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ORIGINAL ARTICLE

A Note on Knot Floer Homology and Fixed Points of Monodromy

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Abstract
Using an argument of Baldwin–Hu–Sivek, we prove that if K is a hyperbolic fibered knot with fiber F in a closed, oriented 3-manifold Y, and $\widehat{HFK}(Y, K, [F], g(F)-1)$ has rank 1, then the monodromy of K is freely isotopic to a pseudo-Anosov map with no fixed points. In particular, this shows that the monodromy of a hyperbolic L-space knot is freely isotopic to a map with no fixed points.

Keywords Knot Floer homology · Fibered knot · Fixed points · L-space knot

Mathematics Subject Classification 57K18 · 57K20 · 53D40

1 Introduction

Knot Floer homology, defined by Ozsváth–Szabó [21] and Rasmussen [26], is a powerful knot invariant. It contains a lot of information about the topology of the knot. For example, it detects the Seifert genus of a knot K [22], and it determines whether K is fibered [7, 15]. In fact, such information is contained in $\widehat{HFK}(K, g(K))$, the first nontrivial term of the knot Floer homology with respect to the Alexander grading, which is often referred as the "topmost term" in knot Floer homology.

In recent years, it became clear that $\widehat{HFK}(K, g(K)-1)$, which is often called "the second term" or "the next-to-top term" in knot Floer homology, also contains interesting information about the knot. For example, Lipshitz–Ozsváth–Thurston [14] showed that the "second term" of the bordered Floer homology of a surface mapping class completely determines this mapping class. Baldwin and Vela-Vick [2] proved that the second term of the knot Floer homology of a fibered knot is always nontrivial,

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and Ni [18] proved the same result for knots whose topmost term is supported in a single $\mathbb{Z}/2\mathbb{Z}$-grading.

The main theorem in this paper is also of this type.

Theorem 1.1 *Let Y be a closed, oriented 3-manifold, and $K \subset Y$ be a hyperbolic fibered knot with fiber F and monodromy φ. If*

$$\operatorname{rank} \widehat{HFK}(Y, K, [F], g(F) - 1) = 1,$$

then φ is freely isotopic to a pseudo-Anosov map with no fixed points.

Recall that a knot $K \subset S^3$ is an *L-space knot* if a positive surgery on K is an L-space. The most interesting scenario to apply Theorem 1.1 is when $K \subset S^3$ is a hyperbolic L-space knot. In this case it is well known that rank $\widehat{HFK}(S^3, K, g(K)-1) = 1$ [24]. We hope Theorem 1.1 will shed more light on the understanding of L-space knots.

The proof of Theorem 1.1 uses a strategy due to Baldwin–Hu–Sivek [1], who proved Theorem 1.1 when K has the same $\widehat{HFK}$ as the cinquefoil $T_{5,2}$. In [1], the authors made use of the zero surgery formula on alternating knots in Heegaard Floer homology due to Ozsváth–Szabó [19]. We basically replace this result with a more general zero surgery formula. For simplicity, we only state here a formula for knots in S^3. If we are just interested in proving Theorem 1.1 for knots in S^3, this case will suffice. The zero surgery formula used in the proof of Theorem 1.1 is Proposition 3.1.

Given a null-homologous knot $K \subset Y$, let $Y_{p/q}(K)$ be the manifold obtained by $\frac{p}{q}$-surgery on K.

Proposition 1.2 *Let $K \subset S^3$ be a fibered knot with genus $g \geq 3$. Suppose that the monodromy of K is neither left-veering nor right-veering, then*

$$\operatorname{rank} HF^+(S^3_0(K), g-2) = \operatorname{rank} \widehat{HFK}(S^3, K, g-1) - 2.$$

During the course of this work, the author learned from John Baldwin that Theorem 1.1 is a special case of a theorem announced by Ghiggini and Spano [8], which states that $\widehat{HFK}(Y, K, g(K) - 1)$ is isomorphic to $HF^{\#}(\varphi)$, a version of the symplectic Floer homology of the monodromy φ.

This paper is organized as follows. In Section 2, we recall some basic results about mapping classes. In Section 3, we use the zero surgery formula in Heegaard Floer homology to prove Proposition 3.1 and Proposition 1.2. In Section 4, we prove Theorem 1.1.

