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Extending periodic automorphisms of surfaces to 3-manifolds

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In the orientable category, we show that, if a finite group G acts on a closed surface Σ so that each element of it is extendable over the 3-sphere S^3 as a diffeomorphism with respect to a fixed embedding $\Sigma \to S^3$, then G is extendable over S^3 as a group action with respect to some other embedding $\Sigma \to S^3$. Degree one maps are involved in the proof. Based on it, we classify all periodic automorphisms of closed surfaces that are extendable over S^3.

1. Introduction

We work in the smooth category, mainly consider oriented manifolds, and use $Aut(\cdot)$ to denote the orientation-preserving automorphism group of a manifold.

Let Σ be a compact oriented surface and M be an oriented 3-manifold, where Σ and M are possibly disconnected. An element f in $Aut(\Sigma)$ is *extendable* over M with respect to an embedding $e : \Sigma \to M$ if there exists an element f' in $Aut(M)$ such that $f' \circ e = e \circ f$. A subgroup G in $Aut(\Sigma)$ is *extendable* over M with respect to an embedding $e : \Sigma \to M$ if there is a group monomorphism $\phi : G \to Aut(M)$ such that $\phi(h) \circ e = e \circ h$ for any $h \in G$. An element f (respectively subgroup G) in $Aut(\Sigma)$ is *extendable* over M if f (resp. G) is extendable over M with respect to some embedding $\Sigma \to M$. When G is generated by a periodic map f, we also say that f is *periodically extendable*.

Let Σ_g be a closed oriented surface of genus g. We are interested in the following question.

Question 1.1. *If an orientation-preserving automorphism f of order n on Σ_g is extendable w.r.t. some embedding $e_1 : \Sigma_g \to S^3$, is f periodically extendable w.r.t. another embedding $e_2 : \Sigma_g \to S^3$?*

Question 1.1 has the positive answer, indeed we prove a more general result.

Theorem 1.2. *A finite subgroup G in $Aut(\Sigma_g)$ is extendable over S^3 if and only if there is an embedding $e : \Sigma_g \to S^3$ so that each element of G is extendable over S^3 with respect to e.*

Theorem 1.2 follows from Theorem 1.3. Both the statement and the proof of Theorem 1.3 involve degree one maps between 3-manifolds. For closed connected oriented 3-manifolds M and M', M' is *1-dominated* by M, denoted by $M \succeq_1 M'$, if there exists a degree one map $M \to M'$.

Theorem 1.3. *Given an integral homology 3-sphere M and a finite subgroup G in $Aut(\Sigma_g)$, if there is an embedding $e : \Sigma_g \to M$ so that each element of G is extendable over M with respect to e, then there exists an integral homology 3-sphere M' so that $M \succeq_1 M'$ and G is extendable over M'.*

Note that on the set of integral homology 3-spheres the relation "$\succeq_1$" is a partial order relation (see Proposition 2.4). Moreover, for a given M there are only finitely many M' satisfying $M \succeq_1 M'$ (see [BRW] and [Liu]).

In Theorem 1.3, passing to another 3-manifold M' is necessary, and in Question 1.1, S^3 can not be replaced by another 3-manifold. Even we start from S^3, to get the required embedding in the conclusion of Theorem 1.3, degree one maps are still essentially involved. The example below is an illustration of Question 1.1 and Theorem 1.3.

Example 1.4. First recall for each complete hyperbolic 3-manifold M with finite volume, $\mathrm{Iso}(M)$, the isometry group of M, has finite order, and moreover any finite group action on M can be conjugated into $\mathrm{Iso}(M)$ by the Geometrization Theorem and Orbifold Theorem (see [Th, Pe] and [BMP]).

(1) The right side of Figure 1 shows an embedded handlebody H_{g+1} in some 3-ball in S^3, where H_{g+1} is presented as g 1-handles attached on a solid torus H. Let $\Sigma_{g+1} = \partial H_{g+1}$. There is a $2\pi/g$-rotation r_g of S^3 which keeps H, H_{g+1}, Σ_{g+1} invariant setwise and acts freely on them. Let $f_g = r_g|_{\Sigma_{g+1}}$, which is shown in Figure 1. Clearly f_g is periodically extendable over S^3 w.r.t. the inclusion map.

Let $e : H_{g+1} \to S^3$ be the embedding shown on the left side of Figure 1, where $e(H)$ is a regular neighborhood of a figure-8 knot in S^3. One may assume that the handlebody $e(H_{g+1})$ stays in the interior of a thicker regular neighborhood H^* of the figure-8 knot. Since $r_g|_H$ is isotopic to the identity, there is a diffeomorphism F on H^* so that $F \circ e|_{\partial H_{g+1}} = e \circ f_g$ and $F|_{\partial H^*}$ is the identity. So F can extend to the whole S^3 by the identity on the complement of H^*, and f_g is extendable over S^3 w.r.t. e as a diffeomorphism.

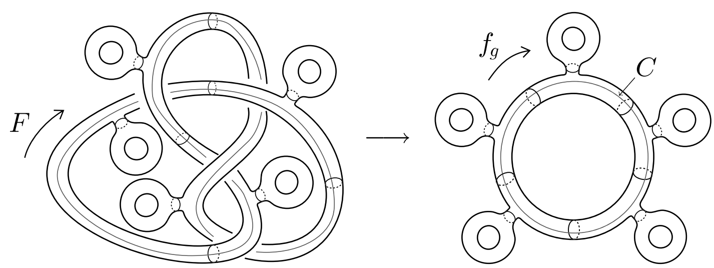

Figure 1: Replace the complement by a simple manifold.

On the other hand, f_g is not periodically extendable w.r.t. e for large g. Otherwise, there is a periodic map f' in $Aut(S^3)$ so that $f' \circ e = e \circ f_g$. Choose an invariant circle C in ∂H_{g+1} as in Figure 1. Then $f'|_{e(C)}$ is a translation of order g along a figure-8 knot in S^3. This can not happen when g is large, because the complement of the figure-8 knot has a complete hyperbolic structure of finite volume.

The periodically extendable embedding on the right side of Figure 1 can be obtained from the embedding on the left side via a degree one map $S^3 \to S^3$, which is the identity on H^* and maps the complement of H^* to a solid torus.

(2) Suppose that f is a periodic map on Σ_g and f is periodically extendable w.r.t. some embedding $\Sigma_g \subset N$. One example of such N is S^3 with $\Sigma_g \subset S^3$ and f being defined as in the first paragraph of Part (1). Another example is $\Sigma_g \times S^1$ for any periodic map f on Σ_g.

For a 3-ball $D \subset N$ missing Σ_g, it is known that f extends to some $\tilde{f}$ on N as a diffeomorphism so that $\tilde{f}|_D$ is the identity. Then, for any 3-manifold M, $\tilde{f}|_{N-D}$ can extend to the connected sum $M\#N = (M - B) \cup (N - D)$ by the identity on $M - B$, where B is the interior of a 3-ball in M. So f is extendable over $M\#N$ as a diffeomorphism.

For simplicity, assume N is irreducible and $M \neq N$. If M is a closed orientable hyperbolic 3-manifold (M can even be an integral homology 3-sphere), then f is not periodically extendable over $M\#N$ when the period n is large. Otherwise, let f' be a periodic extension of f. By the Equivariant Sphere Theorem (see [MY]), f' will induce a periodic map of order n on M. Since M is hyperbolic, the map is conjugate to an isometry of order n. This can not happen when n is large.

Via a degree one map $M\#N \to N$ which is the identity on $N - D$ and sends $M - B$ to D, we get back N, where f is periodically extendable w.r.t. $\Sigma_g \subset N$.

Note that every element in $Aut(S^3)$ is isotopic to the identity (see [Ha]), and every finite subgroup in $Aut(S^3)$ can be conjugated into $SO(4)$ (see [BMP, Pe]). Hence Theorem 1.2 means that, if a finite subgroup in $Aut(\Sigma_g)$ can be realized by topological motions of S^3 (or $\mathbb{R}^3$), then it can be realized by isometric motions of S^3. Also, note that finite subgroups in $Aut(\Sigma_g)$ that are extendable over S^3 with order bigger than $4(g-1)$ can be classified (see [WWZZ]). The following theorem classifies the finite cyclic subgroups (or periodic automorphisms) in $Aut(\Sigma_g)$ that are extendable over S^3 (see Remark 4.4).

Given a finite subgroup G in $Aut(\Sigma_g)$, there is an orbifold Σ_g/G determined by G. Suppose that Σ_g/G has underlying space Σ_r and s singular points of indices $n_1, \dots, n_s$. Then its orbifold fundamental group $\pi_1(\Sigma_g/G)$ has a presentation (see Figure 2 in Lemma 4.2)

$$\left\langle \alpha_1, \beta_1, \dots, \alpha_r, \beta_r, \gamma_1, \dots, \gamma_s \mid \prod_{i=1}^{r}[\alpha_i, \beta_i] \prod_{j=1}^{s} \gamma_j = 1, \gamma_k^{n_k} = 1, 1 \leq k \leq s \right\rangle.$$

The G-action on Σ_g gives an epimorphism $\psi : \pi_1(\Sigma_g/G) \to G$ which is injective on finite subgroups of $\pi_1(\Sigma_g/G)$. We call such ψ a *finitely-injective* epimorphism.

Theorem 1.5. *A finite cyclic subgroup G in $Aut(\Sigma_g)$ is extendable over S^3 if and only if the orbifold Σ_g/G and the finitely-injective epimorphism ψ satisfy:*

(a) *there exist co-prime positive integers p, q such that $n_1, \ldots, n_s \in \{p, q\}$;*

(b) *if $n_i = n_j$ for some $i \neq j$, then either $\psi(\gamma_i) = \psi(\gamma_j)$ or $\psi(\gamma_i\gamma_j) = 1$;*

(c) *$\gamma_1, \ldots, \gamma_s$ can be partitioned into pairs γ_i, γ_j such that $\psi(\gamma_i\gamma_j) = 1$.*

Two such subgroups G and G' are conjugate in $Aut(\Sigma_g)$ if and only if they are isomorphic and Σ_g/G and Σ_g/G' are homeomorphic as orbifolds.

Moreover, the corresponding embedded surface of such a subgroup G can always be a Heegaard surface.

Actually, unless $g = 1$ and G is a free action on Σ_1, the conjugacy class of G is determined by Σ_g/G, which can be enumerated by the Riemann–Hurwitz formula. We also have a standard form of the G-action (see Example 4.3). As a comparison, there exist finite subgroups in $Aut(\Sigma_{21})$ and $Aut(\Sigma_{481})$ which are extendable over S^3, but the embedded surfaces can not be Heegaard surfaces (see [WWZZ]). It is worth mentioning that if an automorphism of Σ_2 is extendable over S^3, then its corresponding embedded surface can always be a Heegaard surface (see [FK]).

After giving some preparations in Section 2, we will prove a stronger version of Theorem 1.3 in Section 3 and generalize the result to general actions on compact manifolds. Then, in Section 4, we will prove Theorem 1.5 and give some intuitive examples which can also be read directly after the introduction. In the appendix, we prove some results about automorphisms of surfaces with boundary.

At the end of the introduction, we mention some literatures related to Question 1.1:

1. There are many results about extending finite group actions on Σ_g to some 3-manifold bounded by Σ_g. For example, finite cyclic group actions are analyzed in the pioneer work [Bo1], and finite abelian group actions are analyzed in [RZ].

2. A result similar to our Theorem 1.2 has been obtained in [Fl] for finite cyclic group actions on finite 3-connected graphs. In this direction, the recent results in [FY] are close to the style of our Theorem 1.3.

