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Fintushel–Stern knot surgery in torus bundles

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Abstract

Suppose that X is a torus bundle over a closed surface with homologically essential fibers. Let X_K be the manifold obtained by Fintushel–Stern knot surgery on a fiber using a knot $K \subset S^3$. We prove that X_K has a symplectic structure if and only if K is a fibered knot. The proof uses Seiberg–Witten theory and a result of Friedl–Vidussi on twisted Alexander polynomials.

1. *Introduction*

One important question in four-dimensional topology is to determine which smooth closed 4-manifolds admit symplectic structures. There are some topological constructions of symplectic 4-manifolds. For example, Thurston [**23**] showed that most surface bundles over surfaces are symplectic, and Gompf [**11**] generalized this result to Lefschetz fibrations. On the other hand, there are obvious obstructions to the existence of symplectic structures from algebraic topology. Moreover, Taubes' results [**20**, **21**] provide more constraints in terms of the Seiberg–Witten invariants of the 4-manifold.

However, very little obstruction to the existence of symplectic structures is known besides the above-mentioned ones. For example, given a symplectic manifold X, a symplectic torus $T \subset X$ with $[T]^2 = 0$, and a knot $K \subset S^3$, Fintushel and Stern [**4**] introduced a construction called knot surgery to get a new manifold X_K. They showed that X_K is symplectic if K is fibered, and X_K can often be proven to be non-symplectic when the Alexander polynomial of K is not monic. (See Section 4 for more details.) However, if the Alexander polynomial of K is monic, the obstruction from Seiberg–Witten theory does not exclude the possibility that X_K has a symplectic structure. Nevertheless, one can mention the following folklore conjecture.

Conjecture 1.1. Suppose that X^4 is a closed 4-manifold admitting a Lefschetz fibration whose regular fibers are tori. Let $T \subset X$ be a regular fiber of the fibration, and suppose that $[T] \neq 0$ in $H_2(X;\mathbb{R})$. (Hence X is symplectic by [**23**].) Let X_K be a manifold obtained by Fintushel–Stern knot surgery on T using a knot $K \subset S^3$. Then X_K has a symplectic structure if and only if K is a fibered knot.

As we remarked before, the 'if' part of the above conjecture was proved by Fintushel and Stern. The most interesting case of Conjecture 1.1 is when $\pi_1(X \setminus T)$ (and hence $\pi_1(X)$ and $\pi_1(X_K)$) is trivial, as X_K is then homeomorphic to X by Freedman's theorem. In this case, the Lefschetz fibration of X must contain singular fibers. Our main result in this paper is the case of the above conjecture when X is a genuine torus bundle, namely, there are no singular fibers in the Lefschetz fibration.

Theorem 1.2. *Conjecture 1.1 is true when the Lefschetz fibration of X^4 is a torus bundle.*

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Friedl and Vidussi [6] proved that a closed 4-manifold $S^1 \times N$ is symplectic if and only if N is a surface bundle over S^1. Their result implies the special case of Theorem 1.2 when X is a trivial torus bundle $T^2 \times F = S^1 \times (S^1 \times F)$, where F is a closed surface. Our proof uses a similar strategy as in [6]. Namely, if X_K has a symplectic structure, then any finite cover of X_K also has a symplectic structure. We can then use the constraints from Seiberg–Witten theory to study the existence of symplectic structures on finite covers of X_K. The Seiberg–Witten invariants of finite covers of X_K can be expressed in terms of twisted Alexander polynomials of K. We then use a vanishing theorem for twisted Alexander polynomials due to Friedl–Vidussi [7] to get our conclusion. Of course, this strategy works only if the fundamental group of the 4-manifold we consider contains many finite index subgroups.

A major difference between [6] and our case is that any finite cover $\widetilde{N} \to N$ gives rise to a finite cover $S^1 \times \widetilde{N} \to S^1 \times N$, but the construction of finite covers of X_K is not so obvious. The main technical part of this paper is devoted to constructing finite covers of X_K. We also need the full strength of the gluing theorem for Seiberg–Witten invariants along essential T^3 from [22].

Throughout this paper, the manifolds we consider are all smooth and oriented. Suppose that M is a submanifold of a manifold N, then $\nu(M)$ denotes a closed tubular neighborhood of M in N, and $\nu^\circ(M)$ denotes the interior of $\nu(M)$.

This paper is organized as follows. In Section 2 we will review the definition of twisted Alexander polynomials and state a vanishing theorem of Friedl–Vidussi [7]. In Section 3 we will review the Seiberg–Witten invariants for 4-manifolds with boundary consisting of copies of T^3, and state the gluing formula for Seiberg–Witten invariants when glued along essential tori. In Section 4 we will review several constructions of symplectic 4-manifolds, and state the constraints on symplectic 4-manifolds from Seiberg–Witten theory. In Section 5, we will analyze the topology of torus bundles and construct certain covers of X_K. Our main theorem will then be proved in Section 6.

2. *Twisted Alexander polynomials*

Twisted Alexander polynomials were introduced by Xiao-Song Lin [15] in 1990. Many authors [2, 12, 13, 24] have since generalized this invariant in various ways. We will follow the treatment in [5].

Let N be a compact 3-manifold with $b_1(N) > 0$,

$$H = H(N) = H^2(N, \partial N)/\mathrm{Tors} \cong H_1(N)/\mathrm{Tors}.$$

Let F be a free abelian group, and $\phi \in Hom(H, F)$. Then $\pi_1(N)$ acts on F by translation via ϕ. Let $\alpha\colon \pi_1(N) \to GL(n, \mathbb{Z})$ be a representation. Then there is an induced representation

$$\alpha \otimes \phi\colon \pi_1(N) \to GL(n, \mathbb{Z}[F])$$

defined as follows. For $g \in \pi_1(N)$, $\alpha \otimes \phi(g)$ sends $\sum_{f\in F} a_f f \in (\mathbb{Z}[F])^n$ to

$$\sum_{f\in F} (\alpha(g)(a_f))(f\phi(g)),$$

where each $a_f \in \mathbb{Z}^n$, and the elements in F are written multiplicatively. Thus $(\mathbb{Z}[F])^n$ is a left $\mathbb{Z}[\pi_1(N)]$-module, whose left $\mathbb{Z}[\pi_1(N)]$ multiplication commutes with the right $\mathbb{Z}[F]$-module structure.

Let $\widetilde{N}$ be the universal cover of N, then $\pi_1(N)$ acts on the left of $\widetilde{N}$ as group of deck transformations. The chain group $C_*(\widetilde{N})$ is a right $\mathbb{Z}[\pi_1(N)]$-module, with the right action defined via $\sigma \cdot g := g^{-1}(\sigma)$. We can form the chain complex

$$C_*(\widetilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} (\mathbb{Z}[F])^n,$$

and define $H_*(N;\alpha\otimes\phi)$ by its homology group, which is also a $\mathbb{Z}[F]$-module. We call $H_1(N;\alpha\otimes\phi)$ the (*first*) *twisted Alexander module*.