2 Preliminaries on Mapping Classes

In this section, we recall Thurston's classification of mapping classes and the concept of right-veering diffeomorphisms.

Given a compact oriented surface F with boundary, Thurston [27] classified automorphisms of F as follows. Every automorphism φ falls into exactly one of 3 classes:

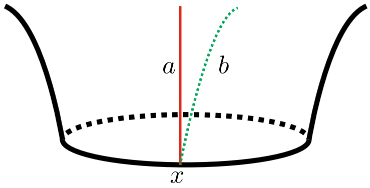

Fig. 1 The arc b is to the right of a

- Periodic: φ is freely isotopic to a periodic map $\widetilde{\varphi}$ with $\widetilde{\varphi}^n = \text{id}$ for some integer $n \geq 1$.
- Pseudo-Anosov: φ is freely isotopic to a pseudo-Anosov map $\widetilde{\varphi}$. That is, there exist two singular measured foliations $(\mathcal{F}^u, \mu^u)$, $(\mathcal{F}^s, \mu^s)$ of F, which are transverse everywhere except at the singular points, such that

$$\widetilde{\varphi}(\mathcal{F}^u, \mu^u) = \lambda(\mathcal{F}^u, \mu^u), \quad \widetilde{\varphi}(\mathcal{F}^s, \mu^s) = \lambda^{-1}(\mathcal{F}^s, \mu^s)$$

  for a fixed real number $\lambda > 1$.
- Reducible: φ is freely isotopic to a reducible map $\widetilde{\varphi}$. That is, there exists a collection $\mathcal{C}$ of mutually disjoint essential simple closed curves, such that $\widetilde{\varphi}(\mathcal{C}) = \mathcal{C}$. Moreover, $F \setminus \mathcal{C}$ can be divided into (possibly disconnected) subsurfaces $F_1, \dots, F_m$, such that the mapping class of $\widetilde{\varphi}|F_i$ is either periodic or pseudo-Anosov. We also require that $\mathcal{C}$ is the minimum collection of curves with this property, and $\mathcal{C} \neq \emptyset$.

Thurston proved that the mapping torus of φ has a complete, finite volume hyperbolic structure in its exterior if and only if the mapping class of φ is pseudo-Anosov.

Let $\varphi : F \to F$ be a diffeomorphism such that $\varphi(\partial F) = \text{id}_{\partial F}$. By Thurston's classification, φ is freely isotopic to a standard representative $\widetilde{\varphi}$, whose restriction to ∂F may not be the identity. For each component C of ∂F, one can define a *fractional Dehn twist coefficient (FDTC)* $c(\varphi) \in \mathbb{Q}$, which is the rotation number of $\widetilde{\varphi}$ on C compared with φ. This concept essentially appeared in work of Gabai [6, Remark 8.7 ii)], and the term FDTC was coined by Honda–Kazez–Matić [10].

Honda–Kazez–Matić [10] also defined right-veering diffeomorphisms.

Definition 2.1 Let F be a compact surface with boundary, $a, b \subset F$ be two properly embedded arcs with $a(0) = b(0) = x$. We isotope a, b with endpoints fixed, so that $|a \cap b|$ is minimal. We say b is *to the right of* a, denoted $a \leq b$, if either b is isotopic to a with endpoints fixed, or $(b \cap U) \setminus \{x\}$ lies in the "right" component of $U \setminus a$, where $U \subset F$ is a small neighborhood of x. See Fig. 1.

Definition 2.2 Let $\varphi\colon F \to F$ be a diffeomorphism that restricts to the identity map on ∂F. Let C be a component of ∂F. Then φ is *right-veering* with respect to C if for every $x \in C$ and every properly embedded arc $a \subset F$ with $x \in a$, the image $\varphi(a)$ is to the right of a at x. Similarly, we can define *left-veering* with respect to C. If φ is right-veering with respect to every component of ∂F, we say φ is a *right-veering* diffeomorphism. Similarly, we can define *left-veering* diffeomorphisms.

