Phi \times 0$, $\Phi \times 1$ are incompressible subsurfaces of $R_-(\gamma)$, $R_+(\gamma)$, respectively.

There exists a product region $E \times I$, such that if $\Phi \times I$ is any product region for M, then there is an ambient isotopy of M which takes $\Phi \times I$ into $E \times I$. $E \times I$ is called a *characteristic product region* for M.

The theory of characteristic product regions is actually a part of JSJ theory [9,10], the version that we need in the current paper can be found in [1].

The following theorem can be abbreviated as: if $\widehat{HFS}(M, \gamma) \cong \mathbb{Z}$, then the characteristic product region carries all the homology. A version of this theorem is also proved by Ian Agol via a different approach.

Theorem 6.2 *Suppose (M, γ) is an irreducible balanced sutured manifold, γ has only one component, and (M, γ) is vertically prime. Let $E \times I \subset M$ be the characteristic product region for M.*

If $\widehat{HFS}(M, \gamma) \cong \mathbb{Z}$, then the map

$$i_* \colon \mathrm{H}_1(E \times I) \to \mathrm{H}_1(M)$$

is surjective.

By Proposition 3.1, (M, γ) is a homology product.

Let G be a genus-1 compact surface with one boundary component. Glue the two sutured manifolds (M, γ) and $G \times I$ together along their vertical boundaries, we get a sutured manifold N with empty suture. N has two boundary components $\Sigma = \Sigma_- = R_-(\gamma) \cup (G \times 0)$, $\Sigma_+ = R_+(\gamma) \cup (G \times 1)$. N is also a homology product, thus there is a natural isomorphism

$$\partial_* \colon \mathrm{H}_2(N, \partial N) \to \mathrm{H}_1(\Sigma).$$

Remark 6.3 Since (M, γ) is a homology product, we can glue $R_-(\gamma)$ to $R_+(\gamma)$, so that the resulting manifold is the complement of a knot K in a homology 3-sphere $Y \not\approx S^3$. Suppose J is a genus-1 fibred knot in a homology sphere $Z \not\approx S^3$, G' is a fibre. Consider the knot $K \# J \subset Y \# Z$. Let Y_0' be the manifold obtained by 0-surgery on $K \# J$. Let $F = R_-(\gamma) \subset Y$, H be the boundary connected sum of F and G', and $\widehat{H}$ be the extension of H in Y_0'.

If we cut Y_0' open along $\widehat{H}$, then we get the manifold N. [4, Theorem 8.9] shows that Y_0' admits a taut foliation, with $\widehat{H}$ as a compact leaf. Thus this foliation induces a foliation of N.

Assume that the map

$$i_* \colon \mathrm{H}_1(E \times I) \to \mathrm{H}_1(M)$$

is not surjective. We can find a simple closed curve $\omega \subset R_-(\gamma)$, such that $[\omega]$ is not in $i_*(\mathrm{H}_1(E \times I))$.

Let $\omega_- = \omega \subset \Sigma_-$, and let $\omega_+ \subset \Sigma_+$ be a circle homologous to ω. We fix an arc δ connecting Σ_- to Σ_+. Let $\mathcal{S}_m(+\omega)$ be the set of properly embedded surfaces $S \subset N$, such that $\partial S = \omega_- \sqcup (-\omega_+)$, and the algebraic intersection number of S with δ is m. Here $-\omega_+$ denotes the curve ω_+, but with opposite orientation. Similarly, let $\mathcal{S}_m(-\omega)$ be the set of properly embedded surfaces $S \subset N$, such that $\partial S = (-\omega_-) \sqcup \omega_+$, and the algebraic intersection number of S with δ is m. Let $x(\mathcal{S}_m(\pm\omega))$ be the minimal value of $x(S)$ for all $S \in \mathcal{S}_m(\pm\omega)$. It is obvious that

$$x(\mathcal{S}_{m+1}(\pm\omega)) \leq x(\mathcal{S}_m(\pm\omega)) + x(\Sigma).$$

The next fact is implicitly contained in [2, Theorem 3.13].

