

## CLOSED 3-BRAIDS ARE NEARLY FIBRED

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### ABSTRACT

We classify fibred closed 3-braids. In particular, given a nontrivial closed 3-braid, either it is fibred, or it differs from a fibred link by adding a crossing. The proof uses Gabai's method of disk decomposition. The topmost term in the knot Floer homology of closed 3-braids is also computed.

*Keywords:* Closed 3-braids; fibred; knot Floer homology; disk decomposition.

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### 1. Introduction

It is well known that every link in  $S^3$  is the closure of a braid, so one can study knots and links via their braid forms. In this paper, we will study the question whether a link is fibred via this approach for closed 3-braids. An oriented link  $L$  is *fibred*, if the complement of  $L$  fibres over the circle, and  $L$  is the boundary of the fibre. Our geometric result is as follows.

**Theorem 1.1.** *Suppose a link  $L \in S^3$  is the closure of a 3-braid, then exactly one of the following 3 cases happens:*

- (i)  $L$  is the 3-component trivial link;
- (ii)  $L$  is fibred;
- (iii)  $L$  or its mirror image is the closure of a nondecreasing positive word  $P$ . Moreover, either  $P$  is a power of one of  $a_1, a_2, a_3$ , or  $P$  is started with  $a_1$  and ended with  $a_3$ . Hence after adding a crossing in the braid diagram, one gets a fibred link.

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The exact meaning of case (iii) will become clear after Definition 2.4.

**Remark 1.2.** We are informed by Stoimenow that the classification of fibred closed 3-braids has also been obtained in [12], with the assistance of Hirasawa and Murasugi.

The original motivation of this paper was to check a conjecture of Ozsváth and Szabó, which asserts that knot Floer homology detects fibred knots, in the case of closed 3-braids. A few months after the first version of the paper was posted online, this conjecture was proved by Ghiggini [3] and the author [6]. So the original interest of this paper has gone. However, our explicit classification of fibred closed 3-braids presented in this paper is still interesting.

The paper is organized as follows. In Sec. 2, we will compute the topmost terms in the knot Floer homology of closed 3-braids. The computation uses a result of Xu [14]. In Sec. 3, we apply Gabai’s method of disk decomposition to prove Theorem 1.1. In Sec. 4, we show that given an arbitrary braid, one can get fibred braids from it by adding sufficiently many full twists.

## 2. Knot Floer Homology of Closed 3-Braids

In this section, we will compute the topmost term in the knot Floer homology of closed 3-braids. Although the result can be deduced from our main theorem, the computation here has its own interest. The computation also motivates our main theorem.

### 2.1. Preliminaries on knot Floer homology

Knot Floer homology was introduced by Ozsváth–Szabó [7] and independently by Rasmussen [10]. When  $K$  is a knot (or link) in  $S^3$ , its knot Floer homology is a finitely generated bigraded abelian group

$$\widehat{HFK}(K) = \bigoplus_{i,j} \widehat{HFK}_j(K, i).$$

Here  $i$  is the *Alexander grading*,  $j$  is the *Maslov grading*, their values lie in  $\mathbb{Z}$  or  $\mathbb{Z} + \frac{1}{2}$ . The Euler characteristic of knot Floer homology gives rise to the Alexander–Conway polynomial.

Knot Floer homology contains a lot of information about the knot or link. For example, it detects the genera of classical links. Namely, we have the following theorem due to Ozsváth and Szabó ([9], see also [4]).

**Theorem 2.1.** *Suppose  $L$  is an oriented link in  $S^3$ . Let  $\chi(L)$  be the maximal Euler characteristic of the Seifert surfaces bounded by  $L$ , and  $i(L) = \frac{|L| - \chi(L)}{2}$ , where  $|L|$  is the number of components of  $L$ . Then*

$$i(L) = \max\{i \mid \widehat{HFK}(L, i) \neq 0\}.$$

We always refer  $\widehat{HFK}(L, i(L))$  as the *topmost term* in the knot Floer homology. We say the knot Floer homology of a link is *monic*, if the topmost term is isomorphic to  $\mathbb{Z}$ . Ozsváth and Szabó proved that if the link is fibred, then the knot Floer homology is monic ([8], see also [4]).

Knot Floer homology satisfies the following skein relation.