3. To get intuition about the symmetries on surfaces, a sequence of papers on embedding symmetries of Σ_g into those of S^3 appeared recently, including [WWZZ] and [GWWZ]. The first one is devoted to maximum order

problems and the second one lists all extendable finite cyclic group actions on Σ_2. The present research is inspired by those papers.

2. Relation "$\succeq$" and property "Σ_g-splittable"

In this section, we introduce two concepts that will be used in our main result Theorem 3.1. Let $\mathcal{M}$ denote the set of closed connected oriented 3-manifolds. We first define the relation "$\succeq$" on $\mathcal{M}$, and then "Σ_g-splittable" for manifolds in $\mathcal{M}$. Some properties and examples about the concepts are given after the definitions, among which only Proposition 2.3 and Proposition 2.8 will be used later.

For M in $\mathcal{M}$ and a compact connected 3-manifold N embedded in M, let $\overline{M-N}$ denote the closure of $M-N$, and let ∂N denote the common boundary of N and $\overline{M-N}$.

If ∂N is a 2-sphere, then we can obtain a closed 3-manifold M' from N by gluing a 3-ball into ∂N. This process of obtaining M' from M is often called a "pinch".

If ∂N is a torus, then by the following "half-live half-die" lemma there exists a unique simple closed curve c in ∂N up to isotopy and orientation reversal such that $[c] \neq 0$ in $H_1(\partial N, \mathbb{Q})$ but $[c] = 0$ in $H_1(\overline{M-N}, \mathbb{Q})$, where $[c]$ denotes the homology class represented by c. Hence, we can obtain a closed 3-manifold M' from N by filling a solid torus into ∂N, mapping the meridian to c. This process of obtaining M' from M is also called a "pinch" in the literature.

Lemma 2.1. *For a compact orientable 3-manifold X, the dimension of the kernel of $H_1(\partial X, \mathbb{Q}) \to H_1(X, \mathbb{Q})$ is half of the dimension of $H_1(\partial X, \mathbb{Q})$.*

Definition 2.2. For M and M' in $\mathcal{M}$, if M' can be obtained from M as in the above construction, then define the relation $M \succeq_s M'$ ("s" for "surgery").

In general, we say $M \succeq M'$, if there is a sequence of manifolds

$$M = M_0, M_1, \ldots, M_n = M'$$

such that $M_i \succeq_s M_{i+1}$, $i = 0, 1, \ldots, n-1$. This defines a reflexive and transitive relation "$\succeq$" on $\mathcal{M}$.

Note that if M' can be obtained from M by pinching an embedded compact 3-manifold whose boundary is a sphere or a torus, then $M \succeq_s M'$. We also have a relation "$\succeq_H$" on $\mathcal{M}$, where $M \succeq_H M'$ if and only if there is an epimorphism $H_1(M,\mathbb{Q}) \to H_1(M',\mathbb{Q})$. It is well known that $M \succeq_1 M'$ implies $M \succeq_H M'$.

Proposition 2.3.

(1) *For M and M' in $\mathcal{M}$, $M \succeq M'$ implies $M \succeq_H M'$.*

(2) *If M is an integral homology 3-sphere, then $M \succeq M'$ implies $M \succeq_1 M'$ and M' is also an integral homology 3-sphere.*

Proof. Suppose that $M \succeq_s M'$, N is the compact 3-manifold embedded in M, c is the simple closed curve in ∂N, and T is the solid torus filled into ∂N along c.

(1) By the Mayer–Vietoris sequence of homology groups,

$$\begin{aligned}\dim H_1(M,\mathbb{Q}) = {} & \dim H_1(N,\mathbb{Q}) + \dim H_1(\overline{M-N},\mathbb{Q}) \\ & - \dim Im\big(H_1(\partial N,\mathbb{Q}) \to H_1(N,\mathbb{Q}) \oplus H_1(\overline{M-N},\mathbb{Q})\big),\end{aligned}$$

where "$Im(\cdot)$" denotes the image of the map. There is a similar equality for M' with $\overline{M-N}$ replaced by T. Let c' be a simple closed curve in ∂N such that $[c]$ and $[c']$ generate $H_1(\partial N,\mathbb{Q})$. For M and M', $[c]=0$ in $H_1(\overline{M-N},\mathbb{Q})$ and $H_1(T,\mathbb{Q})$ respectively, so the images of $[c]$ belong to $H_1(N,\mathbb{Q})$, and by Lemma 2.1, the images of $[c']$ are nonzero in the second factors. Hence for M and M' the last summands are equal. Then, by Lemma 2.1, we have

$$\dim H_1(M,\mathbb{Q}) - \dim H_1\big(M',\mathbb{Q}\big) = \dim H_1(\overline{M-N},\mathbb{Q}) - \dim H_1(T,\mathbb{Q}) \geq 0.$$

The general case when $M \succeq M'$ can be obtained by induction.

(2) Since M is an integral homology 3-sphere, by the Mayer–Vietoris sequence of homology groups, N and $\overline{M-N}$ are integral homology solid tori and $[c]=0$ in $H_1(\overline{M-N},\mathbb{Z})$. Then c bounds a connected surface in $\overline{M-N}$, and there is a map from $\overline{M-N}$ to T, which maps the surface to a meridian disk in T bounded by c and is the identity on ∂N. The map clearly extends to a degree one map from M to M' by the identity on N. So $M \succeq_1 M'$. Note that $H_1(M',\mathbb{Z})=0$. Hence M' is also an integral homology 3-sphere.

The general case when $M \succeq M'$ can be obtained by induction, because on each step we have an integral homology 3-sphere. □

Proposition 2.4. *On the set of integral homology 3-spheres the relation "$\succeq_1$" is a partial order relation, hence "$\succeq$" is also a partial order relation.*

Proof. We only need to show that "$\succeq_1$" is antisymmetric.

Suppose that M and M' are integral homology 3-spheres so that $M \succeq_1 M'$ and $M' \succeq_1 M$. Since the fundamental groups of manifolds in $\mathcal{M}$ are residually finite (see [He2] for Haken manifolds, the general case can be proved similarly based on Thurston's geometrization conjecture [Pe]), they are hopfian groups. Since degree one maps induce epimorphisms between fundamental groups, $\pi_1(M)$ and $\pi_1(M')$, as hopfian groups, must be isomorphic. Let $f : M \to M'$ be the degree one map, and let $f_* : \pi_1(M) \to \pi_1(M')$ be the induced isomorphism.

For a 3-manifold N with ∂N consisting of 2-spheres, let $\widehat{N}$ denote the manifold obtained by capping off ∂N with 3-balls. By the prime decomposition theorem of 3-manifolds, there is a compact 3-manifold N_0 embedded in M so that ∂N_0 consists of 2-spheres, $\widehat{N}_0$ is homeomorphic to S^3, and for each component N of $\overline{M - N_0}$ the 3-manifold $\widehat{N}$ is an irreducible integral homology 3-sphere. Suppose that $\overline{M - N_0}$ has m components, denoted by $N_1, N_2, \dots, N_m$. Let M_i be $\widehat{N}_i$, where $0 \le i \le m$. Similarly, for M' we have N'_j and M'_j, where $0 \le j \le m'$. By the positive solution of Poincaré conjecture (see [Pe]), we can assume that for $i, j \ge 1$ neither $\pi_1(M_i)$ nor $\pi_1(M'_j)$ is trivial. Also, note that each of these groups is not a nontrivial free product of its subgroups (see [He1]).

Assume that the base points of M and M' lie in N_0 and N'_0 respectively, and f preserves the base points. Then, according to the decompositions given by N_i and N'_j, $\pi_1(M_i)$ and $\pi_1(M'_j)$ can be embedded in $\pi_1(M)$ and $\pi_1(M')$, respectively. By the Kurosh subgroup theorem (see [He1, 8.3 and 8.4]), we have $m = m'$, and there exists a permutation σ of $\{1, 2, \dots, m\}$ such that $f_*(\pi_1(M_i))$ and $\pi_1(M'_{\sigma\cdot i})$ are conjugate in $\pi_1(M')$ for each $1 \le i \le m$.

By the positive solution of Thurston's geometrization conjecture and Borel conjecture in dimension three, an irreducible integral homology 3-sphere is determined by its fundamental group. Moreover, all these 3-manifolds are aspherical, except S^3 and the Poincaré homology sphere S^3_P. Hence M_i and $M'_{\sigma\cdot i}$ are homeomorphic for $1 \le i \le m$. Note that M_i and $M'_{\sigma\cdot i}$ have the induced orientations from M and M' respectively. We need to show that the homeomorphisms between M_i and $M'_{\sigma\cdot i}$ can be chosen to preserve the orientations. Then, as connected sums, the oriented 3-manifolds M and M' are homeomorphic.

For each fixed $1 \le i \le m$, there is a degree one map $f'_i : M' \to M'_{\sigma\cdot i}$, which is the identity on $N'_{\sigma\cdot i}$ and maps $\overline{M' - N'_{\sigma\cdot i}}$ to the 3-ball in $M'_{\sigma\cdot i}$ bounded by $\partial N'_{\sigma\cdot i}$. Since $\pi_2(M'_{\sigma\cdot i})$ is trivial, $f'_i \circ f$ is null-homotopic on each 2-sphere in ∂N_0. Hence, the degree one map $f'_i \circ f$ induces a map $f_{ji} : M_j \to$

$M'_{\sigma\cdot i}$ for each $0 \le j \le m$, and we have the equality $\sum_{j=0}^{m} \deg f_{ji} = \deg(f'_i \circ f) = 1$, or $1 - \deg f_{ii} = \sum_{j\ne i} \deg f_{ji}$, where "deg" denotes "degree".

For $j \ne i$, $f_*(\pi_1(M_j))$ is conjugate to the free factor $\pi_1(M'_{\sigma\cdot j})$ of $\pi_1(M')$, which lies in the kernel of the map $\pi_1(M') \to \pi_1(M'_{\sigma\cdot i})$, so $(f_{ji})_* : \pi_1(M_j) \to \pi_1(M'_{\sigma\cdot i})$ is trivial. Hence, if $M'_{\sigma\cdot i}$ is aspherical, then f_{ji} with $j \ne i$ are all null-homotopic, and we have $\deg f_{ii} = 1$; if $M'_{\sigma\cdot i}$ is S^3_P, then $120 \mid \deg f_{ji}$ for all $j \ne i$, and there exists a degree one map from M_i to $M'_{\sigma\cdot i}$. In each case, the degree one map is homotopic to a homeomorphism (see [Sun]). □

Example 2.5. Neither "$\succeq_1$" nor "$\succeq$" is antisymmetric on the set of lens spaces.

Let N be the 3-manifold $A \times S^1$, where A is an oriented annulus and S^1 is an oriented circle. Let C_1 and C_2 be the two components of ∂A. Let D be a disk in the interior of A, and let C_3 be ∂D. For $1 \le i \le 3$, choose a point P_i in C_i. Let S_i be the circle $\{P_i\} \times S^1$ in N. Then C_i and S_i have the induced orientations from $\overline{A - D}$ and S^1, respectively. Let N' be the submanifold $D \times S^1$ of N.