Lemma 6.4 *When m is sufficiently large, there exist connected surfaces $S_1 \in \mathcal{S}_m(+\omega)$ and $S_2 \in \mathcal{S}_m(-\omega)$, such that they give taut decompositions of N.*

Proof Let $D(N)$ be the double of N along ∂N. $a = \partial_*^{-1}([\omega]) \in H_2(N, \partial N)$ is the homology class whose intersection with Σ is $[\omega]$, $D(a)$ is its double in $H_2(D(N))$. There exists $C \geq 0$, such that if $k > C$, then $x(D(a) + (k+1)[\Sigma]) = x(D(a) + k[\Sigma]) + x(\Sigma)$. As in the proof of [2, Theorem 3.13], if Q is a Thurston norm minimizing surface in the homology class $D(a) + k[\Sigma]$, and $Q \cap N$ has no disk or sphere components, then $Q \cap N$ gives a taut decomposition of N.

We can do oriented cut-and-paste of Q with copies of Σ, to get a new surface Q', such that $Q' \cap N$ has positive intersection number with δ. Of course, $Q' \cap N$ still gives a taut decomposition of N. The not-so-good thing is that $\partial(Q' \cap N)$ is not necessarily $\omega_- \sqcup (-\omega_+)$. What we can do is to apply [3, Lemma 0.6]. Note that in the proof of [3, Lemma 0.6], one gets a new decomposition surface with prescribed boundary by gluing subsurfaces W_i of $\Sigma_\pm$ to the original decomposition surface. And by [2, Lemma 3.10], W_i has the same orientation as $\Sigma_\pm$. So the algebraic intersection number of this new decomposition surface with δ is no less than $(Q' \cap N) \cdot \delta > 0$.

Denote the new decomposition surface by S_0, $\partial S_0 = \omega_- \sqcup (-\omega_+)$. Suppose S_1 is the component of S_0 which contains ω_-. For homological reason, S_1 should also contain $-\omega_+$. Thus other components of S_0 are closed surfaces which do not separate Σ_- from Σ_+. Hence the algebraic intersection number of other components with δ is 0. S_1 also gives a taut decomposition of N, by [3, Lemma 0.4]. So S_1 is the surface we need. Similarly, we can prove the result for $\mathcal{S}_m(-\omega)$. $\qquad \square$

We also need the following key lemma.

Lemma 6.5 *For any positive integers* p, q,

$$x(\mathcal{S}_p(+\omega)) + x(\mathcal{S}_q(-\omega)) > (p + q)x(\Sigma).$$

Suppose $S_1 \in \mathcal{S}_p(+\omega)$, $S_2 \in \mathcal{S}_q(-\omega)$. Isotope S_1, S_2 so that they are transverse. Since N is irreducible and S_1, S_2 are incompressible, we can assume $S_1 \cup S_2 - S_1 \cap S_2$ has no disk components. Perform oriented cut-and-paste to S_1, S_2, we get a closed surface $P \subset \text{int}(N)$, with $x(P) = x(S_1) + x(S_2)$. P has no sphere components, otherwise $S_1 \cup S_2 - S_1 \cap S_2$ would have disk components.

Now we will deal with the possible toral components of P. To this end, we need the following lemma.

Lemma 6.6 *If* $T \subset \text{int}(N)$ *is a torus, then the algebraic intersection number of* T *and* δ *is 0.*

Proof Since N is a homology product, we have

$$H_2(D(N)) \cong H_2(\Sigma) \oplus H_1(\Sigma).$$

T is disjoint from Σ, so $[T]$ must be a multiple of $[\Sigma]$ in $H_2(D(N))$. By Remark 6.3, Σ is Thurston norm minimizing. Since $x(\Sigma) > 0 = x(T)$, we must have $[T] = 0$. Hence $T \cdot \delta = 0$. $\qquad \square$

Suppose T is a toral component of P, then T is the union of $2m$ annuli $A_1, A_2, \ldots, A_{2m}$, where $A_{2i-1} \subset S_1$, $A_{2i} \subset S_2$. Let

$$S_1' = \left(S_1 - \bigcup_{i=1}^{m} A_{2i-1} \right) \cup \bigcup_{i=1}^{m} (-A_{2i}),$$

$$S_2' = \left(S_2 - \bigcup_{i=1}^{m} A_{2i} \right) \cup \bigcup_{i=1}^{m} (-A_{2i-1}).$$

Here $-A_j$ means A_j with opposite orientation.