**Theorem 2.2.** *Suppose  $L_+, L_-, L_0$  are three oriented links that differ at exactly one crossing as in the usual skein relation. When the two strands of  $L_+$  at the special crossing belong to the same component, we have an exact triangle as follows.*

$$\begin{array}{ccc} \widehat{HFK}(L_-) & \longrightarrow & \widehat{HFK}(L_0) \\ & \nearrow & \\ \widehat{HFK}(L_+) & & \end{array}$$

When the two strands of  $L_+$  at the crossing belong to different components, the exact triangle is as follows.

$$\begin{array}{ccc} \widehat{HFK}(L_-) & \longrightarrow & \widehat{HFK}(L_0) \otimes V \\ & \nearrow & \\ \widehat{HFK}(L_+) & & \end{array}$$

The  $V$  in the triangle is isomorphic to  $\mathbb{Z}_{(-1,-1)} \oplus \mathbb{Z}_{(0,0)} \oplus \mathbb{Z}_{(1,1)}$ , where the subscripts denote the bigrading  $(i, j)$  of the summands.

Moreover, the maps in the exact triangles preserve the Alexander grading.

**2.2. A standard form for closed 3-braids**

The computation of the topmost term in the knot Floer homology becomes easier if we already know what the genus of a 3-braid is. Fortunately, this problem was solved by Xu [14]. In order to explain her result, we need some preparation.

**Notation 2.3.**  $B_3$  denotes the group of 3-braids,  $\sigma_1$  and  $\sigma_2$  are the standard generators of  $B_3$ . Instead of the standard presentation, we use three generators  $a_1 = \sigma_1, a_2 = \sigma_2, a_3 = \sigma_2\sigma_1\sigma_2^{-1}$ , and the presentation

$$B_3 = \langle a_1, a_2, a_3 \mid a_2a_1 = a_3a_2 = a_1a_3 \rangle.$$

The advantage of such presentation is, one can draw  $a_1, a_2, a_3$  cyclically on a cylinder, thus we can permute the roles of  $a_1, a_2, a_3$  cyclically. The reader is encouraged to figure this out by himself/herself.

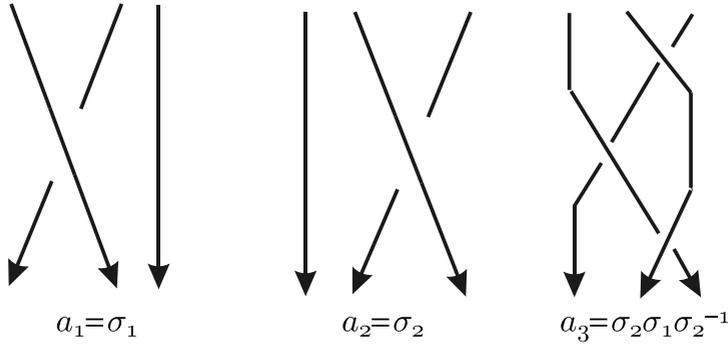


Fig. 1. The generators of  $B_3$ .

If  $w \in B_3$ , then  $\bar{w}$  denotes its inverse. Let  $\alpha = a_2 a_1 = a_3 a_2 = a_1 a_3$ . The following relations will be useful to us:

$$a_i \bar{a}_j = \bar{a}_{i+1} a_{j+1}, \quad \alpha \bar{a}_i = a_{i+1}, \quad \bar{a}_i \alpha = a_{i-1}.$$

**Definition 2.4.** Suppose  $P = a_{\varepsilon_1} \cdots a_{\varepsilon_n}$  is a positive word. We say  $P$  is *nondecreasing* if for each  $j \in \{1, \dots, n - 1\}$ ,  $\varepsilon_{j+1} = \varepsilon_j$  or  $\varepsilon_j + 1$ , where the subscript for  $a$  is understood cyclically.  $P$  is *strictly increasing*, if for each  $j \in \{1, \dots, n - 1\}$ ,  $\varepsilon_{j+1} = \varepsilon_j + 1$ .

**Theorem 2.5 (P. J. Xu).** *Every conjugacy class in  $B_3$  can be represented by a shortest word in  $a_1, a_2, a_3$  which is unique up to symmetries, such that the word has one of the following forms:*

- (i)  $\alpha^k P$ ;
- (ii)  $N \alpha^{-k}$ ;
- (iii)  $NP$ .

Here  $k \geq 0$ ,  $\bar{N}$  and  $P$  are nondecreasing positive words,  $P$  or  $N$  may be empty.

Moreover, the minimal Seifert surface of the corresponding closed braid can be constructed from this word.

We briefly explain how to construct the Seifert surface from a word  $w$ . We first resolve the braid to a 3-component trivial link, bounding 3 disjoint disks. Then for each letter in  $w$ , one attaches a twisted band to connect two of the 3 disks. This surface is called the *Bennequin surface* of the word  $w$ , denoted by  $B_w$ . It has Euler characteristic  $3 - l(w)$ , where  $l(w)$  is the length of  $w$ .

From now on, we also use the word  $w$  to denote the corresponding 3-braid, if there is no confusion. Xu's theorem says that, for a shortest word  $w$  as above,  $\chi(w) = 3 - l(w)$ .