Clearly $[C_i]$ and $[S_i]$ generate $H_1(C_i \times S_i, \mathbb{Z})$ for $1 \le i \le 3$, and in $H_1(\overline{N - N'}, \mathbb{Z})$ we have $[C_1] + [C_2] + [C_3] = 0$. Let a_1, b_1, a_2, b_2 be integers so that the greatest common divisors $\gcd(a_1, b_1)$, $\gcd(a_2, b_2)$, $\gcd(a_1, a_2)$ are equal to 1. For $1 \le i \le 2$, let c_i be an oriented simple closed curve in $C_i \times S_i$ so that $[c_i] = a_i[C_i] + b_i[S_i]$ in $H_1(C_i \times S_i, \mathbb{Z})$. Note that $\gcd(a_1a_2, a_1b_2 + a_2b_1) = 1$. Let c_3 be an oriented s.c.c. in $C_3 \times S_3$ so that $[c_3] = -a_1a_2[C_3] + (a_1b_2 + a_2b_1)[S_3]$ in $H_1(C_3 \times S_3, \mathbb{Z})$. Then a closed 3-manifold M can be obtained from N by filling solid tori into ∂N, mapping the meridians to c_1 and c_2, respectively. And a closed M' can be obtained from N' by filling a solid torus into $\partial N'$, mapping the meridian to c_3.

Let m and n be integers such that $ma_1 - nb_1 = 1$. One can check that M is the lens space $L(a_1b_2 + a_2b_1, ma_2 + nb_2)$ and M' is the lens space $L(a_1b_2 + a_2b_1, a_1a_2)$. Since in $H_1(\overline{N - N'}, \mathbb{Z})$ we have $[c_3] = a_2[c_1] + a_1[c_2]$, the curve c_3 in $\partial N'$ bounds a surface in $\overline{M - N'}$, whose intersections with ∂N are parallel copies of c_1 and c_2. Hence we have $M \succeq_1 M'$ and $M \succeq M'$. Below we will show that if integers p, q_1, q_2 satisfy $\gcd(p, q_1) = \gcd(p, q_2) = 1$ and $q_1q_2 \equiv r^2 \pmod p$ for some integer r, then there exist a_1, b_1, a_2, b_2, m, n such that M is $L(p, q_1)$ and M' is $L(p, q_2)$.

Because $\gcd(p, q_1) = \gcd(p, q_2) = 1$, we have $\gcd(p, r) = 1$, and there exists an integer q_1^* so that $q_1q_1^* \equiv 1 \pmod p$. Let $a_1 = rq_1^* + p$, $a_2 = r$, then $\gcd(a_1, a_2) = 1$. Hence there exist two integers b'_1 and b'_2 such that $a_1b'_2 + a_2b'_1 = 1$. Let $b_1 = b'_1p$, $b_2 = b'_2p$, then we have $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$ and $a_1b_2 + a_2b_1 = p$. Clearly $a_1a_2 \equiv r^2q_1^* \equiv q_2 \pmod p$. Finally, since

$\gcd(a_1, b_1) = 1$, there exist two integers m and n such that $ma_1 - nb_1 = 1$. So $ma_2 + nb_2 \equiv mr \equiv ma_1q_1 \equiv q_1 \pmod p$.

Remark 2.6. The fact that "$\succeq_1$" is not antisymmetric on the set of lens spaces follows from the homotopy-type classification of lens spaces, which is a classical result of Whitehead [Wh].

Definition 2.7. A 3-manifold M in $\mathcal{M}$ is *Σ_g-splittable* if every embedded Σ_g in M separates M into two parts.

Proposition 2.8. *For M in $\mathcal{M}$, M is Σ_g-splittable for $g \leq k$ if M is Σ_k-splittable; M is Σ_g-splittable for any g if and only if M is a rational homology 3-sphere.*

Proof. If an embedded Σ_g does not separate M, then locally add 1-handles to Σ_g. Clearly M is not Σ_k-splittable for $k \geq g$. M is not a rational homology 3-sphere if and only if there is an epimorphism $H_1(M, \mathbb{Z}) \to H_1(S^1, \mathbb{Z})$, which can always be induced by a map from M to S^1. The latter condition is equivalent to that M contains a closed two sided surface which does not separate M (see [He1]). $\square$

Proposition 2.9. *A 3-manifold M in $\mathcal{M}$ is Σ_g-splittable if and only if every prime factor of M is Σ_g-splittable.*

Proof. When $g = 0$, this is equivalent to that M does not contain $S^1 \times S^2$ as a prime factor. Assume that it is true for $g = k - 1$. When $g = k$, we only need to show the "if" part. By induction, we can assume that the embedded Σ_k in M is incompressible. For a sphere in M intersecting Σ_k transversely, an innermost disk in the sphere together with a disk in Σ_k will form a sphere separating M. Then we can remove the intersection by an isotopy or reduce the problem to a 3-manifold with fewer prime factors. Hence the proof can be finished by induction. $\square$

Example 2.10. Any spherical 3–manifold is a rational homology 3-sphere, hence is Σ_g-splittable for any g by Proposition 2.8. Any hyperbolic 3-manifold in $\mathcal{M}$ contains no incompressible torus, hence is Σ_1-splittable. Any irreducible 3-manifold is Σ_0-splittable. If the mapping torus of an automorphism of Σ_g has first Betti number 1, then it is not Σ_g-splittable, but it is Σ_k-splittable for any $k < g$, because any homologically essential surface must represent a generator of $H_2 \cong \mathbb{Z}$, so the minimal genus of such a surface is g [Th2].

3. Extension of periodic automorphisms

In this section, we prove our main result Theorem 3.1. By Proposition 2.3 and Proposition 2.8, it implies Theorem 1.3. We also discuss several generalizations of Theorem 3.1, summarized as Theorem 3.11 and Theorem 3.12.

Theorem 3.1. *Given a Σ_1-splittable M in $\mathcal{M}$ and a finite subgroup G in $Aut(\Sigma_g)$, if there is an embedding $e : \Sigma_g \to M$ so that each element of G is extendable over M with respect to e, then there exists M' in $\mathcal{M}$ so that $M \succeq M'$ (and so $M \succeq_H M'$) and G is extendable over M'.*

Its proof uses similar ideas as [Bo1], as well as [Fl]. According to the sphere and torus/annulus decompositions of $M - \Sigma_g$, we can change complicated 3-manifolds embedded in $M - \Sigma_g$ into 3-balls or solid tori such that G is extendable, and each replacement corresponds to a relation $\succeq_s$. We first list several fundamental results in 3-manifold theory that will be used in the proof.

In what follows, Lemma 3.2 is based on Kneser–Milnor's sphere decomposition theorem (see [Bo1, Lemma A.1]). Theorem 3.3 can be found in [Bo2, Theorem 3.7]. Theorem 3.4 is based on the torus/annulus decomposition theorem and Thurston's hyperbolizaion theorem (see [Bo2]). Lemma 3.5 and Lemma 3.6 for finite cyclic G and I-bundles over closed surfaces can be found in [Bo1, Propositions 4.1 and 4.3], where "I" denotes the unit interval. Theorem 3.7 is based on Mostow's hyperbolic rigidity theorem and Waldhausen's isotopy theorem (see [Bo2] and [Wa]).

Lemma 3.2. *Let X be a compact connected oriented 3-manifold with $\partial X \neq \emptyset$. As in Proposition 2.4, there exists a collection $\mathcal{S}$ of disjoint spheres in $X - \partial X$, which decomposes X into a punctured S^3 and several one-punctured prime factors. If $\mathcal{S}$ and $\mathcal{S}'$ are two such collections, then there exists an element f in $Aut(X)$ such that f maps the spheres in $\mathcal{S}$ to the spheres in $\mathcal{S}'$ and f fixes ∂X.*

Let Σ be a closed oriented surface. A compression body V can be obtained from $\Sigma \times [0, 1]$ by attaching 2-handles along $\Sigma \times \{1\}$ and capping off boundary 2-spheres appeared in the process with 3-balls. The external boundary $\partial_e V$ is the part of ∂V corresponding to $\Sigma \times \{0\}$, and the internal boundary $\partial_i V$ is $\partial V - \partial_e V$.

Theorem 3.3. *Let X' be a compact connected oriented 3-manifold with $\partial X' \neq \emptyset$, which is irreducible. Then, up to isotopy, X' contains a unique* *compression body V such that the external boundary $\partial_e V$ is equal to $\partial X'$, $\overline{X' - V}$ contains no essential compression disk for its boundary $\partial_i V$, and no component of $\overline{X' - V}$ is a 3-ball.*

Theorem 3.4. *Let Y be a compact connected oriented 3-manifold with $\partial Y \neq \emptyset$, which is irreducible and boundary irreducible. Then, up to isotopy, there is a unique minimal collection $\mathcal{T}$ of disjoint properly embedded essential tori and annuli in Y, such that for each piece Z obtained by cutting Y along surfaces in $\mathcal{T}$, either*

(i) *Z is an I-bundle over a compact surface with negative Euler characteristic, where the corresponding ∂I-bundle is equal to $Z \cap \partial Y$, or*

(ii) *Z is a Seifert manifold, where $Z \cap \partial Y$ is a union of fibers, or*

(iii) *with boundaries from $\mathcal{T}$ and tori in $Z \cap \partial Y$ removed, Z admits a complete hyperbolic structure with totally geodesic boundary and with finite volume.*

Lemma 3.5. *Let V be an oriented compression body, possibly disconnected, and G be a finite subgroup in $Aut(\partial_e V)$. If each element h of G is extendable over V with respect to the inclusion map, then the corresponding element h' in $Aut(V)$ can be deformed to an element $\phi(h)$ in $Aut(V)$ by an isotopy*[1] *fixing $\partial_e V$ so that the map $\phi : G \to Aut(V)$ is a group monomorphism.*

Lemma 3.6. *Let Z be a connected oriented I-bundle over a compact surface with negative Euler characteristic. Let $\partial_I Z$ denote the corresponding ∂I-bundle, and G be a finite subgroup in $Aut(\partial_I Z)$. If each element h of G is extendable over Z with respect to the inclusion map, then there is an element $\phi(h)$ in $Aut(Z)$ extending h so that the map $\phi : G \to Aut(Z)$ is a group monomorphism.*

Theorem 3.7. *Let Z' be a connected oriented complete hyperbolic 3-manifold with totally geodesic boundary and with finite volume. Then each element in $Aut(Z')$ can be deformed to a unique isometry in $Aut(Z')$ by an isotopy keeping $\partial Z'$ invariant setwise. Moreover, this correspondence induces an isomorphism between the mapping class group of Z' and the isometry group of Z', which is finite.*

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[1]Throughout this paper, when we say "an isotopy fixing N", we mean that the restriction of the isotopy to N is a fixed map.

We will give proofs of Lemma 3.5 and Lemma 3.6 according to [Bo1, Section 4]. The following lemma will be used in the proofs, as well as in the proof of Theorem 3.1. In what follows, extensions will be respect to the inclusion maps.

Lemma 3.8. *Let W be a compact oriented 3-manifold consisting of diffeomorphic components $W_1, \dots, W_m$ and $\partial W \neq \emptyset$. Let G be a finite subgroup in $Aut(\partial W)$, H be the setwise stabilizer of ∂W_1 in G, and assume that $G = \bigcup_{i=1}^m h_i H$ with $h_i(\partial W_1) = \partial W_i$ for $1 \le i \le m$. If the H-action on ∂W_1 extends to W_1 and each h_i on ∂W_1 extends to a diffeomorphism $h_i' : W_1 \to W_i$, then the G-action on ∂W extends to W.*

Proof. Note that the G-action on ∂W induces permutations on $\{\partial W_1, \dots, \partial W_m\}$, because for each h in G and $1 \le i \le m$ we have

$$h(\partial W_i) = hh_ih_i^{-1}(\partial W_i) = hh_i(\partial W_1) = h_j\tilde{h}(\partial W_1) = h_j(\partial W_1) = \partial W_j,$$

for some $j \in \{1, \dots, m\}$ and $\tilde{h} \in H$. Let σ_h denote the permutation on $\{1, \dots, m\}$ determined by h, then we have $j = \sigma_h(i)$ and $hh_i \in h_{\sigma_h(i)}H$.