A small isotopy will arrange that $|S_1' \cap S_2'| < |S_1 \cap S_2|$. Moreover, $x(S_1') = x(S_1)$, $x(S_2') = x(S_2)$. We want to show that $S_1' \in \mathcal{S}_p(\omega)$, $S_2' \in \mathcal{S}_q(-\omega)$. Obviously, $\partial S_1' = \partial S_1 = \omega_- \sqcup (-\omega_+)$, $\partial S_2' = \partial S_2 = (-\omega_-) \sqcup \omega_+$. Lemma 6.6 shows that $S_1' \cdot \delta = S_1 \cdot \delta$. Thus $S_1' \in \mathcal{S}_p(+\omega)$. Similarly, $S_2' \in \mathcal{S}_q(-\omega)$. Therefore, we can replace S_1, S_2 with S_1', S_2', then continue our argument.

Now we can assume P has no toral components, and proceed to the proof of Lemma 6.5. Our approach to this lemma was suggested by David Gabai. In fact, this argument is similar to the argument in [4, Lemma 8.22].

Proof of Lemma 6.5 If $x(\mathcal{S}_p(+\omega)) + x(\mathcal{S}_q(-\omega)) \leq (p+q)x(\Sigma)$, then we can get a surface $P \subset \text{int}(N)$ as above, $x(P) \leq (p+q)x(\Sigma)$. Define a function $\varphi \colon (N - P) \to \mathbb{Z}$ as follows. When $z \in \Sigma_-$, $\varphi(z) = 0$. In general, given $z \in N - P$, choose a path from Σ_- to z, φ is defined to be the algebraic intersection number of this path with P.

N has the homology type of Σ, thus any closed curve in N should have zero algebraic intersection number with any closed surface. Thus φ is well-defined. Moreover, the value of φ on Σ_+ is $p + q$.

Let J_i be the closure of $\{ x \in (N - P) \mid \varphi(x) = i \}$, $P_i = J_{i-1} \cap J_i$. Thus $P = \bigsqcup_{i=1}^{p+q} P_i$, and $\bigcup_{k=0}^{i-1} J_k$ gives a homology between Σ and P_i. Since $x(P) \leq (p+q)x(\Sigma)$, and Σ is Thurston norm minimizing in $D(N)$, we must have $x(P_i) = x(\Sigma)$ for each i.

P_i has only one component. Otherwise, suppose $P_i = Q_1 \sqcup Q_2$, then

$$x(Q_1), x(Q_2) < x(P_i) = x(\Sigma).$$

As in the proof of Lemma 6.6, we find that $[Q_1]$, $[Q_2]$ are multiples of $[\Sigma]$, which gives a contradiction.

We can isotope P, so that $P_i \cap M$ is a genus g surface with one boundary component. In fact, after an isotopy, we can arrange that $P \cap \gamma_1$ consists of parallel essential curves in γ_1. Since F and G are Thurston norm minimizing in $H_2(M, \gamma)$ and $H_2(G \times I, \partial G \times I)$, respectively, we must have

$$x(P_i \cap M) = x(F), \quad x(P_i \cap (G \times I)) = x(G) = 1.$$

If an annulus A is a component of $P_i \cap (G \times I)$, then we can isotope A inside $G \times I$ into $\partial G \times I$, a further isotopy of P will decrease the number

of components of $P \cap (G \times I)$. So we can assume that $P_i \cap (G \times I)$ has no annular components. Now the fact that $x(P_i \cap (G \times I)) = 1$ indicates that $P_i \cap (G \times I)$ is either a thrice-punctured sphere or a once-punctured torus. In the former case there would be an essential sphere in $\widehat{G} \times I$, where $\widehat{G}$ is the torus obtained by capping off ∂G, which is impossible. Hence $P_i \cap (G \times I)$ is a once-puncture torus, and $P_i \cap M$ is diffeomorphic to F.