Since $\mathbb{Z}[F]$ is Noetherian, $H_1(N;\alpha\otimes\phi)$ is a finitely generated module over $\mathbb{Z}[F]$. There exists a free resolution

$$(\mathbb{Z}[F])^m \xrightarrow{\ f\ } (\mathbb{Z}[F])^n \longrightarrow H_1(N;\alpha\otimes\phi) \longrightarrow 0,$$

where m,n are positive integers. We can always arrange that $m \geqslant n$. Let A be an $n\times m$ matrix over $\mathbb{Z}[F]$ representing f.

DEFINITION 2.1. The *twisted Alexander polynomial* of (N,α,ϕ), denoted by $\Delta^{\alpha}_{N,\phi}$, is the greatest common divisor of all the $n\times n$ minors of A. It is well defined only up to multiplication by a unit in $\mathbb{Z}[F]$.

When $F=H$ and ϕ is the identity on H, we simply write Δ^{α}_{N}. When α is the trivial representation to $GL(1,\mathbb{Z})$, we omit the superscript α. In particular, $\Delta_N\in\mathbb{Z}[H]$ is the usual Alexander polynomial of N. When $\alpha\colon \pi_1(N)\to G$ is a representation into a finite group, we get an induced representation into $\mathrm{Aut}(\mathbb{Z}[G])$, which is denoted by α as well. In that case, the twisted Alexander polynomials are essentially determined by the untwisted Alexander polynomials of the covers of N corresponding to $\ker\alpha$. More precisely, we recall [**5**, Proposition 3.6]:

PROPOSITION 2.2 (Friedl–Vidussi). *Let N be a 3-manifold with $b_1(N)>0$ and let $\alpha\colon \pi_1(N)\to G$ be an epimorphism onto a finite group. Let N_G be the covering space of N corresponding to $\ker\alpha$. Let $\pi_*\colon H(N_G)\to H(N)$ be the map induced by the covering map. Then Δ^{α}_{N} and Δ_{N_G} satisfy the following relations.*

- *If $b_1(N_G)>1$, then*

$$\Delta^{\alpha}_{N}=\begin{cases}\pi_*(\Delta_{N_G}), & \text{if } b_1(N)>1;\\ (a-1)^2\pi_*(\Delta_{N_G}), & \text{if } b_1(N)=1,\ \mathrm{im}\pi_*=\langle a\rangle.\end{cases}$$

- *If $b_1(N_G)=1$, then $b_1(N)=1$ and*

$$\Delta^{\alpha}_{N}=\pi_*(\Delta_{N_G}).$$

Given $\phi\in H^1(N)$, we say ϕ is *fibered* if ϕ is dual to a fiber of a fibration of N over S^1. A key ingredient in this paper is the following vanishing theorem of Friedl–Vidussi [**7**] concerning non-fibered cohomology classes.

THEOREM 2.3 (Friedl–Vidussi). *Let N be a compact, orientable, connected 3-manifold with (possibly empty) boundary consisting of tori. If $\phi\in H^1(N)$ is not fibered, then there exists an epimorphism $\alpha\colon\pi_1(N)\to G$ onto a finite group G such that $\Delta^{\alpha}_{N,\phi}=0$.*

3. *The Seiberg–Witten invariants and gluing formula along essential tori*

In this section, we will review the Seiberg–Witten theory for 4-manifolds with boundary consisting of tori, and the gluing formula for cutting along essential tori. We will follow the treatment in [**22**].

First, let us recall the usual Seiberg–Witten invariants for closed 4-manifolds [**26**]. Given a closed, oriented, connected, smooth, 4-manifold X with $b_2^+(X)>0$, let $\mathrm{Spin}^c(X)$ be the set of

Spin^c structures on X. One can define the Seiberg–Witten invariant $sw_X : \mathrm{Spin}^c(X) \to \mathbb{Z}/\{\pm 1\}$. The sign can be fixed by choosing an orientation on

$$L_X = \Lambda^{\mathrm{top}} H^1(X;\mathbb{R}) \otimes \Lambda^{\mathrm{top}} H^{2+}(X;\mathbb{R}).$$

In order to construct sw_X, we need to start with a Riemannian metric on X. It turns out that sw_X does not depend on the choice of the metric when $b_2^+(X) > 1$. When $b_2^+(X) = 1$, there are two chambers in the space of metrics corresponding to two orientations on $H_2^+(X;\mathbb{R})$, sw_X only depends on the chamber the metric lies in.

From now on in this section, we assume that X is a compact, oriented, connected, smooth 4-manifold such that ∂X is a (possibly empty) disjoint union of T^3, and there exists a cohomology class $\varpi \in H^2(X;\mathbb{R})$ whose pull-back is non-zero in the cohomology of each component of ∂X. When $\partial X = \emptyset$, we do not need such ϖ to define sw_X, but we still assume the existence of ϖ in order to state the gluing formula. Moreover, we assume $b_2^+(X) > 0$ when $\partial X = \emptyset$.

Let $\mathrm{Spin}^c(X)$ be the set of Spin^c structures on X, and $\mathrm{Spin}^c_0(X) \subset \mathrm{Spin}^c(X)$ be the subset consisting of $\mathfrak{s}$ such that the pull-back of $c_1(\mathfrak{s})$ is zero in $H^2(\partial X)$. By the exact sequence

$$H^2(X, \partial X) \xrightarrow{\ \pi^*\ } H^2(X) \xrightarrow{\ \iota^*\ } H^2(\partial X),$$

if $\mathfrak{s} \in \mathrm{Spin}^c_0(X)$, then $c_1(\mathfrak{s}) \in H^2(X)$ is in the image of π^*. Let

$$\mathrm{Spin}^c_0(X, \partial X) = \left\{ (\mathfrak{s}, z) \,\middle|\, \mathfrak{s} \in \mathrm{Spin}^c_0(X), z \in H^2(X, \partial X), \pi^*(z) = c_1(\mathfrak{s}) \right\}.$$

One can define the relative Seiberg–Witten invariant

$$sw_X : \mathrm{Spin}^c_0(X, \partial X) \to \mathbb{Z}/\{\pm 1\}.$$

The sign can be fixed by choosing an orientation on

$$L_X = \Lambda^{\mathrm{top}} H^1(X, \partial X;\mathbb{R}) \otimes \Lambda^{\mathrm{top}} H^{2+}(X, \partial X;\mathbb{R}).$$

When $\partial X = \emptyset$, sw_X is just the usual Seiberg–Witten invariant. When $\partial X \neq \emptyset$, sw_X is an invariant of the pair (X, ϖ), and it is unchanged under continuous deformation of ϖ in $H^2(X;\mathbb{R})$ through classes with non-zero restriction in the cohomology of each component of ∂X.