Let $\phi_H : H \to Aut(W_1)$ be the group homomorphism given by the extension of the H-action. Then, for each h in G and $1 \le i \le m$, define $\phi(h)$ on W_i by

$$\phi(h)|_{W_i} = h'_{\sigma_h(i)} \circ \phi_H\big(h^{-1}_{\sigma_h(i)} hh_i\big) \circ h_i'^{-1}.$$

Clearly $\phi(h)|_{W_i} : W_i \to W_{\sigma_h(i)}$ is a diffeomorphism and it extends $h|_{\partial W_i}$. Moreover, one can check that $\phi : G \to Aut(W)$ is a group monomorphism. □

Note that Lemma 3.8 is essentially algebraic. Under the settings of Lemma 3.8, the H-action on ∂W_1 can be non-faithful, but in applications it will be faithful.

Proof of Lemma 3.5. By classical results in differential topology and Lemma 3.8, Lemma 3.5 holds when V is a disjoint union of 3-balls or an I-bundle over $\partial_e V$.

Below we assume that each component of $\partial_e V$ has genus bigger than zero and it contains an essential simple closed curve that bounds a disk in V. We can choose a G-invariant Euclidean or hyperbolic structure on each component of $\partial_e V$, by the geometrization theorem for 2-dimensional orbifolds. Then, among all the essential curves that bound disks in V there is a curve c having the minimum length. Since each element h in G is extendable over V w.r.t. the inclusion map, the curve $h(c)$ also bounds a disk in V.

We have $h(c) = c$ or $h(c) \cap c = \emptyset$. Otherwise, let D and D' be disks bounded by c and $h(c)$, respectively, which intersect transversely. Let α be an arc in $D \cap D'$ which is outermost in D, then α co-bounds a disk in D with an arc $a \subset c$, where a does not intersect $h(c)$ in the interior of a. We can choose α so that

$$\text{length}(a) \leq \frac{1}{2}\,\text{length}(c).$$

Now ∂a separates $h(c)$ into two arcs a' and a''. Let $c' = a \cup a'$ and $c'' = a \cup a''$, then both c' and c'' are null-homotopic in V. Since each of c' and c'' is the union of two geodesic arcs, none of them is null-homotopic in $\partial_e V$. By Dehn's Lemma, both c' and c'' bound disks in V. We may assume $\text{length}(a') \leq \text{length}(a'')$. So

$$\text{length}\big(a'\big) \leq \frac{1}{2}\,\text{length}\big(h(c)\big) = \frac{1}{2}\,\text{length}(c),$$

and $\text{length}(c') \leq \text{length}(c)$. Then we can get a strictly shorter loop by rounding off the corners of c'. This contradicts the choice of c.

Let $\{c_1, \dots, c_m\}$ be the orbit of c. Each c_i bounds a disk D_i in V. By applying surgeries, we can assume that $D_1, \dots, D_m$ are disjoint from each other. Then the G-action on $\bigcup_{i=1}^m c_i$ can extend to $\bigcup_{i=1}^m D_i$, by the method in Lemma 3.8. And the G-action on $\partial_e V$ can extend to a regular neighborhood U of $\partial_e V \cup (\bigcup_{i=1}^m D_i)$. Let $\varphi(h)$ in $Aut(U)$ be the extension of h. Since V is irreducible, each extension h' in $Aut(V)$ can be deformed to preserve the collection of the disks by an isotopy fixing $\partial_e V$. It can be further deformed to coincide with $\varphi(h)$ on U.

Note that $\overline{V - U}$ is still a compression body, possibly disconnected, and G acts on its external boundary $\partial U - \partial_e V$. Hence on $\overline{V - U}$ we can use the results about 3-balls and I-bundles, or repeat the above procedure. The proof can be finished by induction on the genera of the components of the external boundary. □

To prove Lemma 3.6, we will need some results about automorphisms of surfaces with boundary. The versions of these results for closed surfaces are well known. The versions we need can be found in the Appendix.

Proof of Lemma 3.6. Let Σ denote the base space of the I-bundle Z.

Case 1. Σ is orientable, no element in G swaps the two components of $\partial_I Z$. In this case $Z = \Sigma \times [0,1]$, and $\partial_I Z = \Sigma \times \{0,1\}$.

For each element h in G and $i = 0, 1$, let $\rho_i(h) : \Sigma \to \Sigma$ be the map induced by the restriction of h on $\Sigma \times \{i\}$. Then ρ_0 and ρ_1 are two representations of G into $Aut(\Sigma)$. Since G is finite, for $i = 0, 1$, we can equip Σ

with a hyperbolic structure m_i for which the $\rho_i(G)$-action is isometric, and $\partial\Sigma$ is geodesic in each m_i.

We will use Proposition A.4. Let $\mathrm{id} : \Sigma \to \Sigma$ denote the identity map. Since $\rho_0(h)$ is an isometry w.r.t. m_0, $\tau_{\mathrm{id}} \circ \rho_0(h)$ has constant dilatation and is homotopic to $\rho_0(h)$. So $\tau_{\mathrm{id}} \circ \rho_0(h) = \tau_{\rho_0(h)}$. Similarly $\rho_1(h) \circ \tau_{\mathrm{id}} = \tau_{\rho_1(h)}$. Since h is extendable over Z w.r.t. the inclusion map, $\rho_0(h)$ and $\rho_1(h)$ are homotopic. So $\tau_{\rho_0(h)} = \tau_{\rho_1(h)}$.

Thus τ_{id} realizes a conjugacy from $\rho_0(h)$ to $\rho_1(h)$ for each h in G, and it induces an orbifold homeomorphism $\overline{\tau_{\mathrm{id}}} : \Sigma/\rho_0(G) \to \Sigma/\rho_1(G)$. In general, the map τ_{id} is not smooth, but via deforming $\overline{\tau_{\mathrm{id}}}$, we can obtain a diffeomorphism τ'_{id} which also realizes the conjugacy between $\rho_0(G)$ and $\rho_1(G)$ and is isotopic to τ_{id}. Then τ'_{id} is smoothly isotopic to id. Let $\phi_t : \Sigma \to \Sigma$ be the isotopy with $\phi_0 = \mathrm{id}$ and $\phi_1 = \tau'_{\mathrm{id}}$. For each h in G, define $\phi(h) : Z \to Z$ by the formula

$$\phi(h)(x,t) = \big(\phi_t \circ \rho_0(h) \circ \phi_t^{-1}(x), t\big), (x,t) \in \Sigma \times [0,1].$$

Then $\phi(h)$ is in $Aut(Z)$, and $\phi : G \to Aut(Z)$ is a group monomorphism.

Case 2. Σ is orientable, and there exist elements in G swapping $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$.

By the method in [Wa, Section 3] (where the paper works in the PL category), each extension of h can be deformed to an element $\varphi(h)$ in $Aut(Z)$ by an isotopy fixing $\Sigma \times \{0,1\}$ so that $\varphi(h)$ keeps $\Sigma \times \{1/2\}$ invariant setwise. So the map $\varphi : G \to Aut(Z)$ induces a group homomorphism $\overline{\varphi} : G \to \pi_0(Aut(\Sigma))$. Note that h exchanges the two components of ∂Z if and only if $\overline{\varphi}(h)$ is an orientation-reversing mapping class in $\pi_0(Aut(\Sigma))$. Since $\chi(\Sigma) < 0$, the map $\overline{\varphi}$ is injective.

By Proposition A.3, $\overline{\varphi}$ can be lifted to a group monomorphism $G \to Aut(\Sigma)$. Hence, each $\varphi(h)$ can be deformed to some $\widetilde{\varphi}(h)$ in $Aut(Z)$ by an isotopy fixing $\Sigma \times \{0,1\}$ so that $\widetilde{\varphi}(h)$ keeps $\Sigma \times [1/4, 3/4]$ invariant setwise and all the $\widetilde{\varphi}(h)$ together give a G-action on it. Let G_0 be the subgroup of G consisting of elements that preserve $\Sigma \times \{0\}$. Then, by Case 1, the G_0-action on $\Sigma \times \{0, 1/4\}$ can extend to $\Sigma \times [0, 1/4]$. By Lemma 3.8, the G-action can extend to Z.

Case 3. Σ is non-orientable. So Z is the orientable twisted I-bundle over Σ and $\partial_I Z$ is an orientable double cover of Σ.

Let Π be the section of Z which meets each fiber $[0,1]$ at $\frac{1}{2}$. As in Case 2, for each h we get an extension $\varphi(h)$ in $Aut(Z)$ so that $\varphi(h)$ keeps Π invariant setwise. So $\varphi : G \to Aut(Z)$ induces a group homomorphism $\overline{\varphi} : G \to \pi_0(Aut(\Pi))$. Let U be a neighborhood of Π which has an induced

I-bundle structure. We can assume that $\varphi(h)$ preserves U. Then we see that $G \to \overline{\varphi}(G)$ is 1-to-1 or 2-to-1.

By Proposition A.3, $\overline{\varphi}(G)$ can be lifted to $Aut(\Pi)$. So each $\varphi(h)$ can be deformed to some $\widetilde{\varphi}(h)$ by an isotopy fixing $\partial_I Z$ so that $\widetilde{\varphi}(h)$ preserves U and the induced I-bundle, and all the $\widetilde{\varphi}(h)$ together give a G-action on Π, possibly non-faithful. They also give a G-action on U, which must be faithful, because $\widetilde{\varphi}(h)|_{\partial U}$ and $\widetilde{\varphi}(h)|_{\partial_I Z}$ are cobordant via $\widetilde{\varphi}(h)$ on $\overline{Z-U}$, which is an I-bundle over $\partial_I Z$. Then the proof can be finished by applying Case 1. $\square$

Lemma 3.9. *Let Z be an I-bundle over a compact surface Σ with $\chi(\Sigma) < 0$. Let $h \in Aut(\partial_I Z)$. Suppose that $h_1, h_2 \in Aut(Z)$ are two extensions of h, then h_1, h_2 are (topologically) isotopic rel $\partial_I Z$.*

Proof. We use a procedure similar to the proof of Lemma 3.6. By composing both h_1, h_2 with h_2^{-1}, we may assume that $h = \mathrm{id}_{\partial_I Z}$ and $h_2 = \mathrm{id}_Z$.

First, we consider the case Σ is orientable. By [Wa, Section 3], we may assume h_1 is level-preserving. Thus h_1 defines a loop in $Aut(\Sigma)$. By [EE, ES], every component of $Aut(\Sigma)$ is contractible, so this loop is null-homotopic. So we can construct the desired isotopy using the null homotopy in $Aut(Z)$.

When Σ is non-orientable, let Π be the section of Z which meets each fiber $[0,1]$ at $\frac{1}{2}$. After an isotopy, we may assume h_1 preserves Π. Since $h_1|_\Pi$ is homotopic to id_Π, they are also isotopic. We can isotope h_1 so that $h_1|_\Pi = \mathrm{id}_\Pi$. We can further isotope h_1 to get a new homeomorphism h_1', still extending h, and $h_1' = \mathrm{id}$ on an I–bundle neighborhood U of Π. Now we can apply the last paragraph to get an isotopy connecting h_1' and id_Z. $\square$

To give the proof of Theorem 3.1, we also need the following lemma.