Since M is vertically prime, $P_i \cap M$ is parallel to either $R_-(\gamma)$ or $R_+(\gamma)$ in M. Now the picture of $P_1, \ldots, P_{p+q}$ is clear: there exists a number $r \in \{0, 1, \ldots, p+q\}$, such that $P_1, \ldots, P_r$ are parallel to Σ_-, and $P_{r+1}, \ldots, P_{p+q}$ are parallel to Σ_+. Here we let $P_0 = \Sigma_-$, $P_{p+q+1} = \Sigma_+$.

Since P is gotten by doing cut-and-paste to S_1, S_2, we can isotope S_1 so that $S_1 \cap J_i$ consists of vertical annuli. See Fig. 9 for the local picture. We denote $S_1 \cap J_i$ by $C_i \times I$, where C_i is the collection of some circles in P_i. Obviously, $[C_i]$ is homologous to $[\omega]$.
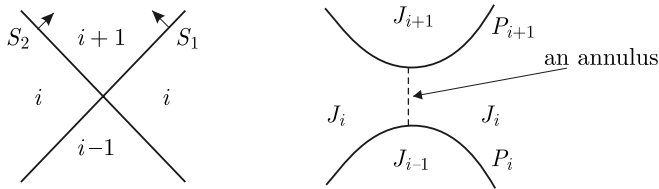

Fig. 9: Doing oriented cut-and-paste to S_1, S_2

Consider J_r, which is bounded by P_r, P_{r+1}. J_r is homeomorphic to N. Since

$$[\omega] \notin \text{im}(i_* : H_1(E \times I) \to H_1(N)),$$

one component of $C_r \times I$ must lie outside $\text{im}\, i_*$. Thus this vertical annulus can not be homotoped into $E \times I$, which contradicts to the definition of characteristic product region. □

Lemma 6.7 *Let K be a knot in a homology 3-sphere Y, Y_p be the manifold obtained by p-surgery on K. Let $g > 1$ be the genus of K. Suppose $\widehat{HFK}(Y, K, g; \mathbb{Q}) \cong \mathbb{Q}$, then*

$$HF^+(Y_0, [g-1]; \mathbb{Q}) \cong \mathbb{Q}.$$

Proof We will use $\mathbb{Q}$ coefficients in the homologies. As in [13, Corollary 4.5], when p is sufficiently large, we have two exact triangles

$$\xrightarrow{\delta} \widehat{HFK}(Y, K, g) \xrightarrow{\sigma} HF^+(Y_p, [g-1]) \xrightarrow{\psi} HF^+(Y) \xrightarrow{\delta} ,$$

$$\xrightarrow{\delta'} HF^+(Y_0, [g-1]) \xrightarrow{\sigma'} HF^+(Y_p, [g-1]) \xrightarrow{f} HF^+(Y) \xrightarrow{\delta'} .$$

And f has the form $\psi + \iota$, where ι is a sum of homogeneous maps which have lower orders than ψ.

Since $\widehat{HFK}(Y, K, g; \mathbb{Q}) \cong \mathbb{Q}$, either δ is surjective or σ is injective. Therefore, either ψ is injective, or ψ is surjective. For simplicity, denote $HF^+(Y_p, [g-1])$ by A, and $HF^+(Y)$ by B.

If ψ is injective, B can be written as $\psi(A) \oplus C$ for some subgroup C of B. If $b \in B$ is in the form of $(\psi(a), c)$, then let $\rho(b) = a$. Thus ρ is a homomorphism, $\rho\psi = \mathrm{id}$, and $\iota\rho\colon B \to B$ is a homomorphism which strictly decreases degree. Now

$$\mathrm{id} - \iota\rho + (\iota\rho)^2 - (\iota\rho)^3 + \cdots$$

is a well-defined homomorphism and

$$\psi = (\mathrm{id} - \iota\rho + (\iota\rho)^2 - (\iota\rho)^3 + \cdots)(\psi + \iota).$$

Hence $f = \psi + \iota$ is also injective, and $HF^+(Y_0, [g-1]) \cong B/f(A)$.

It is easy to check that $\mathrm{id} + \iota\rho$ induces a homomorphism from $B/\psi(A)$ to $B/f(A)$, whose inverse is induced by

$$\mathrm{id} - \iota\rho + (\iota\rho)^2 - (\iota\rho)^3 + \cdots.$$

Thus $B/f(A) \cong B/\psi(A)$. So $\mathrm{rank}(HF^+(Y_0, [g-1])) = 1$.