Suppose that $M \subset X$ is a 3-torus such that the restriction of ϖ to $H^2(M;\mathbb{R})$ is nontrivial. We will consider the gluing formula for sw when X is cut open along M. There are two cases. In the first case, X is split by M into two parts X_1, X_2. In the second case, $X_1 = X \setminus \nu^\circ(M)$ is connected.

When M is separating, there is a canonical isomorphism

$$L_X \cong L_{X_1} \otimes L_{X_2}. \tag{1}$$

One can define a map

$$\wp\colon \mathrm{Spin}^c_0(X_1, \partial X_1) \times \mathrm{Spin}^c_0(X_2, \partial X_2) \to \mathrm{Spin}^c_0(X, \partial X).$$

When M is non-separating, there is a canonical isomorphism

$$L_X \cong L_{X_1}. \tag{2}$$

One can define

$$\wp\colon \mathrm{Spin}^c_0(X_1, \partial X_1) \to \mathrm{Spin}^c_0(X, \partial X).$$

In any case, if $(\mathfrak{s}, z) \in \mathrm{im}\wp$, then $c_1(\mathfrak{s})|_M = 0$.

Theorem 3.1 (Taubes). *Let $M \subset X$ be a three-dimensional torus satisfying that the pull-back of ϖ in $H^2(M;\mathbb{R})$ is nontrivial.*

- *If M splits X into two parts X_1, X_2, we orient L_X using (1). Then*

$$sw_X(\mathfrak{s}, z) = \sum_{((\mathfrak{s}_1,z_1),(\mathfrak{s}_2,z_2))\in\wp^{-1}(\mathfrak{s},z)} sw_{X_1}(\mathfrak{s}_1, z_1) sw_{X_2}(\mathfrak{s}_2, z_2).$$

- *If M does not split X, let $X_1 = X \setminus \nu^\circ(M)$, we orient L_X using (2). Then*

$$sw_X(\mathfrak{s}, z) = \sum_{(\mathfrak{s}_1,z_1)\in\wp^{-1}(\mathfrak{s},z)} sw_{X_1}(\mathfrak{s}_1, z_1).$$

Theorem 3.1 implies the more general case of the gluing formula when we cut X open along more than one tori. Suppose that $M = M_1 \cup M_2 \cup \cdots \cup M_m$ is a disjoint union of three-dimensional tori in X such that the restriction of ϖ to $H^2(M_i; \mathbb{R})$ is nontrivial for every i. Let $X_1, \dots, X_n$ be the components of $X \setminus \nu^\circ(M)$. Let $\mathcal{G}$ be the graph with vertices $v_1, \dots, v_n$ and edges $e_1, \dots, e_m$. The incidence relation in $\mathcal{G}$ is as follows: if M_k is adjacent to X_i and X_j, the edge e_k connects v_i and v_j. Let $\mathcal{T}$ be a spanning tree of $\mathcal{G}$, then $\mathcal{T}$ has exactly $n-1$ edges. Without loss of generality, we may assume the edges in $\mathcal{G} \setminus \mathcal{T}$ are $e_1, \dots, e_{m-n+1}$. We consider a sequence of manifolds $X^{(i)}$, $i = 0, \dots, m$:

$$X^{(0)} = X, \quad X^{(i)} = X^{(i-1)} \setminus \nu^\circ(M_i), i > 0.$$

Clearly, M_i is non-separating in $X^{(i-1)}$ when $1 \leqslant i \leqslant m-n+1$, and M_i is separating in $X^{(i-1)}$ when $m-n+2 \leqslant i \leqslant m$. Thus we can apply Theorem 3.1 inductively to get the gluing formula when we cut open along M.

More precisely, applying (2) and (1) consecutively, we get a canonical isomorphism

$$L_X \cong \bigotimes_{i=1}^{n} L_{X_i},$$

which will be used to orient L_X. We can also define a map

$$\wp\colon \prod_{i=1}^{n} \mathrm{Spin}_0^c(X_i, \partial X_i) \to \mathrm{Spin}_0^c(X, \partial X).$$

Theorem 3.2. *Under the above settings, we have*

$$sw_X(\mathfrak{s}, z) = \sum_{((\mathfrak{s}_1,z_1),\dots,(\mathfrak{s}_n,z_n))\in\wp^{-1}(\mathfrak{s},z)} \prod_{i=1}^{n} sw_{X_i}(\mathfrak{s}_i, z_i).$$

In practice, it is more convenient to consider the following version of Seiberg–Witten invariant:

$$SW_X : H^2(X, \partial X) \to \mathbb{Z}$$

defined by letting

$$SW_X(z) = \sum_{(\mathfrak{s},z)\in\mathrm{Spin}_0^c(X,\partial X)} sw_X(\mathfrak{s}, z).$$

Let

$$\rho_i\colon H^2(X_i, \partial X_i) \to H^2(X, \partial X), \quad i = 1, \dots, n$$

be the natural maps, and

$$\rho = \rho_1 + \cdots + \rho_n\colon \bigoplus_{i=1}^{n} H^2(X_i, \partial X_i) \to H^2(X, \partial X).$$

Then Theorem 3.2 implies

Theorem 3.3. *Under the condition of Theorem 3.2, we have*

$$SW_X(z) = \sum_{(z_1,\ldots,z_n)\in\rho^{-1}(z)} \prod_{i=1}^{n} SW_{X_i}(z_i).$$

It is often convenient to represent the Seiberg–Witten invariants in the following more compact form.

Let $H(X) = H^2(X,\partial X)/\mathrm{Tors}$. Given $z \in H^2(X,\partial X)$, let $[z] \in H(X)$ be the reduction of z. We define

$$\underline{SW}_X = \sum_{z\in H^2(X,\partial X)} SW_X(z)[z],$$

which lies either in $\mathbb{Z}[H(X)]$, or, in certain cases, an extension of this group ring which allows semi-infinite power series.

For example, let $t \in H(D^2\times T^2)$ be the Poincaré dual to the fundamental class of the torus, then

$$\underline{SW}_{D^2\times T^2} = \frac{t}{1-t^2} = t + t^3 + \cdots . \tag{3}$$

The invariant $\underline{SW}_X$ is related to the Alexander polynomial of a 3-manifold. Let N be a compact, oriented, connected 3-manifold with $b_1(N) > 0$ such that ∂N is a (possibly empty) disjoint union of T^2. Let $p^*: H^2(N,\partial N) \to H^2(S^1\times N, \partial(S^1\times N))$ be the map on cohomology induced by the projection $p\colon S^1\times N \to N$. Let

$$\Phi_2 : \mathbb{Z}[H(N)] \to \mathbb{Z}[H(S^1\times N)]$$

be the map induced by $2p^*$. Meng and Taubes [**17**] proved the following theorem.