Lemma 3.10. *Let Z be a piece in the case (iii) of Theorem 3.4. Let $\partial_Y Z$ denote the part of $Z \cap \partial Y$ without tori. Assume that $\partial_Y Z \neq \emptyset$ and G is a finite subgroup in $Aut(\partial_Y Z)$. If each h in G extends to an element h' in $Aut(Z)$, then the h' can be (topologically) deformed to an element $\phi(h)$ in $Aut(Z)$ by an isotopy fixing $\partial_Y Z$ such that the map $\phi : G \to Aut(Z)$ is a group monomorphism.*

Proof. Note that each component of $\partial_Y Z$ is an oriented compact surface with negative Euler characteristic, and the surface $\overline{\partial Z - \partial_Y Z}$ consists of boundaries from the decomposition tori/annuli in $\mathcal{T}$ and tori in $Z \cap \partial Y$.

Let U be a regular neighborhood of $\partial_Y Z$, which has an I-bundle structure over $\partial_Y Z$. Let $Z_0 = \overline{Z - U}$ and $Z' = Z_0 - \partial Z$. Then Z_0 is diffeomorphic to Z, and Z' is diffeomorphic to $Z - \overline{\partial Z - \partial_Y Z}$. So Z' admits a complete hyperbolic structure with totally geodesic boundary and with finite volume, by the assumption.

For each h in G, we can assume that the h' keeps Z_0 invariant setwise. By Theorem 3.7, the $h'|_{Z'}$ in $Aut(Z')$ can be deformed to a unique isometry $\varphi(h)$ in $Aut(Z')$ by an isotopy keeping $\partial Z'$ invariant setwise. If there are $f_1, \dots, f_s$ in G such that $\prod_{i=1}^{s} f_i$ is the identity, then $\prod_{i=1}^{s} f_i'|_{\partial_I U - \partial_Y Z}$ is isotopic to the identity map. Since components of $\partial_Y Z$ have negative Euler characteristic, the finite order isometry $\prod_{i=1}^{s} \varphi(f_i)$ is the identity. By the same reason, if $\varphi(h)$ is the identity, then h is the identity. So the map $\varphi : G \to Aut(Z')$ is a group monomorphism.

By passing to the orbifold $Z'/\varphi(G)$, we can construct a $\varphi(G)$-invariant compact manifold Z_0' in Z' so that $\overline{Z_0 - Z_0'}$ is an I-bundle over $Z_0 \cap \partial Z$. Then, each $h'|_{Z_0}$ is isotopic to some $\widetilde{\varphi}(h)$ in $Aut(Z_0)$ relative to $Z_0 \cap \partial Z$ so that $\widetilde{\varphi}(h)|_{Z_0'} = \varphi(h)|_{Z_0'}$. Via further isotopy that preserves $Z_0 \cap \partial Z$, we can assume that $\widetilde{\varphi} : G \to Aut(Z_0)$ is a group monomorphism. Then the proof can be finished by Lemmas 3.6 and 3.9. □

Proof of Theorem 3.1. Since M is Σ_1-splittable, M is Σ_0-splittable. If $g \le 1$, then $e(\Sigma_g)$ bounds a manifold on each side, which can be replaced by a 3-ball or a solid torus, according to Definition 2.2 and the note following it. The result M' satisfies $M \succeq M'$, and each element of G is extendable over M' w.r.t. the new embedding $e' : \Sigma_g \to M'$ (see Step 3 below). Then, by Lemma 3.5, G is extendable over M'.

Below we assume that $g \ge 2$. The surface $e(\Sigma_g)$ is two-sided in M. We cut M along $e(\Sigma_g)$ and choose a connected component of the resulting manifold, denoted by X. Since the extension of each h in G preserves the two sides of $e(\Sigma_g)$, G can be naturally embedded in $Aut(\partial X)$, so that each h is extendable over X w.r.t. the inclusion map. We will find a compact 3-manifold W in $X - \partial X$ with ∂W being a union of spheres or tori, and replace it by 3-balls or solid tori as in Section 2, such that the G-action on ∂X can extend to the modified manifold. We will do this for each possible component X. When the boundaries of the modified components are glued back to $e(\Sigma_g)$, we will obtain a manifold M', which will satisfy $M \succeq M'$ by definition, and G will be extendable over M'.

We have three steps to modify X and extend the G-action to it. As before, for h in G, we use h' to denote its extension. It will be modified in each step.

Step 1. If X admits a nontrivial prime decomposition as in Lemma 3.2, since M is Σ_0-splittable, ∂X lies in a single prime factor X'. For each h in

G, the images of the decomposition spheres under h' give X another prime decomposition. Hence, by Lemma 3.2, there exists an element f in $Aut(X)$ such that $f \circ h'$ preserves the original decomposition spheres and f fixes ∂X. So $f \circ h'$ is also an extension of h. According to the note following Definition 2.2, we replace X by X', and replace $f \circ h'$ by an extension of h in $Aut(X')$. It will still be denoted by h'. Now G is embedded in $Aut(\partial X')$. Note that X' is irreducible.

Step 2. By Theorem 3.3, we have a compression body V in X'. Moreover, for each h in G, the h' in $Aut(X')$ can be deformed to keep V invariant setwise, by an isotopy fixing $\partial X'$. Note that V is connected if and only if $\partial_e V = \partial X'$ is connected, and G acts on each component of $\partial_e V$. By Lemma 3.5, the h' can be further deformed so that the map given by $h \mapsto h'|_V$ from G to $Aut(V)$ is a group monomorphism. So the G-action extends to V. Below we assume that $V \neq X'$.

By Theorem 3.3, $\overline{X' - V}$ is irreducible and boundary irreducible. Its boundary $\partial_i V$ consists of oriented closed surfaces with genera bigger than zero. Because the singular set of the orbifold V/G is empty or 1-dimensional, the action on each component of $\partial_i V$ given by the stabilizer is faithful. Now G is embedded in $Aut(\partial_i V)$, and each h in G is extendable over $\overline{X' - V}$ w.r.t. the inclusion map.

Step 3. Let $Y_1, \ldots, Y_n$ be all the components of $\overline{X' - V}$. Because each h in G extends to some h' in $Aut(\overline{X' - V})$, the G-action on $\partial_i V$ induces permutations on $\{\partial Y_1, \ldots, \partial Y_n\}$. Let H be the setwise stabilizer of ∂Y_1 in G, $\{\partial Y_1, \ldots, \partial Y_m\}$ be the orbit of ∂Y_1, and $G = \bigcup_{i=1}^m h_i H$ with $h_i(\partial Y_1) = \partial Y_i$ for $1 \leq i \leq m$. Note that G and H act faithfully on $\bigcup_{i=1}^m \partial Y_i$ and ∂Y_1, respectively. Let $W = \bigcup_{i=1}^m Y_i$, then G and H are embedded in $Aut(\partial W)$, and each h in G extends to some h' in $Aut(W)$.

Since $h_i'|_{Y_1} : Y_1 \to Y_i$ is a diffeomorphism, which extends $h_i|_{\partial Y_1}$, if the H-action on ∂Y_1 extends to Y_1, then by Lemma 3.8, the G-action on ∂W extends over W. In general, we will show that the H-action on ∂Y_1 can extend over a 3-manifold W_1 obtained from Y_1 by surgeries. Then, since $h_i'|_{Y_1}$ identifies Y_1 with Y_i, there is a W_i obtained from Y_i and a diffeomorphism $W_1 \to W_i$ that also extends $h_i|_{\partial Y_1}$. Hence, the G-action on ∂W can extend over the modified 3-manifold $\bigcup_{i=1}^m W_i$.

For simplicity, below we denote Y_1 by Y, and we need to show that the H-action on ∂Y extends over a modified 3-manifold.

Case 1. ∂Y contains a torus.

Because M is Σ_1-splittable, ∂Y is the torus. By Lemma 2.1, we have an essential simple closed curve c in ∂Y so that $[c] = 0$ in $H_1(Y, \mathbb{Q})$. Then $[h(c)] = 0$ for each h in H, because $h'(Y) = Y$. Hence, by Lemma 2.1, $h(c)$

is isotopic to c in ∂Y. According to Definition 2.2, we replace Y by a solid torus P. Since both c and $h(c)$ bound disks in P, $h|_{\partial Y}$ extends over P for each h in H. So, by Lemma 3.5, the H-action on ∂Y extends over P.

Case 2. Each component of ∂Y has genus bigger than one.

Note that Y is irreducible and boundary irreducible. By Theorem 3.4, we have a collection $\mathcal{T}$ of properly embedded tori and annuli in Y, so that each piece Z obtained by cutting Y along surfaces in $\mathcal{T}$ belongs to one of the three types: (i) Hyperbolic; (ii) I-bundle; (iii) Seifert fibered space.

Subcase 2.1. $\mathcal{T}$ contains only tori. In this case, let T be the union of the tori in $\mathcal{T}$. (Note that $\mathcal{T}$ is a set of tori, while T is the union of the tori in $\mathcal{T}$.)

By Theorem 3.4, for each h in H, the h' in $Aut(Y)$ can be deformed to keep T invariant setwise, by an isotopy fixing ∂Y. Because M is Σ_1-splittable, ∂Y lies in a single piece Z. Since components of ∂Y have genera bigger than one, Z is not a Seifert manifold. If Z is an I-bundle, then it is an I-bundle over a closed surface. So it is Y, and by Lemma 3.6, the H-action on ∂Y can extend over Y. If Z is hyperbolic, then according to Lemma 3.10, $\partial_Y Z = \partial Y$, and the h' can be further deformed so that the map given by $h \mapsto h'|_Z$ from H to $Aut(Z)$ is a group monomorphism. So the H-action extends over Z. If $T \neq \emptyset$, then H is embedded in $Aut(\partial Z - \partial Y)$ as in Step 2. Since $\partial Z - \partial Y$ is a union of tori, by the argument in Case 1, $\overline{Y - Z}$ can be replaced by a union of solid tori so that the H-action can extend further over it.

Subcase 2.2. $\mathcal{T}$ contains annuli. Let A denote the union of all the annuli in $\mathcal{T}$, let N be a tubular neighborhood of the union of surfaces in $\mathcal{T}$.

Because H is finite, we can choose a H-invariant hyperbolic structure on ∂Y. Each circle in ∂A is isotopic to a closed geodesic in ∂Y, where non-isotopic ones correspond to disjoint geodesics. For each such closed geodesic c, suppose that there are $k(c)$ circles in ∂A isotopic to c, then we can choose $k(c)$ circles in ∂Y, parallel and sufficiently close to c, so that the union of all these circles is invariant under the H-action. Hence, we can choose A and N so that $h(N \cap \partial Y) = N \cap \partial Y$ for each h in H.

By Theorem 3.4, for each h in H, the h' in $Aut(Y)$ can be deformed to keep N invariant setwise, by an isotopy fixing ∂Y. Let Z_1 (which is possibly empty) be the union of all hyperbolic pieces Z in $Y - N$ with $\partial_Y Z \neq \emptyset$. By Lemma 3.10, we can (topologically) deform $h'|_{Z_1}$ to a diffeomorphism h'_1 of Z_1 relative to $\partial_Y Z_1$, so that the map defined by $h|_{\partial_Y Z_1} \mapsto h'_1$ is a group monomorphism. Hence the H-action on V extends to a new H-action $\phi_1 : H \to Aut(V \cup Z_1)$. Moreover, each $\phi_1(h)$ extends to a diffeomorphism of $V \cup Y$, still called h'.

Let Z_2 (which is possibly empty) be the union of all I-bundles in $Y - N$. By Lemma 3.6, we can extend each $\phi_1(h)$ to get a group monomorphism $\phi_2 : H \to Aut(V \cup Z_1 \cup Z_2)$. By Lemma 3.9, for each $h \in H$, we can construct an isotopy H_t between $\phi_2(h)$ and $h'|_{V\cup Z_1\cup Z_2}$, so that H_t is equal to $\phi_1(h)$ on $(V \cup Z_1) \times \{t\}$.