If the mapping class of φ is pseudo-Anosov, then φ is right-veering with respect to C if and only if $c(\varphi) > 0$ for C [10, Proposition 3.1]. If the mapping class of φ is periodic, then φ is right-veering with respect to C if $c(\varphi) > 0$ for C [10, Proposition 3.2].

3 Knot Floer Homology and the Zero Surgery Formula

In this section, we will use arguments in [16, 18] to prove Propositions 1.2 and 3.1. We assume the readers are reasonably familiar with Heegaard Floer homology.

Proposition 3.1 *Let Y be a closed, oriented 3-manifold, and $K \subset Y$ be a fibered knot with fiber F. Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ be the underlying Spin^c structure for the open book decomposition of Y with binding K and page F. Let $\widehat{F} \subset Y_0(K)$ be the closed surface obtained by capping off ∂F with a disk, and let $\mathfrak{t}_k \in \mathrm{Spin}^c(Y_0(K))$ be the Spin^c structure satisfying*

$$\mathfrak{t}_k|(Y \setminus K) = \mathfrak{s}|(Y \setminus K), \quad \langle c_1(\mathfrak{t}_k), \widehat{F}\rangle = 2k, \quad k \in \mathbb{Z}.$$

Suppose that either $\mathfrak{s}$ is torsion, or $HF^+(Y, \mathfrak{s}) = 0$. Suppose also that the monodromy of K is neither left-veering nor right-veering. If

$$\mathrm{rank}\, \widehat{HFK}(Y, K, [F], g(F) - 1) = 2$$

and $g(F) \geq 3$, then

$$\mathrm{rank}\, HF^+(Y_0(K), \mathfrak{t}_{g-2}) = 0.$$

Let Y be a closed, oriented 3-manifold, $K \subset Y$ be a fibered knot with a fiber F of genus g, and let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ be the Spin^c structure of the open book decomposition with binding K and page F. By [23], there exists a Heegaard diagram for (Y, K), such that the topmost knot Floer chain complex $\widehat{CFK}(Y, K, [F], g)$ has a single generator. Let

$$C = CFK^\infty(Y, K, \mathfrak{s}, [F])$$

be the knot Floer chain complex. Let ∂ be the differential on C, and let ∂_0 be the summand of ∂ which preserves the (i, j)-grading.

We will consider the chain complex $C\{i < 0, j \geq g - 2\}$, which has the form

$$\begin{array}{ccc} & & C(-1, g-1) \\ & \swarrow^{\partial_{zw}} & \downarrow{\scriptstyle \partial_z} \\ C(-2, g-2) & \xleftarrow{\partial_w} & C(-1, g-2), \end{array} \tag{1}$$

where

$$C(-i, g - i) \cong \widehat{CFK}(Y, K, \mathfrak{s}, [F], g) \cong \mathbb{Z} \quad \text{for all } i \in \mathbb{Z}, \tag{2}$$

and

$$C(-1, g-2) \cong \widehat{CFK}(Y, K, \mathfrak{s}, [F], g-1).$$

We also let $\partial_{zw} : C(-2, g-2) \to C(-3, g-3)$ denote the summand of ∂. By (2), $\partial_0 = 0$ on $C(-1, g-1)$ and $C(-2, g-2)$. Since $\partial^2 = 0$, we see that

$$\partial_w \partial_z = 0 \tag{3}$$

and $\partial_{zw}^2 = 0$, where ∂_{zw}^2 is a map $C(-1, g-1) \to C(-3, g-3)$. We can use (2) to identify every $C(-i, g-i)$ with $\mathbb{Z}$. Under this identification, the two maps

$$\partial_{zw} : C(-1, g-1) \to C(-2, g-2)$$

and

$$\partial_{zw} : C(-2, g-2) \to C(-3, g-3)$$

are the same map $\mathbb{Z} \to \mathbb{Z}$, since the chain complex C is U-equivariant. Since $\partial_{zw}^2 = 0$, we must have

$$\partial_{zw} = 0.$$

Lemma 3.2 *If the monodromy of K is neither left-veering nor right-veering, then the rank of the homology of the mapping cone (1) is*

$$\operatorname{rank} \widehat{HFK}(Y, K, \mathfrak{s}, [F], g-1) - 2.$$

Proof As we have showed above, the mapping cone (1) becomes the chain complex

$$\begin{array}{ccc} & & \mathbb{Z} \\ & & \downarrow{\scriptstyle \partial_z} \\ \mathbb{Z} & \xleftarrow{\ \partial_w\ } & C(-1, g-2). \end{array} \tag{4}$$