Theorem 3.4 (Meng–Taubes). *Let N be a compact, oriented, connected 3-manifold with $b_1(N) > 0$ such that ∂N is a (possibly empty) disjoint union of T^2. When $b_1(N) = 1$, let t be a generator of $H(N) \cong \mathbb{Z}$, and let $|\partial N| = 0$ or 1 be the number of boundary components of ∂N. Then, there exits an element $\xi \in \pm p^*(H(N))$, such that*

$$\underline{SW}_{S^1\times N} = \begin{cases} \xi\Phi_2(\Delta_N), & \text{if } b_1(N) > 1;\\ \xi\Phi_2((1-t)^{|\partial N|-2}\Delta_N), & \text{if } b_1(N) = 1.\end{cases}$$

As a corollary, we prove the following gluing result for the Alexander polynomial.

Corollary 3.5. *Let N be as in Theorem 3.4, and $K \subset N$ be a knot such that $[K]$ is nontorsion. Let $\kappa \in H[N]$ be the coset of the Poincaré dual of $[K]$. Let $M = N \setminus \nu^\circ(K)$, and let*

$$\pi^* : H^2(M,\partial M) \cong H^2(N, \nu(K)\cup\partial N) \to H^2(N,\partial N)$$

be the natural map induced by the inclusion $(N,\partial N) \subset (N, \nu(K)\cup\partial N)$. We also use π^* to denote the induced map $\mathbb{Z}[H(M)] \to \mathbb{Z}[H(N)]$. Then there exists an element $\xi \in \pm H(N)$, such that

$$\Delta_N = \begin{cases} \xi\pi^*(\Delta_M), & \text{if } b_1(N) = 1;\\ \xi(1-\kappa)^{-1}\pi^*(\Delta_M), & \text{if } b_1(N) > 1.\end{cases}$$

Proof. Let $p_N^*: H^2(N, \partial N) \to H^2(S^1 \times N, S^1 \times \partial N)$, and define p_M^* similarly. We first consider the case $b_1(N) > 1$. By Theorem 3.4, there exist $\zeta \in \pm p_N^*(H(N))$ and $\eta = \pm p_M^*(H(M))$ such that

$$\zeta \Phi_2(\Delta_N) = \sum_{z \in H^2(N,\partial N)} SW_{S^1 \times N}(p_N^*(z))[p_N^*(z)], \tag{4}$$

and

$$\eta \Phi_2(\Delta_M) = \sum_{w \in H^2(M,\partial M)} SW_{S^1 \times M}(p_M^*(w))[p_M^*(w)]. \tag{5}$$

Let $a \in H^2(T^2 \times D^2, T^2 \times \partial D^2)$ be the positive generator. Using Theorem 3.3 and (3), we get

$$\begin{aligned} &SW_{S^1 \times N}(p_N^*(z)) \\ &= \sum_{n \in \mathbb{Z}} SW_{T^2 \times D^2}((2n+1)a) \sum_{\substack{w \in H^2(M,\partial M) \\ \rho((2n+1)a), p_M^*(w)) = p_N^*(z)}} SW_{S^1 \times M}(p_M^*(w)) \\ &= \sum_{n \geqslant 0} \sum_{\substack{w \in H^2(M,\partial M) \\ \rho((2n+1)a, p_M^*(w)) = p_N^*(z)}} SW_{S^1 \times M}(p_M^*(w)). \end{aligned}$$

Using (4), (5), and the fact that

$$\rho((2n+1)a, p_M^*(w)) = p_N^*((2n+1)\mathrm{PD}([K]) + \pi^*(w)),$$

we get

$$\begin{aligned} &\zeta \Phi_2(\Delta_N) \\ &\quad = \sum_{z \in H^2(N,\partial N)} \sum_{n \geqslant 0} \sum_{\substack{w \in H^2(M,\partial M) \\ \rho((2n+1)a, p_M^*(w)) = p_N^*(z)}} SW_{S^1 \times M}(p_M^*(w))[p_N^*(z)] \\ &\quad = \sum_{n \geqslant 0} \sum_{w \in H^2(M,\partial M)} SW_{S^1 \times M}(p_M^*(w)) \kappa^{2n+1} [\pi^*(w)] \\ &\quad = \frac{\kappa}{1 - \kappa^2} \pi^*(\eta \Phi_2(\Delta_M)). \end{aligned}$$

So our result holds.

When $b_1(N) = 1$, the proof is similar. □

4. *Symplectic geometry*

In this section, we will review some topological constructions of symplectic 4-manifolds, and state the constraints on the Seiberg–Witten invariants of symplectic manifolds.

Thurston [**23**] found a very general topological construction of symplectic manifolds:

THEOREM 4.1 (Thurston). *Let $M^{2n+2} \to N^{2n}$ be a fiber bundle over a symplectic manifold. If the homology class of the fiber is nonzero in $H_2(M; \mathbb{R})$, then M has a symplectic structure such that each fiber is a symplectic submanifold. Moreover, if $\rho\colon N \hookrightarrow M$ is a section, then the image of ρ is a symplectic submanifold.*

In dimension 4, Thurston's construction was generalized by Gompf [**11**] to the extent that if a 4-manifold X admits a Lefschetz fibration (or a Lefschetz pencil) such that the homology class of the fiber is nontorsion, then X has a symplectic structure. This construction, together with the celebrated theorem of Donaldson [**3**] that all closed symplectic manifolds have Lefschetz pencils, gives us a topological characterization of closed symplectic 4-manifolds.

The above characterization of symplectic 4-manifolds is not always practical. When we construct symplectic 4-manifolds, we often need the following construction due to Gompf [**10**] and McCarthy–Wolfson [**16**]. Suppose that X_1, X_2 are two smooth 4-manifolds, $F_i \subset X_i$, $i = 1, 2$, are two two-dimensional closed connected submanifolds such that F_1 is homeomorphic to F_2 and $[F_1]^2 = -[F_2]^2$. Let $N(F_i), \nu(F_i)$ be two tubular neighborhoods of F_i in X_i, $i = 1, 2$, such that $\nu(F_i)$ is contained in the interior of $N(F_i)$. Let $W_i = N(F_i) \setminus \nu^\circ(F_i)$, $i = 1, 2$, regarded as an annulus bundle over F_i. Suppose that $f: F_1 \to F_2$ is a diffeomorphism, then there exists an orientation preserving diffeomorphism $\bar{f}: W_1 \to W_2$ such that $\bar{f}(\partial N(F_1)) = \partial\nu(F_2)$, and $\bar{f}$ is a bundle map covering f. Let X be the manifold obtained by gluing $X_1 \setminus \nu^\circ(F_1)$ and $X_2 \setminus \nu^\circ(F_2)$ together via the diffeomorphism $\bar{f}$. Then X is called the *normal connected sum* of (X_1, F_1) and (X_2, F_2), denoted $X_1 \#_f X_2$. If X_i is symplectic, F_i is a symplectic submanifold, $i = 1, 2$, and $f, \bar{f}$ are chosen to be symplectomorphisms, then X also has a symplectic structure, and the operation is called a *symplectic normal connected sum* or simply *symplectic sum*.