Let T_2 be the union of the torus component of $\partial(V \cup Z_1 \cup Z_2)$. Then $\phi_2(h)$ and $h'|_{V\cup Z_1\cup Z_2}$ induce isotopic maps on T_2. By the argument in Case 1, we can fill T_2 with solid tori, so that $\phi_2(H)$ extends over these solid tori. Our conclusion follows by Lemma 3.5. $\square$

In what follows, we discuss some generalizations of Theorem 3.1.

1. We can replace Σ_g by any compact connected 3-manifold N. The extendable automorphisms and extendable subgroups in $Aut(N)$ can be defined as the surface case. The proof is almost the same, except that now we choose X as a component of $\overline{M - N}$, and for each possible X we consider the setwise stabilizer of ∂X in G, where G is naturally embedded in $Aut(\partial N)$.

We can also replace Σ_g by any compact connected surface. Let Σ denote such an oriented surface. We can first extend the G-action to an oriented I-bundle over Σ embedded in M, then extend the G-action to a regular neighborhood of Σ. This leads to the case of compact 3-manifolds.

Let Π be a compact connected nonorientable surface embedded in an oriented 3-manifold M, and $Aut(\Pi)$ be its automorphism group. Then we can define extendable automorphisms and extendable subgroups in $Aut(\Pi)$ similarly. Since M is oriented the surface $e(\Pi)$ is one-sided. In M, we can choose an oriented I-bundle Z over Π and assume that for each h in G the h' in $Aut(M)$ preserves Z and the bundle structure. Because the ∂I-bundle $\partial_I Z$ is an orientable double cover of Π, each h has two lifts in $Aut(\partial_I Z)$, and they differ by the covering transformation. Since h' preserves the two sides of $\partial_I Z$, the restriction of h' on $\partial_I Z$ is the orientation-preserving lift of h. Hence the G-action can extend to a regular neighborhood of Π.

Clearly we can replace Σ_g by S^1 or I. Hence we have the following result.

Theorem 3.11. *Given a Σ_1-splittable M in $\mathcal{M}$, a compact connected manifold N, and a finite subgroup G in $Aut(N)$, if there is an embedding $e : N \to M$ so that each element of G is extendable over M with respect to e, then there exists M' in $\mathcal{M}$ so that $M \succeq M'$ and G is extendable over M'.*

2. Theorem 3.11 will still hold if orientation-reversing automorphisms are added into $Aut(N)$ and $Aut(M)$, when we define "extendable" suitably. Note

that all the results listed after Theorem 3.1 are still valid when orientation-reversing elements are added into the groups involved. The proofs of Lemma 3.5 and Lemma 3.6 will also hold, after some small changes. For example, in the proof of Lemma 3.6, $\overline{\varphi}$ in Case 2 can be 2-to-1 now, and we can use arguments as in Case 3 to deal with it. The proofs of other lemmas are almost the same.

In the case of compact 3-manifolds, the proof is also valid. Note that in each step in the proof, h is orientation-reversing if and only if h' is orientation-reversing. In the cases of compact surfaces, we need more discussions.

For the oriented surface Σ, we can define a map $\rho : G \to \mathbb{Z}_2$ so that $\rho(h) = 0$ if and only if the element h' corresponding to h preserves the two sides of $e(\Sigma)$. For the nonorientable surface Π, we can define a map $\rho : G \to \mathbb{Z}_2$ so that $\rho(h) = 0$ if and only if the element h' corresponding to h preserves the orientation of M. Note that for Π we have an oriented I-bundle Z over it, and h' is orientation-preserving on M if and only if the restriction of h' on $\partial_I Z$ is orientation-preserving.

If the map ρ is a group homomorphism, then the G-action extends to a regular neighborhood of $e(\Sigma)$ (resp. $e(\Pi)$). This leads to the case of compact 3-manifolds. Otherwise, the identity on Σ (resp. Π) can extend to an automorphism of M that exchanges the two sides (resp. reverses the orientation). Then every element in G can extend to an automorphism of M that preserves the two sides (resp. preserves the orientation), and the proof can be finished as the previous case.

For periodic automorphisms, similar arguments can show the following theorem, where when N is a surface, it is possible that f has odd order while f' reverses the orientation of M, in which case $\varphi(f)$ has order twice the order of f.

Theorem 3.12. *If a periodic automorphism f of a compact connected manifold N extends to an automorphism f' of a Σ_1-splittable M in $\mathcal{M}$, then there exists M' in $\mathcal{M}$ so that $M \succeq M'$, f extends to a periodic automorphism $\varphi(f)$ of M', and $\varphi(f)$ preserves the orientation of M' if and only if f' preserves the orientation of M.*

4. Classification of extendable cyclic actions

In this section, we first prove Theorem 1.5, then we give some examples.

According to the orbifold theory, which is developed and carefully discussed in [Th] and [BMP], two subgroups G and G' in $Aut(\Sigma_g)$ are conjugate in the extended automorphism group $Aut^{\pm}(\Sigma_g)$, the group of all (possibly orientation-reversing) automorphisms of Σ_g, if and only if there exist an isomorphism $\eta : G \to G'$ and an orbifold homeomorphism $\tau : \Sigma_g/G \to \Sigma_g/G'$ so that $\eta \circ \psi = \psi' \circ \tau_*$, where $\psi : \pi_1(\Sigma_g/G) \to G$ and $\psi' : \pi_1(\Sigma_g/G') \to G'$ are the epimorphisms corresponding to G and G' respectively, and $\tau_* : \pi_1(\Sigma_g/G) \to \pi_1(\Sigma_g/G')$ is the induced map of τ. Moreover, G and G' are conjugate in $Aut(\Sigma_g)$ if and only if τ is orientation-preserving.

We write the group law of the cyclic group $\mathbb{Z}_m$ multiplicatively. So the identity element in $\mathbb{Z}_m$ will be 1. The following Lemma 4.1 is a consequence of the Chinese remainder theorem. We leave its proof to the reader.

Lemma 4.1. *Let m, p, q be positive integers with $\gcd(p, q) = 1$. Given a generator h of the group $\mathbb{Z}_{mpq}$ and elements a and b in $\mathbb{Z}_{mpq}$ with orders p and q respectively, there is an automorphism η of $\mathbb{Z}_{mpq}$ so that $\eta(a) = h^{mq}$ and $\eta(b) = h^{mp}$.*

Lemma 4.2. *Let $\mathcal{F}$ be the 2-orbifold having underlying space Σ_r and s singular points of indices $n_1, \ldots, n_s$. According to Figure 2, $\pi_1(\mathcal{F})$ has the presentation*

$$\left\langle \alpha_1, \beta_1, \ldots, \alpha_r, \beta_r, \gamma_1, \ldots, \gamma_s \;\middle|\; \prod_{i=1}^{r}[\alpha_i, \beta_i] \prod_{j=1}^{s} \gamma_j = 1, \gamma_k^{n_k} = 1, 1 \le k \le s \right\rangle.$$

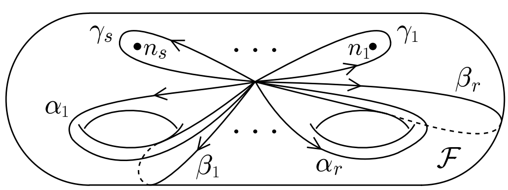

Figure 2: Generators of $\pi_1(\mathcal{F})$.

Let $\psi : \pi_1(\mathcal{F}) \to \mathbb{Z}_m$ be a finitely-injective epimorphism, and h be a generator of $\mathbb{Z}_m$. Then there is an automorphism τ of $\mathcal{F}$ so that $\psi \circ \tau_*(\alpha_1) = h$, $\psi \circ \tau_*(\alpha_i) = 1$ for $2 \le i \le r$, $\psi \circ \tau_*(\beta_i) = 1$ for $1 \le i \le r$, and $\psi \circ \tau_*(\gamma_j) = \psi(\gamma_j)$ for $1 \le j \le s$.

Proof. If $r = 0$, then let τ be the identity. Below we assume that $r \ge 1$.

Note that we do not need to consider the base point, because $\mathbb{Z}_m$ is abelian. In fact, ψ factors through $H_1(\mathcal{F})$, so we often abuse the notation by regarding ψ as a map $H_1(\mathcal{F}) \to \mathbb{Z}_m$. We will also use α_i, β_i, γ_j, $1 \le i \le r$, $1 \le j \le s$ to denote the loops presenting them.

Let n be the least common multiple of $n_1, \dots, n_s$. Since ψ is injective on finite subgroups of $\pi_1(\mathcal{F})$, the subgroup of $\mathbb{Z}_m$ generated by $\psi(\gamma_1), \dots, \psi(\gamma_s)$ has order n. Below we will consider slides and Dehn twists on $\mathcal{F}$ along the loops α_i and β_i for $1 \le i \le r$. Figure 3 shows a sketch of slides, where we omit the indices of the singular points. The left two pictures indicate the slide of singular points, and the right two pictures indicate the slide of handles.

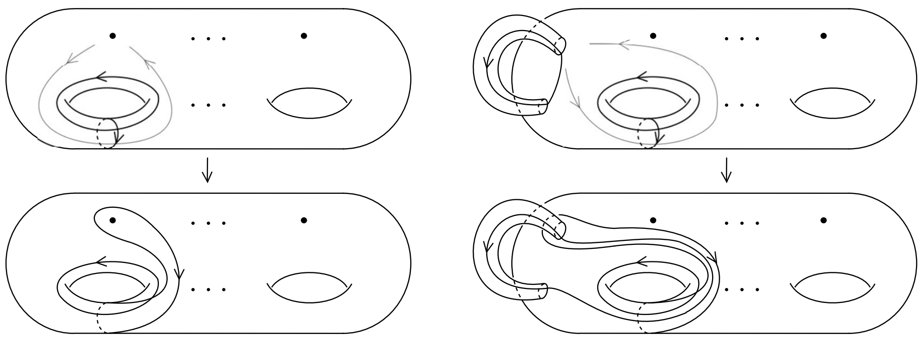

Figure 3: Slide of singular points and slide of handles.

If $m = n$, then ψ is surjective on $\langle \gamma_1, \dots, \gamma_s \rangle$. Each singular point of $\mathcal{F}$ can slide along the loop α_i or β_i. Consider the singular point corresponding to γ_k. The slide of it along α_i will change $\psi(\beta_i)$ to $\psi(\beta_i \gamma_k^{\pm 1})$ and not affect the $\psi(\beta_j)$ with $j \ne i$, $\psi(\alpha_j)$ with $1 \le j \le r$ and $\psi(\gamma_j)$ with $1 \le j \le s$. Similarly, the slide of it along β_i will only affect $\psi(\alpha_i)$. Hence, we can change $\psi(\alpha_i)$ and $\psi(\beta_i)$ for $1 \le i \le r$ to the required values, and τ can be a composition of slides of singular points.

If $m > n$, then we also need slides of handles and Dehn twists. The Dehn twist along α_i will change $\psi(\beta_i)$ to $\psi(\beta_i \alpha_i^{\pm 1})$ and not affect the $\psi(\beta_j)$ with $j \ne i$, $\psi(\alpha_j)$ with $1 \le j \le r$ and $\psi(\gamma_j)$ with $1 \le j \le s$. Similarly, the Dehn twist along β_i will only possibly affect $\psi(\alpha_i)$. Hence, we can change $\psi(\beta_i)$ for $1 \le i \le r$ to 1 by Dehn twists along α_i and β_i. Then, for i and k with $k \ne i$, there is a slide of a handle along the loop β_i, so that it changes $\psi(\alpha_i)$ to $\psi(\alpha_i \alpha_k^{\pm 1})$ and does not affect the $\psi(\alpha_j)$ with $j \ne i$, $\psi(\beta_j)$ with $j \ne k$ and $\psi(\gamma_j)$ with $1 \le j \le s$. The slide will also not affect $\psi(\beta_k)$, because now each $\psi(\beta_i)$ is 1. Hence, we can further change $\psi(\alpha_i)$ for $2 \le i \le r$ to 1 by slides of handles.