A similar argument shows that if ψ is surjective, then f is also surjective, and $\mathrm{rank}(HF^+(Y_0, [g-1])) = 1$. $\square$

Proof of Theorem 6.2 Use the notations in Remark 6.3, we have

$$\mathrm{rank}(\widehat{HFK}(Y\#Z, K\#J, g+1)) = \mathrm{rank}(\widehat{HFK}(Y, K, g)) = 1.$$

Thus $\mathrm{rank}(HF^+(Y_0', g)) = 1$ by Lemma 6.7.

If i_* is not surjective, then the proof of [5, Theorem 1.4], combined with Lemma 6.4 and Lemma 6.5, shows that

$$\mathrm{rank}(HF^+(Y_0', g)) > 1,$$

which gives a contradiction.

More precisely, by Lemma 6.4 and Lemma 6.5, there exist connected surfaces $S_1 \in \mathcal{S}_m(+\omega)$ and $S_2 \in \mathcal{S}_m(-\omega)$, such that they give taut decompositions of N, and $x(S_1) + x(S_2) > 2mx(\Sigma)$. By Gabai's work [2, Sect. 5], there exist two taut foliations $\mathcal{F}_1, \mathcal{F}_2$ of N, such that

$$\chi(S_1) = e(\mathcal{F}_1, S_1) = e(\mathcal{F}_1, S_0) + m\chi(\Sigma),$$
$$\chi(S_2) = e(\mathcal{F}_2, S_2) = e(\mathcal{F}_2, -S_0) + m\chi(\Sigma).$$

Here $e(\mathcal{F}, S)$ is defined in [5, Definition 3.7], S_0 is any surface in $\mathcal{S}_0(+\omega)$.

Now we can conclude that $e(\mathcal{F}_1, S_0) \neq e(\mathcal{F}_2, S_0)$. Hence [5, Theorem 3.8] can be applied. $\square$

7 Proof of the main theorem

Proof of Theorem 1.1 Suppose (M, γ) is the sutured manifold obtained by cutting open $Y - \text{int}(\text{Nd}(K))$ along F, $E \times I$ is the characteristic product region. We need to show that M is a product. By Proposition 3.1, M is a homology product. Moreover, by Theorem 4.1, we can assume M is vertically prime.

If M is not a product, then $M - E \times I$ is nonempty. Thus there exist some product annuli in (M, γ), which split off $E \times I$ from M. Let (M', γ') be the remaining sutured manifold. By Theorem 6.2, $R_\pm(\gamma')$ are planar surfaces, and $M' \cap (E \times I)$ consists of separating product annuli in M. Since we assume that M is vertically prime, M' must be connected. (See the first paragraph in the proof of Theorem 5.1.) Moreover, M' is also vertically prime, and there are no nontrivial product disks or product annuli in M'. By Theorem 5.1, $\widehat{HFS}(M', \gamma') \cong \mathbb{Z}$.

We add some product 1-handles to M' to get a new sutured manifold (M'', γ'') with γ'' connected. By Proposition 2.9, $\widehat{HFS}(M'', \gamma'') \cong \mathbb{Z}$. It is easy to see that M'' is also vertically prime. Proposition 3.1 shows that M'' is a homology product.

In the manifold M'', the characteristic product region $E'' \times I$ is the union of the product 1-handles and $\text{Nd}(\gamma')$. Obviously $i_* \colon H_1(E'') \to H_1(M'')$ is not surjective, which contradicts to Theorem 6.2. □

Proof of Corollary 1.2 Cut $Y - \text{int}(\text{Nd}(L))$ open along F, we get a sutured manifold (M, γ), $\widehat{HFS}(M, \gamma) \cong \mathbb{Z}$. By adding product 1-handles with feet at γ, we can get a new sutured manifold (M', γ'), where γ' has only one component. We have $\widehat{HFS}(M', \gamma') \cong \mathbb{Z}$. By Theorem 1.1, M' is a product, hence M is also a product. So the desired result holds. □

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