Suppose that X is a smooth 4-manifold containing a smooth 2-torus T with $[T]^2 = 0$. Let $K \subset S^3$ be a knot, and let $K' \subset S^3_0(K)$ be the dual knot in the zero surgery. We can perform the normal connected sum of (X, T) and $(S^1 \times S^3_0(K), S^1 \times K')$ to get a new manifold X_K. (This X_K is usually not unique, since it depends on the choice of a homeomorphism f and $\bar{f}$.) This procedure was investigated by Fintushel and Stern [**4**], who called it *knot surgery*. By Theorems 3.1 and 3.4, we know that

$$\underline{SW}_{X_K} = \underline{SW}_X \cdot \Delta_K(\mathrm{PD}([T])^2), \tag{6}$$

where Δ_K is the Alexander polynomial of K. (Clearly, X_K has the same homology type as X, so we can identify $H(X_K)$ with $H(X)$.) This construction is particularly interesting when $\pi_1(X \setminus T) = 1$, since X_K is then homeomorphic to X by Freedman's theorem, but X_K is not diffeomorphic to X if $\Delta_K \neq 1$.

When K is fibered, $S^1 \times S^3_0(K)$ is a surface bundle over T^2 with $S^1 \times K'$ being a section, and the fiber is homologically essential. Theorem 4.1 implies that $S^1 \times S^3_0(K)$ has a symplectic structure such that $S^1 \times K'$ is a symplectic submanifold. Hence the symplectic sum construction implies the following theorem.

THEOREM 4.2 (Fintushel–Stern). *Suppose that X is a symplectic 4-manifold, $T \subset X$ is a symplectic torus with $[T]^2 = 0$. Then X_K is symplectic if K is fibered.*

It is natural to guess that a converse to Theorem 4.2 should be true in many cases. More precisely, one can mention the folklore Conjecture 1.1. Evidence to this conjecture is a famous theorem of Taubes [**20**, **21**].

THEOREM 4.3 (Taubes). *Suppose that (X, ω) is a closed symplectic 4-manifold with $b_2^+ > 1$, $\mathfrak{k}$ is the canonical Spin^c structure on X, and $\bar{\mathfrak{k}}$ is the conjugate of $\mathfrak{k}$. Then*

$$SW_X(\mathfrak{k}) = \pm 1.$$

Moreover, if $\mathfrak{s} \in \mathrm{Spin}^c(X)$ satisfies that $SW_X(\mathfrak{s}) \neq 0$, then

$$|c_1(\mathfrak{s}) \smile [\omega]| \leqslant c_1(\mathfrak{k}) \smile [\omega],$$

and the equality holds if and only if $\mathfrak{s} = \mathfrak{k}$ or $\bar{\mathfrak{k}}$.

In particular, if X is the K3 surface, and Δ_K is not monic, Taubes' theorem implies that X_K is not symplectic.

We will also need the following theorem proved by Bauer [1] and Li [14].

THEOREM 4.4 (Bauer, Li). *Suppose that X is a closed symplectic 4-manifold with $c_1(\mathfrak{k})$ torsion. Then $b_1(X) \leqslant 4$.*

5. *Constructing covering spaces of X_K*

Before we state the main result in this section, we set up the basic notations we will use. Let X be a torus bundle over a closed surface F. Let T be a fiber of X, and let $E = X \setminus \nu^\circ(T)$. Let $K \subset S^3$ be a nontrivial knot, $N = S^3 \setminus \nu^\circ(K)$, $N_0 = S^3_0(K)$ be the zero surgery on K, and $K' \subset N_0$ be the dual knot of the surgery. Let $f \colon S^1 \times K' \to T$ be a diffeomorphism, and let $X_K = X \#_f (S^1 \times N_0)$.

The goal in this section is to construct covering spaces of X_K. More precisely, we will prove the following proposition.

PROPOSITION 5.1. *Suppose that $\alpha \colon \pi_1(N_0) \to G$ is an epimorphism, where G is a finite group. Let $p_0 \colon \widetilde{N}_0 \to N_0$ be the covering map corresponding to $\ker \alpha$, and let $\widetilde{N} = p_0^{-1}(N)$. Suppose that $p_0^{-1}(K')$ has r components. Since p_0 is a regular cover, the restriction of p_0 on each component of $p_0^{-1}(K')$ has the same degree l. If the genus of F is positive, then there exists an rl^3-fold cover $\widetilde{X}_K$ of X_K, such that $\widetilde{X}_K$ contains a submanifold diffeomorphic to $S^1 \times \widetilde{N}$, and $\widetilde{X}_K$ admits a retraction onto the complete bipartite graph $K_{r,l}$.*

In order to prove this proposition, we will need some preliminary material. We start by analyzing the topology of torus bundles. The structural group of a torus bundle is $\mathrm{Diff}^+(T^2)$, which is homotopy equivalent to its subgroup $\mathrm{Aff}^+(T^2) \cong T^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$. If the structural group is contained in $\mathrm{SL}(2,\mathbb{Z})$, we say this torus bundle is an $\mathrm{SL}(2,\mathbb{Z})$-*bundle*.

Each torus bundle $X \to F$ is uniquely determined up to isomorphism by the homotopy type of its classifying map $F \to B\mathrm{Diff}^+(T^2) \simeq B\mathrm{Aff}^+(T^2)$. From the short split exact sequence

$$1 \to T^2 \to \mathrm{Aff}^+(T^2) \to \mathrm{SL}(2,\mathbb{Z}) \to 1$$

we get a fiber bundle

$$BT^2 \to B\mathrm{Aff}^+(T^2) \to B\mathrm{SL}(2,\mathbb{Z})$$

which has a section. Since $BT^2 = \mathbb{C}P^\infty \times \mathbb{C}P^\infty = K(\mathbb{Z}^2, 2)$ and $B\mathrm{SL}(2,\mathbb{Z}) = K(\mathrm{SL}(2,\mathbb{Z}), 1)$, we have

$$\pi_1(B\mathrm{Aff}^+(T^2)) \cong \mathrm{SL}(2,\mathbb{Z}),$$
$$\pi_2(B\mathrm{Aff}^+(T^2)) \cong \mathbb{Z}^2.$$

Hence the homotopy type of a map $F \to B\mathrm{Aff}^+(T^2)$ is determined by a representation $\rho \colon \pi_1(F) \to \pi_1(B\mathrm{Aff}^+(T^2)) \cong \mathrm{SL}(2,\mathbb{Z})$ (called the *monodromy*) and a pair of integers $(m,n) \in H^2(F; \pi_2(B\mathrm{Aff}^+(T^2))) \cong \mathbb{Z}^2$ (called the *Euler class*).