We also need to change $\psi(\alpha_1)$ to h. Since ψ is surjective, the $\psi(\alpha_1)$ and $\psi(\gamma_j)$ with $1 \le j \le s$ together generate $\mathbb{Z}_m$. We can use the Dehn twists along α_1 and β_1 together with the slides of singular points along α_1 and β_1 to change $\psi(\alpha_1)$ to h. This process will possibly change $\psi(\beta_1)$, but we can use the Dehn twists along α_1 to change $\psi(\beta_1)$ to 1. Hence, we can change $\psi(\alpha_i)$ and $\psi(\beta_i)$ for $1 \leq i \leq r$ to the required values, and τ can be a composition of slides of singular points, slides of handles, and Dehn twists. □

Proof of Theorem 1.5. **We first show the "only if" part.**

Let G be a finite cyclic subgroup in $Aut(\Sigma_g)$ that is extendable over S^3. Then, by the orbifold theorem and the geometrization theorem (see [BMP] and [Pe]), we can assume that the monomorphism ϕ maps G into $SO(4)$. Hence, we can get an embedding of the 2-orbifold $\mathcal{F} = \Sigma_g/G$ in the spherical 3-orbifold $\mathcal{O} = S^3/\phi(G)$. Denote this embedding by $\overline{e}$.

We can identify S^3 with the set $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$, and let h be a generator of G. Then we can assume that $\phi(h)$ has the form

$$\phi(h) : (z_1, z_2) \mapsto \big(e^{i\theta_1} z_1, e^{i\theta_2} z_2\big).$$

Suppose that G has order n, then there are integers l_1 and l_2 so that $\theta_1 = 2\pi l_1/n$ and $\theta_2 = 2\pi l_2/n$. Note that the greatest common divisor of l_1, l_2, n is 1, otherwise the order of G will be smaller that n. Now assume that $z_1 z_2 \neq 0$, and there exists a positive integer k so that $\phi(h)^k((z_1, z_2)) = (z_1, z_2)$. Then $n \mid k$, so the image of (z_1, z_2) in $\mathcal{O}$ is a regular point. For $(z_1, 0)$ or $(0, z_2)$, the image in $\mathcal{O}$ has index $\gcd(l_1, n)$ or $\gcd(l_1, n)$, respectively. Hence, such an image is a regular point if and only if the corresponding greatest common divisor is 1.

Let $p = \gcd(l_1, n)$ and $q = \gcd(l_2, n)$, then $\gcd(p, q) = 1$. The singular set of $\mathcal{O}$ consists of at most two circles with indices p and q. So clearly $n_1, \ldots, n_s$ belong to $\{p, q\}$. Since $\overline{e}(\mathcal{F})$ separates $\mathcal{O}$, each of p and q is valued even times. Then, the singular points of $\overline{e}(\mathcal{F})$ in each singular circle of $\mathcal{O}$ can be partitioned into pairs so that the conditions (b) and (c) in Theorem 1.5 are satisfied.

Then we show the "if" part.

Suppose that the orbifold $\mathcal{F} = \Sigma_g/G$ and the finitely-injective epimorphism ψ satisfy the three conditions (a), (b), (c) in Theorem 1.5. Then, by (a), each $\psi(\gamma_k)$ is either an element of order p or an element of order q. By (b), all the $\psi(\gamma_k)$ with order p (resp. q) have at most two possible values. So, by (c) and Lemma 4.1, the values of $\psi(\gamma_k)$ can be determined, up to automorphisms of G. Note that for any two singular points in $\mathcal{F}$ with the same index, there exists an automorphism of $\mathcal{F}$ which exchanges them and keeps the other singular points fixed. Hence, combined with Lemma 4.2, we can assume that ψ satisfies the following conditions.

(1) $\psi(\alpha_1) = h$, $\psi(\alpha_i) = 1$ for $2 \le i \le r$, $\psi(\beta_i) = 1$ for $1 \le i \le r$;

(2) $\psi(\gamma_j) = h^{mq}$ for $1 \le j \le s_1$, $\psi(\gamma_j) = h^{-mq}$ for $s_1 + 1 \le j \le 2s_1$, $\psi(\gamma_j) = h^{mp}$ for $2s_1 + 1 \le j \le 2s_1 + s_2$, $\psi(\gamma_j) = h^{-mp}$ for $2s_1 + s_2 + 1 \le j \le s$.

Here h is a generator of G; $s_1, s_2 \ge 0$ are two integers such that $2s_1 + 2s_2 = s$. Since ψ is finitely-injective, we have an integer m such that mpq equals the order of G, where $p = 1$ (resp. $q = 1$) if $s_1 = 0$ (resp. $s_2 = 0$).

Hence, when $\mathcal{F}$ is a torus with no singular points, up to conjugacy, the G-action on Σ_1 is determined by the order of G, and it is extendable over S^3; otherwise, up to conjugacy, the G-action on Σ_g is completely determined by Σ_g/G, and we need to show that such a G-action is extendable over S^3.

For any 2-orbifold $\mathcal{F}$ and finitely-injective epimorphism $\psi : \pi_1(\mathcal{F}) \to \mathbb{Z}_n$ that satisfy conditions (a), (b) and (c), we will construct an extendable cyclic action of order n in Example 4.3 such that its corresponding orbifold is $\mathcal{F}$. By the "only if" part, this new action also satisfies the three conditions, and hence is conjugate to the G-action by the above arguments. Since in Example 4.3 all the surfaces in S^3 are Heegaard surfaces, we have the "moreover" part of Theorem 1.5. $\square$

Example 4.3. Let $\mathcal{F}$ be the 2-orbifold having underlying space Σ_r, $2s_1$ singular points of index p, and $2s_2$ singular points of index q, where s_1, s_2 are nonnegative integers, p, q are co-prime positive integers, and $s_1 = 0$ (resp. $s_2 = 0$) if and only if $p = 1$ (resp. $q = 1$). There exists a finitely-injective epimorphism $\psi : \pi_1(\mathcal{F}) \to \mathbb{Z}_n$ if and only if one of the following conditions holds.

(1) $r = 0$ and $n = pq$;

(2) $r \ge 1$ and $n = mpq$ for some positive integer m.

In each case, we will construct a $\mathbb{Z}_n$-action on a closed surface below such that its corresponding orbifold is orbifold homeomorphic to $\mathcal{F}$. The action is extendable over S^3 and its corresponding embedded surface is a Heegaard surface.

We identify S^3 with the set $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. Let $\theta^q_m = 2\pi/mq$ and $\theta^p_m = 2\pi/mp$ where $m = n/(pq)$. Let $h^{p,q}_m$ be the isometry of S^3 defined by

$$h^{p,q}_m : (z_1, z_2) \mapsto \big(e^{i\theta^q_m} z_1, e^{i\theta^p_m} z_2\big).$$

Clearly $h^{p,q}_m$ has order mpq. Let T be the set $\{(z_1, z_2) \in S^3 \mid |z_1| = |z_2|\}$. Then T is a torus in S^3, which is invariant setwise under the action of $\langle h^{p,q}_m \rangle$. Let ϵ

be a sufficiently small positive number, for example $\epsilon < (100n(s_1+1)(s_2+1))^{-1}$.

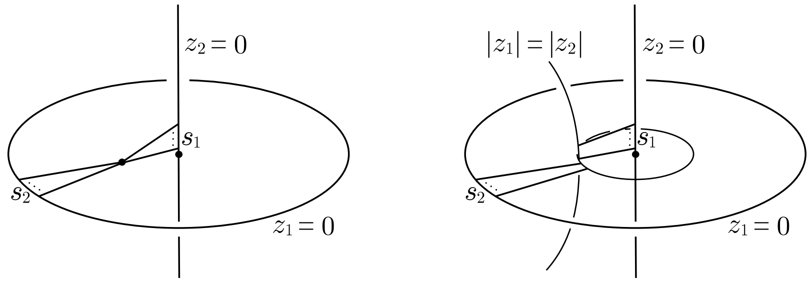

Figure 4: Part of $\Gamma^{p,q}_{s_1,s_2}$ and $\Gamma^{p,q}_{s_1,s_2,m}$ in $S^3 - \{(-1,0)\}$.

In the case of (1), we connect the point $(\sqrt{2}/2, \sqrt{2}/2)$ to the points $(e^{ik\epsilon}, 0)$ and $(0, e^{il\epsilon})$ for $1 \le k \le s_1$ and $1 \le l \le s_2$ by the shortest geodesics in S^3. The left picture in Figure 4 shows a sketch of the construction. The union of the orbits of these geodesics under the action of $\langle h_1^{p,q} \rangle$ forms a graph $\Gamma^{p,q}_{s_1,s_2}$, and we can choose an invariant regular neighborhood of $\Gamma^{p,q}_{s_1,s_2}$. Denote its boundary by $\Sigma^{p,q}_{s_1,s_2}$. Then the orbifold $\Sigma^{p,q}_{s_1,s_2}/\langle h_1^{p,q} \rangle$ is orbifold homeomorphic to $\mathcal{F}$.

In the case of (2), we connect $(e^{ik\epsilon}\sqrt{2}/2, \sqrt{2}/2)$ to $(e^{ik\epsilon}, 0)$ for $1 \le k \le s_1$, and connect $(\sqrt{2}/2, e^{il\epsilon}\sqrt{2}/2)$ to $(0, e^{il\epsilon})$ for $1 \le l \le s_2$, by the shortest geodesics in S^3. The right picture in Figure 4 shows a sketch of the construction. The union of the orbits of these geodesics under the action of $\langle h_m^{p,q} \rangle$ forms a graph $\Gamma^{p,q}_{s_1,s_2,m}$, which is not connected in general. We choose sufficiently small disjoint spheres centred at the vertices of degree bigger than one. For each edge of the graph we can make a tube along it connecting the small spheres to the torus T. Then we can obtain a closed surface which is invariant setwise under the action of $\langle h_m^{p,q} \rangle$. At each point in the orbit of $(\sqrt{2}/2, \sqrt{2}/2)$ we can add $r-1$ local handles to the surface equivariantly. Denote the result by $\Sigma^{p,q,r}_{s_1,s_2,m}$. Then $\Sigma^{p,q,r}_{s_1,s_2,m}/\langle h_m^{p,q} \rangle$ is orbifold homeomorphic to $\mathcal{F}$.

Note that the surfaces $\Sigma^{p,q}_{s_1,s_2}$ and $\Sigma^{p,q,r}_{s_1,s_2,m}$ are all Heegaard surfaces. Hence the $\langle h_1^{p,q} \rangle$-action on $\Sigma^{p,q}_{s_1,s_2}$ and the $\langle h_m^{p,q} \rangle$-action on $\Sigma^{p,q,r}_{s_1,s_2,m}$ give all extendable finite cyclic actions on Σ_g stated in Theorem 1.5. By the Riemann–Hurwitz formula,

$$2 - 2g = n\left(2 - 2r - 2s_1\left(1 - \frac{1}{p}\right) - 2s_2\left(1 - \frac{1}{q}\right)\right).$$

Since $\gcd(p, q) = 1$ and $pq \mid n$, we can find all solutions (n, p, q, r, s_1, s_2) for a given g by enumeration. Especially, when $s_1 = s_2 = 0$, we get the standard forms of the free finite cyclic actions on Σ_g, which correspond to the factors of $g - 1$.