**2.3. Computation of the topmost term**

**Theorem 2.6.** *Suppose  $L$  is the closure of a 3-braid  $w$ , where  $w$  is in the form in Theorem 2.5. We consider the word  $UT(w)$ . If  $UT(w)$  is in the form of  $N$  or  $P$ , and  $l(UT(w)) = 3t + 1$  or  $3t + 3$ , ( $t \geq 0$ ), then*

$$\widehat{HFK}(L, i(L)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

*In other cases,  $\widehat{HFK}(L)$  is monic, except when  $L$  is the 3-component trivial link.*

**Lemma 2.7.** *If the subword  $a_i a_i$  appears in  $w$ , one can replace it by a single  $a_i$  to get a new word  $w'$ . Then the top terms in the knot Floer homology of  $w$  and  $w'$  are isomorphic as abelian groups. Moreover, the closure of  $w'$  is fibred with fibre  $B_{w'}$  if and only if the closure of  $w$  is fibred with fibre  $B_w$ .*

**Proof.** The Bennequin surface of  $w$  is the plumbing of the Bennequin surface of  $w'$  with a Hopf band. The statement about knot Floer homology is a special case of [5, Theorem 1.1]. (It can also be proved by using the argument in Lemma 2.9.) The last statement is a result of Gabai [1]. □

Given a reduced word  $w$  in Xu’s form, we can apply the “untwisting” operation in the previous lemma repeatedly, until we get a word also in Xu’s form, but now the  $\bar{N}$  and/or  $P$  are strictly increasing. We denote this new word by  $UT(w)$ .

**Proposition 2.8.** *Suppose  $w = \alpha^k P$  is a word in Xu’s form,  $k > 0$ .  $L$  is the closure of  $w$ , then  $L$  is fibred with fibre  $B_w$ . Here  $B_w$  is the Bennequin surface of  $w$ .*

**Proof.** Suppose the first letter in  $P$  is  $a_1$ , then  $P = a_1 P'$ .  $w = \alpha^k a_1 P' = \alpha^{k-1} a_2 a_1 a_1 P'$ . Hence  $B_w$  is the plumbing of a Hopf band with the Bennequin surface of  $\alpha^{k-1} a_2 a_1 a_1 P' = \alpha^k P'$ . By Lemma 2.7, we can reduce our problem to  $\alpha^k P'$ , hence to  $\alpha^k$  by induction. Our conclusion holds since  $\alpha^k$  is a torus link. □

**Lemma 2.9.** *If  $L$  is in the form  $NP$ ,  $1 = l(N) \leq l(P)$ , then  $\widehat{HFK}(L)$  is monic.*

**Proof.** We can assume  $N = \bar{a}_2$ . We will prove our result by induction on  $l(P)$ . When  $l(P) = 1$ ,  $NP = \bar{a}_2 a_1$  or  $\bar{a}_2 a_3$ , hence  $L$  is the unknot. Now assume  $l(P) > 1$ , and  $P$  is strictly increasing.

If the last letter in  $P$  is  $a_3$ , then  $P$  can be written as  $P' a_1 a_2 a_3$ . We have the skein relation for

$$L_- = \bar{a}_2 P' a_1 a_2 a_3, \quad L_0 = \bar{a}_2 P' a_1 a_3, \quad L_+ = \bar{a}_2 P' a_1 \bar{a}_2 a_3.$$

And we have  $\bar{a}_2 P' a_1 a_3 \sim \alpha \bar{a}_2 P' = a_3 P'$ , (“ $\sim$ ” denotes conjugacy relation in  $B_3$ ), hence  $\chi(L_0) \geq \chi(L_-) + 3$ . In the local picture of the skein relation, if the two strands in  $L_-$  belong to the same component, then  $|L_0| = |L_-| + 1$ , and  $i(L_0) < i(L_-)$ , hence  $\widehat{HFK}(L_0, i_+) = 0$  by Theorem 2.1; if the two strands in  $L_-$  belong to different components, then  $|L_0| = |L_-| - 1$ , and  $i(L_0) + 1 < i(L_-)$ , hence  $\widehat{HFK}(L_0, i_+ -$

1) = 0. In any case, using the exact triangle (Theorem 2.2) at the Alexander grading  $i(L_-) = i(L_+)$ , we get an isomorphism between  $\widehat{HFK}(L_-, i(L_-))$  and  $\widehat{HFK}(L_+, i(L_+))$  since the third term is 0.

As for  $L_+$ , we have  $\bar{a}_2 P' a_1 \bar{a}_2 a_3 = \bar{a}_2 P' a_1 a_1 \bar{a}_2 \sim \bar{a}_2^2 P' a_1^2$ . As we already mentioned in Lemma 2.7, its topmost term of the knot Floer homology is the same as the one of  $\bar{a}_2 P' a_1$ , to which we can apply the inductive hypothesis.