REMARK 5.2. In particular, when $F = S^2$, X is completely determined by the Euler class. It is easy to see $[T] \neq 0 \in H_2(X;\mathbb{R})$ if and only if $(m,n) = (0,0)$. In this case, $X = T^2 \times S^2$. As we mentioned before, this case is covered by Friedl and Vidussi's work [6]. Hence, in order to prove Theorem 1.2, we only need to consider the case when the genus of F is positive.

REMARK 5.3. In general, $[T] \neq 0 \in H_2(X;\mathbb{R})$ if and only if $E^\infty_{0,2} \cong \mathbb{Z}$, where $\{E^i_{*,*}\}^\infty_{i=1}$ is the Leray–Serre spectral sequence for the fiber bundle $X \to F$. (See [9, Section 4] or [25, Lemma 4.6] for more detail.) When F is a torus, Geiges explicitly described the cases when $[T] \neq 0$ [**9**, Theorem 1], using Sakamoto–Fukuhara's classification of torus bundles over torus [**19**].

DEFINITION 5.4. Let $T \subset Y^4$ be a torus with trivial neighborhood. We fix a product structure $S^1 \times S^1$ on T and identify S^1 with $\mathbb{R}/\mathbb{Z}$. We can remove a neighborhood $\nu(T) \cong T^2 \times D^2$ then glue it back to $Y \setminus \nu^\circ(T)$ via the homeomorphism $f: T^2 \times \partial D^2 \to \partial(Y \setminus \nu^\circ(T))$ which sends (x, y, θ) to $(x + m\theta, y + n\theta, \theta)$. This procedure is called the (m, n)*-framed surgery* on T.

Given ρ and (m, n), as in [**25**, Section 4], we can reconstruct $X \to F$ by first constructing an $\mathrm{SL}(2, \mathbb{Z})$-bundle over F using the monodromy ρ then doing (m, n)-framed surgery on a fiber. Suppose that

$$\pi_1(F) = \left\langle a_1, a_2, \ldots, a_{2g-1}, a_{2g} \,\middle|\, \prod_{k=1}^{g} [a_{2k-1}, a_{2k}] \right\rangle$$

and that $\rho(a)$ acts on $\mathbb{Z}^2 = \langle s_1, s_2 | [s_1, s_2] \rangle$ for every $a \in \pi_1(F)$. We can write down a presentation of $\pi_1(X)$ from the construction of X as follows:

$$\pi_1(X) = \left\langle s_1, s_2, t_1, \ldots, t_{2g} \,\middle|\, \begin{array}{l} [s_1, s_2], \\ t_i s_j t_i^{-1} (\rho(a_i)(s_j))^{-1}, (1 \leqslant i \leqslant 2g, j = 1, 2) \\ s_1^m s_2^n (\prod_{k=1}^{g} [t_{2k-1}, t_{2k}])^{-1} \end{array} \right\rangle. \tag{7}$$

PROPOSITION 5.5. *Let $X \to F$ be a torus bundle over a closed surface with positive genus. For any integer $l > 0$, there exists a torus bundle $\widetilde{X}$ and an l^3-fold cover $p: \widetilde{X} \to X$, such that for any fiber $T \subset X$ and any component $\widetilde{T}$ of $p^{-1}(T)$, the map $p|_{\widetilde{T}}: \widetilde{T} \to T$ is the covering map corresponding to the characteristic subgroup $(l\mathbb{Z}) \times (l\mathbb{Z}) \subset \pi_1(T)$.*

Proof. Let $\overline{F} \to F$ be an l-fold cover, and $\overline{X} \to \overline{F}$ be a torus bundle over $\overline{F}$ which is the pull-back of $X \to F$. Suppose that the genus of $\overline{F}$ is $\bar{g}$, and the monodromy of $\overline{X}$ is $\bar{\rho}$. Suppose that the Euler class of $X \to F$ is (m, n), then the Euler class of $\overline{X} \to \overline{F}$ is (ml, nl). By (7),

$$\pi_1(\overline{X}) = \left\langle s_1, s_2, t_1, \ldots, t_{2\bar{g}} \,\middle|\, \begin{array}{l} [s_1, s_2], \\ t_i s_j t_i^{-1} (\bar{\rho}(a_i)(s_j))^{-1}, (1 \leqslant i \leqslant 2\bar{g}, j = 1, 2) \\ s_1^{ml} s_2^{nl} (\prod_{k=1}^{\bar{g}} [t_{2k-1}, t_{2k}])^{-1} \end{array} \right\rangle.$$

Let Γ_l be the subgroup of $\Gamma = \pi_1(\overline{X})$ generated by $s_1^l, s_2^l, t_1, \ldots, t_{2\bar{g}}$, we claim that $[\Gamma : \Gamma_l] = l^2$. If this claim is true, let $\widetilde{X}$ be the covering space of $\overline{X}$ corresponding to Γ_l, then $\widetilde{X}$ is the covering space of X we want.

The rest of this proof is devoted to proving $[\Gamma : \Gamma_l] = l^2$. Any element in Γ can be written as a word st, where s is a word in $s_1^{\pm 1}, s_2^{\pm 1}$, t is a word in $t_i^{\pm 1}$, $1 \leqslant i \leqslant 2\bar{g}$. Since the subgroup $\Sigma_l = \langle s_1^l, s_2^l \rangle$ of $\langle s_1, s_2 \rangle \cong \mathbb{Z}^2$ is preserved by any $\bar{\rho}(a_i)$, $st \in \Gamma_l$ if and only if $s \in \Sigma_l$. Let $(u_1, v_1), (u_2, v_2) \in \{0, 1, \ldots, l-1\}^2$, then it follows that $s_1^{u_1} s_2^{v_1} \in s_1^{u_2} s_2^{v_2} \Gamma_l$ if and only if $(u_1, v_1) = (u_2, v_2)$. So

$$s_1^u s_2^v \Gamma_l, \quad (u, v) \in \{0, 1, \ldots, l-1\}^2$$

are distinct left cosets of Γ_l in Γ. Clearly, the union of these cosets is Γ, so $[\Gamma : \Gamma_l] = l^2$. □

Proof of Proposition 5.1. By Proposition 5.5, there exists a degree l^3 covering map $p_X: \widetilde{X} \to X$, such that for any fiber $T \subset X$ and any component $\widetilde{T}$ of $p_X^{-1}(T)$, the map

$p_X|_{\widetilde{T}}\colon \widetilde{T} \to T$ is the covering map corresponding to $(l\mathbb{Z}) \times (l\mathbb{Z}) \subset \pi_1(T)$. By the construction of $\widetilde{X}$, $p_X^{-1}(T)$ has l components. Let $\widetilde{E} = p_X^{-1}(E)$.