Remark 4.4. For a finite cyclic subgroup G in $Aut(\Sigma_g)$ that is extendable over S^3, the generators of G are not conjugate to each other in general. For two generators h_1 and h_2, let η be the automorphism of G with $\eta(h_1) = h_2$. Let $\psi : \pi_1(\Sigma_g/G) \to G$ be the finitely-injective epimorphism corresponding to G. By Nielsen's classification of periodic automorphisms of closed surfaces (see [Ni]), h_1 and h_2 are conjugate to each other if and only if $\eta \circ \psi(\gamma_k) = \psi(\gamma_k)$ or $\eta \circ \psi(\gamma_k) = \psi(\gamma_k)^{-1}$ for $1 \le k \le s$, equivalently, there exists an orbifold automorphism τ of Σ_g/G such that $\eta \circ \psi = \psi \circ \tau_*$.

To see the equivalence, if $\eta \circ \psi = \psi \circ \tau_*$, then $\eta \circ \psi(\gamma_k) = \psi \circ \tau_*(\gamma_k)$. Since $\tau_*(\gamma_k)$ is conjugate to some γ_j or γ_j^{-1}, which corresponds to a singular point with order $n_j = n_k$. Since the image of ψ is cyclic, by Theorem 1.5, we have $\psi \circ \tau_*(\gamma_k) = \psi(\gamma_k)$ or $\psi(\gamma_k)^{-1}$. Conversely, by Theorem 1.5, we can first find τ_1 so that $\eta \circ \psi(\gamma_k) = \psi \circ \tau_{1*}(\gamma_k)$ for $1 \le k \le s$, where τ_1 is an orbifold automorphism exchanging singular points. By Lemma 4.2, we can assume that $\psi(\alpha_1) = h_1$, $\psi(\alpha_i) = 1$ for $2 \le i \le r$, and $\psi(\beta_i) = 1$ for $1 \le i \le r$. Then, by the proof of Lemma 4.2, we can find an orbifold automorphism τ_2 so that $\psi \circ \tau_{2*} \circ \tau_{1*} = \eta \circ \psi$, and $\tau = \tau_2 \circ \tau_1$ is the required orbifold automorphism.

Hence, the classification of orientation-preserving periodic automorphisms of Σ_g that are extendable over S^3 needs a little more work.

As the end of the paper, we give some examples and questions as illustrations and supplements to our theorems.

Example 4.5. We give two periodic automorphisms of the handlebody of genus four, which are not extendable over S^3. It is not so easy to prove this fact without Theorem 1.2 and Theorem 1.5.

We first consider the handlebody with two 0-handles V_1, V_2 and five 1-handles E_i for $1 \le i \le 5$. We will construct a periodic automorphism f of order five. It is a $2\pi/5$-rotation on V_1, is a $4\pi/5$-rotation on V_2, and permutes the five 1-handles such that $f(E_i) = E_{i+1}$ for $1 \le i \le 5$, where $E_6 = E_1$. Then we can equivariantly attach the five 1-handles onto V_1 and V_2 such that each E_i is adjacent to each of V_1 and V_2. See the left picture in Figure 5. This gives an order five periodic automorphism of the handlebody of genus four. Intuitively, the map is not extendable over S^3, because the "rotation speeds" of the two 0-handles are different. In view of our theorems, the condition (b) in Theorem 1.5 is not satisfied.

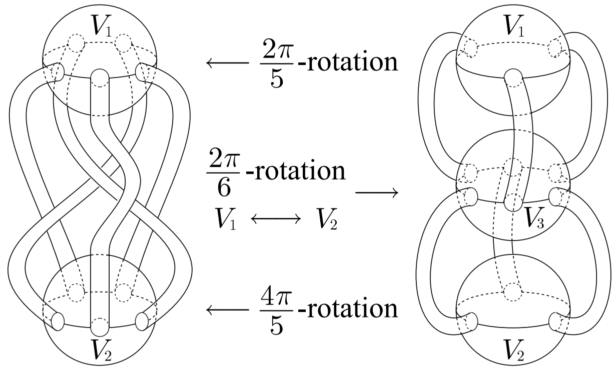

Figure 5: Actions on the handlebody of genus four.

Then, consider the handlebody with three 0-handles V_1, V_2, V_3 and six 1-handles E_i for $1 \leq i \leq 6$. We will construct a periodic automorphism f of order six. It is a $2\pi/6$-rotation on V_3 and it exchanges V_1 and V_2. Its square preserves V_1 and V_2, and is a $2\pi/3$-rotation on them. It permutes the six 1-handles such that $f(E_i) = E_{i+1}$ for $1 \leq i \leq 6$, where $E_7 = E_1$. Then we can equivariantly attach the six 1-handles onto V_1, V_2, V_3 such that each of E_1, E_3, E_5 is adjacent to each of V_1 and V_3, and each of E_2, E_4, E_6 is adjacent to each of V_2 and V_3. See the right picture in Figure 5. This gives an order six periodic automorphism of the handlebody of genus four. Intuitively, it is not extendable over S^3, because V_1 and V_2 can not be exchanged. In the view of our theorems, the condition (a) in Theorem 1.5 is not satisfied.

Example 4.6. By the result in [WWZZ], the maximum order of finite subgroups in $Aut(\Sigma_g)$ that are extendable over S^3 is $6(g-1)$ when g is 21 or 481. Moreover, the corresponding embedded surfaces of such groups cannot be Heegaard surfaces. Hence, the "moreover" part of Theorem 1.5 does not hold for general finite group actions. Below we will give an explicit example about Σ_{21}.

Consider two spheres in $\mathbb{R}^3$ centered at the origin with radii $1/2$ and 2. Project a regular dodecahedron centered at the origin onto the spheres, and denote the two images by P_1 and P_2, where P_2 is the larger one. Clearly the orientation-preserving isometries of the dodecahedron preserves $P_1 \cup P_2$. Consider the composition of the inversion about the unit sphere and a reflection about a plane containing the origin. We can choose the plane such that the composition preserves $P_1 \cup P_2$. Then all these elements preserving $P_1 \cup P_2$ generate a group of order 120, denoted by G.

Clearly G can act on S^3. Let v and w be adjacent vertices in the dodecahedron, and let v_1, w_1 and v_2, w_2 be their corresponding vertices in P_1 and P_2. Then we can choose an arc connecting v_1 and w_2 such that its images under the action of G only meet at vertices of P_1 and P_2. The union of these image arcs is a graph with genus 21. It has a regular neighborhood N which is preserved by the action of G. Then G also preserves ∂N which is homeomorphic to Σ_{21}.

Question 4.7. *If we admit orientation-reversing automorphisms (of Σ_g or S^3), how to classify the periodic automorphisms of Σ_g that is extendable over S^3? Can the corresponding embedded surface always be a Heegaard surface?*

Appendix: Two results about automorphisms of surfaces with boundary

In this section, we will prove two results about automorphisms of surfaces with boundary. These results are well known for closed surfaces, but it is hard to find the versions for surfaces with boundary in the literature. We include the proofs here for the convenience of the reader.

Definition A.1. Let Σ be a compact oriented surface with (possibly empty) boundary, let $D\Sigma$ be the double of Σ along $\partial\Sigma$. Given a hyperbolic structure m on Σ so that $\partial\Sigma$ is geodesic, let Dm be the hyperbolic structure on $D\Sigma$ induced by m. Given a map $f : (\Sigma, \partial\Sigma) \to (\Sigma, \partial\Sigma)$, let $Df : D\Sigma \to D\Sigma$ be the induced map.

Definition A.2. Let Σ be a compact oriented surface with (possibly empty) boundary, and let m_0, m_1 be two hyperbolic structures on Σ so that $\partial\Sigma$ is geodesic in each m_i. We say a homeomorphism $\varphi : (\Sigma, m_0) \to (\Sigma, m_1)$ is *quasi-conformal*, if its double $D\varphi : (D\Sigma, Dm_0) \to (D\Sigma, Dm_1)$ is quasi-conformal.

The following proposition was sketched in [Ke, Section IV], but we did not find an explicit statement in the literature, so we include a proof here.

Proposition A.3 (Nielsen Realization Problem for surface with boundary). *Let Σ be a compact surface with boundary, $\chi(\Sigma) < 0$. Let G be a finite subgroup of $\pi_0(Aut(\Sigma))$, then G can be lifted to a finite subgroup of $Aut(\Sigma)$.*

Proof. By Definition A.1, every element $f \in Aut(\Sigma)$ can be doubled to $Df \in Aut(D\Sigma)$. On the level of mapping classes, G is embedded as a subgroup DG of $\pi_0(Aut(D\Sigma))$. Let $r : D\Sigma \to D\Sigma$ be the reflection across $\partial\Sigma$. Then the mapping class of r and DG generate a finite subgroup $\widetilde{G}$ of $\pi_0(Aut(D\Sigma))$, such that DG is an index-2 subgroup of $\widetilde{G}$. By the positive solution of the Nielson realization problem for closed surfaces [Ke], there exists a hyperbolic metric m on $D\Sigma$ so that there is an embedding $\rho : \widetilde{G} \to \mathrm{Iso}(D\Sigma, m)$ lifting $\widetilde{G}$. (The main theorem in [Ke] is only stated for orientable surfaces, but it also holds true for non-orientable surfaces. See the remark at the end of [Ke, Section IV].)

Let $c_1, \ldots, c_n$ be the components of $\partial\Sigma$, and let $\bar{c}_i$ be the unique geodesic loop homotopic to c_i. Clearly, $\bar{c}_1, \ldots, \bar{c}_n$ are mutually disjoint. Since $r(c_i) = c_i$, we must have $\rho(r)(\bar{c}_i) = \bar{c}_i$. Moreover, since DG permutes the homotopy classes $[c_1], \ldots, [c_n]$, $\rho(DG)$ permutes $\bar{c}_1, \ldots, \bar{c}_n$. So $\rho(DG)$ sends every component of $D\Sigma \setminus (\bar{c}_1 \cup \cdots \cup \bar{c}_n)$ to itself. Thus $\rho(DG)$ acts on Σ, which is isotopic to a component of $D\Sigma \setminus (\bar{c}_1 \cup \cdots \cup \bar{c}_n)$. It is clear that $\rho(DG)$ lifts G. □

Proposition A.4. *Notations are as in Definition A.2. Let $f : \Sigma \to \Sigma$ be an orientation preserving homeomorphism, then there is a unique quasi-conformal homeomorphism $\tau_f : (\Sigma, m_0) \to (\Sigma, m_1)$ between the two hyperbolic surfaces with constant dilatation, so that the map τ_f is homotopic to f.*

Proof. When Σ is closed, this is a standard result in Teichmüller theory, see [FM, Theorems 11.8 and 11.9]. We now consider the general case.

Let $r : D\Sigma \to D\Sigma$ be the reflection across $\partial\Sigma$, then r is an orientation-reversing isometry for each Dm_i. Now $r\tau_{Df}r$ is homotopic to $r(Df)r = Df$, and it has the same constant dilatation as τ_{Df}. The uniqueness of τ_{Df} implies that

$$r\tau_{Df}r = \tau_{Df}.$$

In particular, if $x \in \partial\Sigma$, we have $r(x) = x$, so $r\tau_{Df}(x) = \tau_{Df}(x)$, which implies that $\tau_{Df}(x) \in \partial\Sigma$. Thus τ_{Df} sends $\partial\Sigma$ onto $\partial\Sigma$. Let τ_f be the restriction of τ_{Df} to Σ, then it is an automorphism of Σ. The uniqueness of τ_f follows from the uniqueness of τ_{Df}. □

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