Since the monodromy of K is neither left-veering nor right-veering, by the argument in the proof of [2, Theorem 1.1], the induced map

$$(\partial_z)_* : \mathbb{Z} \to H_*(C(-1, g-2))$$

is injective, and the induced map

$$(\partial_w)_* : H_*(C(-1, g-2)) \to \mathbb{Z}$$

is surjective. Using (3), we see that the rank of the homology of (4) is

$$\operatorname{rank} H_*(C(-1, g-2)) - 2 = \operatorname{rank} \widehat{HFK}(Y, K, \mathfrak{s}, [F], g-1) - 2.$$

□

Proof of Proposition 1.2 By a standard argument, (see, for example, [21, Corollary 4.5],) $HF^+(S^3_0(K), g-2)$ is isomorphic to the homology of (1), so our conclusion follows from Lemma 3.2. □

As in [25], for any $k \in \mathbb{Z}$, let

$$A_k^+ = C\{i \geq 0 \text{ or } j \geq k\}, \quad k \in \mathbb{Z}$$

and $B^+ = C\{i \geq 0\} \cong CF^+(Y, \mathfrak{s})$. There are chain maps

$$v_k^+, h_k^+ \colon A_k^+ \to B^+.$$

Here v_k^+ is the vertical projection, and h_k^+ is essentially the horizontal projection.

Proof of Proposition 3.1 It is well known that $HF^+(Y_0(K), \mathfrak{t}_{g-2})$ is isomorphic to the homology of $MC(v_{g-2}^+ + h_{g-2}^+)$, the mapping cone of

$$v_{g-2}^+ + h_{g-2}^+ \colon A_{g-2}^+ \to B^+.$$

In fact, when $\mathfrak{s}$ is torsion, this result follows from the same argument as in [25, Subsection 4.8]; when $\mathfrak{s}$ is non-torsion, the formula is proved in [17, Theorem 3.1].

By the exact triangle

$$\begin{array}{ccc} H_*(A_{g-2}^+) & \xrightarrow{(v_{g-2}^+)_*} & H_*(B^+) \\ \uparrow & \swarrow & \\ H_*(C\{i<0, j \geq g-2\}) & & \end{array} \tag{5}$$

$C\{i < 0, j \geq g-2\}$ is quasi-isomorphic to $MC(v_{g-2}^+)$, the mapping cone of v_{g-2}^+.

By Lemma 3.2, $H_*(MC(v_{g-2}^+); \mathbb{Q}) = 0$, so $(v_{g-2}^+)_*$ is an isomorphism over $\mathbb{Q}$.

If $\mathfrak{s}$ is torsion, there is an absolute $\mathbb{Q}$-grading on A_{g-2}^+ and B^+. Since $g-2 > 0$, the grading shift of h_{g-2}^+ is strictly less than the grading shift of v_{g-2}^+. Hence $(v_{g-2}^+)_* + (h_{g-2}^+)_*$ is also an isomorphism over $\mathbb{Q}$, which implies that

$$HF^+(Y_0(K), \mathfrak{t}_{g-2}; \mathbb{Q}) = 0. \tag{6}$$

If $HF^+(Y, \mathfrak{s}) = 0$, $H_*(B^+) = 0$, so $H_*(A_{g-2}^+; \mathbb{Q}) = 0$ since $(v_{g-2}^+)_*$ is an isomorphism over $\mathbb{Q}$. We again have (6). □

4 Proof of the Main Theorem

Lemma 4.1 *There exists a hyperbolic fibered knot $L \subset Z = S^1 \times S^2$ with fiber G, such that the monodromy of the fibration is right-veering, and the Spin^c structure of the open book decomposition with binding L and page G is non-torsion.*

Proof By [5], there exists a non-torsion contact structure ξ on Z. Let (S, h) be an open book decomposition supporting ξ. By [3], we can stabilize (S, h) many times to get a new open book (G, ψ) with connected binding L, such that the monodromy ψ is pseudo-Anosov and right-veering. □

Let L be the knot as in Lemma 4.1, and let L' be the $(2n+1, 2)$-cable of L for a sufficiently large integer n. Let E be the fiber of the new fibration of Z with $\partial E = L'$, then

$$E = T \cup G_1 \cup G_2,$$

where T is a genus n surface with 3 boundary components, and G_1, G_2 are two copies of G. Let ρ be the monodromy of L'. Then $\rho|T$ is isotopic to a periodic map of period $4n + 2$, and ρ swaps G_1, G_2.