There is a covering map

$$q_N = q_l \times p_0 \colon S^1 \times \widetilde{N}_0 \to S^1 \times N_0, \tag{8}$$

where $q_l \colon S^1 \to S^1$ is the l-fold cyclic cover. There are r components in $q_N^{-1}(S^1 \times K')$, and the restriction of q_N on each component is the covering map corresponding to $(l\mathbb{Z}) \times (l\mathbb{Z}) \subset \pi_1(S^1 \times K')$.

Since $(l\mathbb{Z}) \times (l\mathbb{Z})$ is a characteristic subgroup of $\mathbb{Z} \times \mathbb{Z}$, for any component $\widetilde{T}$ of $p_X^{-1}(T)$ and any component $\widetilde{S}$ of $q_N^{-1}(S^1 \times K')$, the map $f\colon S^1 \times K' \to T$ lifts to a map $\tilde{f}\colon \widetilde{S} \to \widetilde{T}$. Hence we can use $\tilde{f}$ to perform a normal connected sum of $\widetilde{X}$ and $S^1 \times \widetilde{N}_0$.

Recall that $p_X^{-1}(T) \subset \widetilde{X}$ has l components, and $q_N^{-1}(S^1 \times K') \subset S^1 \times \widetilde{N}_0$ has r components. Take r copies of $\widetilde{X}$ and l copies of $S^1 \times \widetilde{N}_0$. For any copy of $\widetilde{X}$ and any copy of $S^1 \times \widetilde{N}_0$, we can perform a normal connected sum of these two manifolds along a component of $p_X^{-1}(T)$ and a component of $q_N^{-1}(S^1 \times K')$, such that each component of $p_X^{-1}(T)$ or $q_N^{-1}(S^1 \times K')$ is used exactly once. The new manifold we get, denoted by $\widetilde{X}_K$, is clearly an rl^3-fold cover of X_K.

By the construction, $\widetilde{X}_K$ is obtained by gluing r copies of $\widetilde{E}$ and l copies of $S^1 \times \widetilde{N}$ together, such that any copy of $\widetilde{E}$ and any copy of $S^1 \times \widetilde{N}$ are glued along a T^3. Hence there is a retraction of $\widetilde{X}_K$ onto $K_{r,l}$. □

6. *Proof of the main theorem*

In this section, we will prove Theorem 1.2. By Remark 5.2, we only consider the case that X is a torus bundle over a closed surface F with positive genus. Assume that K is a nontrivial knot in S^3 and X_K is a symplectic manifold.

LEMMA 6.1. *There exists a finite cover of X_K with $b_1 > 4$.*

Proof. Let $\widehat{\Sigma} \subset N_0$ be the closed surface obtained from a minimal Seifert surface Σ of K by capping off $\partial\Sigma$ with a disk. By [8], N_0 is irreducible and $\widehat{\Sigma}$ is incompressible in N_0. Since $\pi_1(N_0)$ is residually finite, we can find an epimorphism α from $\pi_1(N_0)$ onto a finite group G, such that $\pi_1(\widehat{\Sigma}) \not\subset \ker\alpha$. Hence $p_0\colon \widetilde{N}_0 \to N_0$, the covering map corresponding to $\ker\alpha$, is not a cyclic covering map. As a result, $p_0^{-1}(K')$ has $r > 1$ components. Suppose that each component of $p_0^{-1}(K')$ is a l-fold cyclic cover of K'. We may assume $l > 5$, since we can always take a large cyclic cover of N_0 first.

We construct a cover $\widetilde{X}_K$ of X_K as in Proposition 5.1. Since there is a retraction of $\widetilde{X}_K$ onto $K_{r,l}$,

$$b_1(\widetilde{X}_K) \geqslant b_1(K_{r,l}) = (r-1)(l-1) \geqslant l-1 > 4.$$

□

COROLLARY 6.2. *Let $\mathfrak{k}$ be the canonical* Spin^c *structure of X_K. Then $c_1(\mathfrak{k})$ is nontorsion.*

Proof. By Lemma 6.1, there exists a finite cover $\widetilde{X}_K$ of X_K with $b_1 > 4$. Assume that $c_1(\mathfrak{k})$ is torsion, then $c_1(\widetilde{X}_K)$ is also torsion since it is the pull-back of $c_1(\mathfrak{k})$ by the covering map. By Theorem 4.4, $b_1(\widetilde{X}_K) \leqslant 4$, a contradiction. □

In order to apply Theorem 4.3, we need the following lemma.

LEMMA 6.3. *If $(r-1)(l-1) > 2$, then $b_2^+(\widetilde{X}_K) > 1$.*

Proof. The Euler characteristic of X is zero since the fiber has zero Euler characteristic. It is well known that the signature of X is zero [**18**]. Since X_K has the same homology type as X, both the Euler characteristic and the signature of X_K are zero, and the same is true for $\widetilde{X}_K$. It follows that

$$b_2^+(\widetilde{X}_K) = b_1(\widetilde{X}_K) - 1 \geqslant (r-1)(l-1) - 1 > 1. \qquad \square$$

Proof of Theorem 1.2. Assume that K is not fibered. By [**8**], N_0 is not fibered. Let ϕ be the positive generator of $H^1(N_0) \cong \mathbb{Z}$, and let $\psi \in H^1(N)$ be the restriction of ϕ. We can regard ϕ as a map $\pi_1(N_0) \to \mathbb{Z}$. By Theorem 2.3, there exists a surjective homomorphism $\alpha\colon \pi_1(N_0) \to G$, where G is a finite group, such that

$$\Delta^{\alpha}_{N_0} = \Delta^{\alpha}_{N_0,\phi} = 0. \tag{9}$$

As in Proposition 5.1, let $p_0\colon \widetilde{N}_0 \to N_0$ be the covering map corresponding to $\ker\alpha$, and let $\widetilde{N} = p_0^{-1}(N)$. We may assume $r > 1, l > 3$. Otherwise, as in the proof of Lemma 6.1, we can take a regular finite cover M_0 of N_0 satisfying $r > 1, l > 3$, and let $\beta\colon\ \pi_1(N_0) \to G_1$ be an epimorphism onto a finite group such that $\ker\beta = \ker\alpha \cap \pi_1(M_0)$. It follows from [**7**, Lemma 2.2] that $\Delta^{\beta}_{N_0} = 0$. So we can use β instead of α.

Since $r > 1, l > 3$, we have $b_2^+(\widetilde{X}_K) > 1$ by Lemma 6.3.

Let $(p_0)_*\colon \pi_1(\widetilde{N}_0) \to \pi_1(N_0)$ be the induced map on π_1, and let

$$\widetilde{\phi} = \phi \circ (p_0)_*\colon \pi_1(\widetilde{N}_0) \to \mathbb{Z}.$$

Let $\widetilde{\phi}_*\colon \mathbb{Z}[H(\widetilde{N}_0)] \to \mathbb{Z}[\mathbb{Z}]$ be the induced ring homomorphism. By Proposition 2.2, (9) implies $(p_0)_*(\Delta_{\widetilde{N}_0}) = 0$, hence

$$\widetilde{\phi}_*(\Delta_{\widetilde{N}_0}) = 0. \tag{10}$$

Let $(p_0|\widetilde{N})_*\colon \pi_1(\widetilde{N}) \to \pi_1(N)$ be the induced map on π_1, and let

$$\widetilde{\psi} = \psi \circ (p_0|\widetilde{N})_*\colon \pi_1(\widetilde{N}) \to \mathbb{Z}.$$

Let $\widetilde{\psi}_*\colon \mathbb{Z}[H(\widetilde{N})] \to \mathbb{Z}[\mathbb{Z}]$ be the induced ring homomorphism. We also regard $\widetilde{\psi}$ as a cohomology class in $H^1(\widetilde{N})$, then $\widetilde{\psi} \in H^1(\widetilde{N})$ is the pull-back of $\psi \in H^1(N)$ by the covering map. Clearly, for any component $\widetilde{K'}$ of $p_0^{-1}(K')$, we have $\widetilde{\psi}([\widetilde{K'}]) \neq 0$. Hence we can use Corollary 3.5 and (10) to conclude

$$\widetilde{\psi}_*(\Delta_{\widetilde{N}}) = 0. \tag{11}$$

We construct a finite cover $\widetilde{X}_K$ as in Proposition 5.1. Suppose that ω is a symplectic form on X_K. Since $[T] \neq 0 \in H_2(X;\mathbb{R}) \cong H_2(X_K;\mathbb{R})$ and $c_1(\mathfrak{k}) \neq 0 \in H^2(X_K;\mathbb{R})$ by Corollary 6.2, we may perturb and rescale ω so that

$$[\omega]([T]) \neq 0, \quad c_1(\mathfrak{k}) \smile [\omega] \neq 0, \tag{12}$$

and $[\omega] \in H^2(X_K;\mathbb{Z})$. Let Ω be the pull-back of ω on $\widetilde{X}_K$, then Ω is also a symplectic form. Moreover, it follows from (12) that

$$[\Omega]([\widetilde{T}]) \neq 0, \quad c_1(\widetilde{X}_K, \Omega) \smile [\Omega] \neq 0, \tag{13}$$

The inclusion map $S_1 \times N \subset X_K$ induces a map

$$\iota_N^*\colon H^2(X_K) \to H^2(S^1 \times N) \cong H^1(S^1) \otimes H^1(N).$$

Let σ be the positive generator of $H^1(S^1)$. Then

$$\iota_N^*([\omega]) = k\sigma \otimes \psi, \text{ for some integer } k \neq 0, \tag{14}$$

by (12).

Let $X_2 \subset \widetilde{X}_K$ be a copy of $S^1 \times \widetilde{N}$, let $X_1 = \widetilde{X}_K \setminus \mathrm{int}(X_2)$, and $M = \partial X_1$. Let $\iota_i^* \colon H^2(\widetilde{X}_K) \to H^2(X_i)$, $i = 1, 2$, be the natural maps induced by the inclusion maps.

Let $p^* \colon H^2(\widetilde{N}, \partial\widetilde{N}) \to H^2(S^1 \times \widetilde{N}, S^1 \times \partial\widetilde{N})$ be the map induced by the projection. Let q_N be the covering map in (8). If $w \in H^2(\widetilde{N}, \partial\widetilde{N})$, using (14), we have

$$\begin{aligned}\iota_2^*[\Omega] \smile p^*(w) &= q_N^*(\iota_N^*[\omega]) \smile p^*(w)\\ &= q_N^*(k\sigma \otimes \psi) \smile p^*(w)\\ &= kl\sigma\widetilde{\psi} \smile p^*(w)\\ &= kl\widetilde{\psi} \smile_3 w,\end{aligned} \tag{15}$$

where $\smile_3$ means the cup product in $(\widetilde{N}, \partial\widetilde{N})$. Here we identify an element $a \cup b \in H^n(Y^n, \partial Y^n)$ with an integer via the isomorphism $H^n(Y^n, \partial Y^n) \cong \mathbb{Z}$.

Let

$$\rho_i \colon H^2(X_i, \partial X_i) \to H^2(\widetilde{X}_K), \quad i = 1, 2,$$

be the natural restriction maps, and let

$$\rho = \rho_1 + \rho_2 \colon H^2(X_1, \partial X_1) \oplus H^2(X_2, \partial X_2) \to H^2(\widetilde{X}_K).$$

Suppose that $z_i \in H^2(X_i, \partial X_i)$, $i = 1, 2$, then it is elementary to check

$$\rho(z_1, z_2) \smile [\Omega] = z_1 \smile \iota_1^*[\Omega] + z_2 \smile \iota_2^*[\Omega]. \tag{16}$$

Suppose that $n = c_1(\widetilde{X}, \Omega) \smile [\Omega]$. Then $n \neq 0$ by (13). Using Theorem 3.3 and (16), we have

$$\begin{aligned}&\sum_{z \in H^2(\widetilde{X}_K), z \smile [\Omega] = n} SW_{\widetilde{X}_K}(z)\\ &= \sum_{\substack{z_1 \in H^2(X_1, \partial X_1)\\ z_2 \in H^2(X_2, \partial X_2)\\ z_1 \smile \iota_1^*[\Omega] + z_2 \smile \iota_2^*[\Omega] = n}} SW_{X_1}(z_1) SW_{X_2}(z_2)\\ &= \sum_{z_1 \in H^2(X_1, \partial X_1)} SW_{X_1}(z_1) \cdot \left(\sum_{\substack{z_2 \in H^2(X_2, \partial X_2)\\ z_2 \smile \iota_2^*[\Omega] = n - z_1 \smile \iota_1^*[\Omega]}} SW_{X_2}(z_2) \right).\end{aligned} \tag{17}$$

It follows from (15) and Theorem 3.4 that the inner sum in (17) is a coefficient in $\widetilde{\psi}_*(\Delta_{\widetilde{N}})$, which is zero by (11). Hence the right-hand side of (17) is zero. This contradicts Theorem 4.3 and the fact that $n \neq 0$. □

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