Lemma 4.2 *The FDTC of ρ on L' is $\frac{1}{4n+2}$.*

Proof The complement of L' is the union of a cable space and $Z \setminus L$. The slope on L' of the Seifert fiber of the cable space is $4n + 2$. Our conclusion follows from the definition of FDTC. □

Proof of Theorem 1.1 The proof of Theorem 1.1 uses a similar argument as [1, Theorem 3.5]. The new input here is to replace [1, Equation (3.3)] with Proposition 3.1.

Since rank $\widehat{HFK}(Y, K, [F], g(F) - 1) = 1$, it follows from [16, Theorem A.1] that φ is either right-veering or left-veering. Without loss of generality, we assume φ is right-veering.

Let L' be as in Lemma 4.2. By [9], we have

$$\widehat{HFK}(Z, L', [E], g(E) - 1) \cong \mathbb{Z}.$$

Consider the connected sum $K\#\overline{L'}$, which is a knot in $Y\#Z$. Let g' be the genus of the Seifert surface $F\natural\overline{E}$. By the Künneth formula,

$$\text{rank}\, \widehat{HFK}(Y\#Z, K\#\overline{L'}, [F\natural\overline{E}], g' - 1) = 2. \tag{7}$$

The monodromy of $K\#\overline{L'}$ is a map σ on $F\natural\overline{E}$. By Lemma 4.2, $\sigma|\overline{E}$ is left-veering, so σ is neither left-veering nor right-veering.

Let $\mathfrak{s} \in \text{Spin}^c(Y\#Z)$ be the Spin^c structure of the open book decomposition with binding $K\#\overline{L'}$ and page $F\natural\overline{E}$. Since the restriction of $\mathfrak{s}$ to $Z \setminus B^3$ is non-torsion, $HF^+(Y\#Z, \mathfrak{s}) = 0$ by the adjunction inequality [20].

Now Proposition 3.1 and (7) imply that

$$\operatorname{rank} HF^+((Y\#Z)_0(K\#\overline{L'}), g'-2) = 0. \tag{8}$$

The manifold $(Y\#Z)_0(K\#\overline{L'})$ is a surface bundle over S^1. Its fiber P is a closed surface which is the union of F and $\overline{E}$. Let $\widehat{\sigma}$ be the monodromy, then $\widehat{\sigma}|F = \varphi$.

The rest of our argument is similar to [1, Theorem 3.5]. Using work of Lee–Taubes [13], Kutluhan–Lee–Taubes [12], Kronheimer–Mrowka [11], one sees that $HF^+((Y\#Z)_0(K\#\overline{L'}), g'-2)$ is isomorphic to the symplectic Floer homology $HF_*^{\text{symp}}(P, \widehat{\sigma})$ of $(P, \widehat{\sigma})$. By (8), $HF_*^{\text{symp}}(P, \widehat{\sigma}; \mathbb{Q}) = 0$.

Recall that we assume φ is right-veering. In the terminology of [4], φ has no Type IIId fixed points. By [4, Theorem 4.16], $HF_*^{\text{symp}}(P, \widehat{\sigma}; \mathbb{Q})$ contains a direct summand which is freely generated by a superset of the fixed points of the pseudo-Anosov representative $\widetilde{\varphi}$ of φ. So $\widetilde{\varphi}$ has no fixed points. □

Remark 4.3 In [1], the authors computed $S^3_0(K\#\overline{K}, g(K\#\overline{K})-2)$ to get the conclusion. This approach will also work in the general case if the underlying Spin^c structure of the open book decomposition corresponding to K is torsion.

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