If the last letter in  $P$  is  $a_1$ , then  $P$  can be written as  $P' a_3 a_1$ . We consider the skein relation for

$$L_- = \bar{a}_2 P' a_3 a_1, \quad L_0 = \bar{a}_2 P' a_3, \quad L_+ = \bar{a}_2 P' a_3 \bar{a}_1.$$

We have  $\bar{a}_2 P' a_3 \bar{a}_1 \sim P' a_3 \bar{a}_1 \bar{a}_2 = P' a_3 \bar{a} = P' \bar{a}_1$ . Length of  $P' \bar{a}_1$  is less than length of  $\bar{a}_2 P' a_3 a_1$ , hence  $i(L_+) < i(L_-)$ , which means  $\widehat{HFK}(L_+, i(L_-)) = 0$  by Theorem 2.1. We have  $\chi(L_0) = \chi(L_-) + 1$ . If the two strands in  $L_-$  belong to the same component, then  $|L_0| = |L_-| + 1$ , and  $i(L_0) = i(L_-)$ ; if the two strands in  $L_-$  belong to different components, then  $|L_0| = |L_-| - 1$ , and  $i(L_0) = i(L_-) - 1$ . In any case, using the exact triangle (Theorem 2.2) at the Alexander grading  $i(L_-)$ , we get an isomorphism between  $\widehat{HFK}(L_-, i(L_-))$  and  $\widehat{HFK}(L_0, i(L_0))$ , since the third term is 0. Now we apply the inductive hypothesis to  $L_0$ . □

**Proposition 2.10.** *If  $L$  is of the type  $NP$ ,  $N$  and  $P$  are nonempty, then  $\widehat{HFK}(L)$  is monic.*

**Proof.** We induct on  $l(N)$ . The case when  $l(N) = 1$  is the lemma above. Now we assume  $l(N) \geq 2$ , we can suppose the first letter in  $N$  is  $\bar{a}_3$ , then  $N = \bar{a}_3 \bar{a}_2 N'$ .

If the last letter in  $P$  is  $a_1$ ,  $P = P' a_1$ . Then we consider the exact triangle for

$$L_- = a_3 \bar{a}_2 N' P' a_1, \quad L_0 = \bar{a}_2 N' P' a_1, \quad L_+ = \bar{a}_3 \bar{a}_2 N' P' a_1.$$

We have  $a_3 \bar{a}_2 N' P' a_1 \sim a_1 a_3 \bar{a}_2 N' P' = a_3 N' P'$ . The same argument as before shows that  $\widehat{HFK}(L_0, i(L_0)) \cong \widehat{HFK}(L_+, i(L_+))$ . We then apply the inductive hypothesis to  $L_0$ .

If the last letter in  $P$  is  $a_2$ ,  $P = P' a_2$ . Consider the exact triangle for

$$L_- = a_3 \bar{a}_2 N' P' a_2, \quad L_0 = \bar{a}_2 N' P' a_2, \quad L_+ = \bar{a}_3 \bar{a}_2 N' P' a_2.$$

$L_0$  can be reduced to  $N' P'$ , hence we get our conclusion as before, by applying the inductive hypothesis to  $L_- = \bar{a}_2 N' P' a_2 a_3$ . □

**Proposition 2.11.** *If  $L$  is in the form  $P$ ,  $P = (a_1 a_2 a_3)^t$  or  $(a_1 a_2 a_3)^t a_1$ ,  $t \geq 1$ . Then*

$$\widehat{HFK}(L, i(L)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

**Proof.** Suppose  $P = (a_1 a_2 a_3)^t$ , consider the skein relation for

$$L_- = a_2 (a_1 a_2 a_3)^t, \quad L_0 = (a_1 a_2 a_3)^t, \quad L_+ = \bar{a}_2 (a_1 a_2 a_3)^t.$$

$L_-$  can be rewritten as  $\alpha P'$ , which was considered in Proposition 2.8, and  $L_+$  is of the type considered in Lemma 2.9. Consider the exact triangle at the Alexander grading  $i(L_+)$ , we get the exact triangle

$$\begin{array}{ccc} \widehat{HFK}(L_-, i(L_-)) & \longrightarrow & \widehat{HFK}(L_0, i(L_0)) \\ \uparrow & \swarrow & \\ \widehat{HFK}(L_+, i(L_+)) & & \end{array}$$

or

$$\begin{array}{ccc} \widehat{HFK}(L_-, i(L_-)) & \longrightarrow & \widehat{HFK}(L_0, i(L_0)) \otimes \mathbb{Z}_{1,1} \\ \uparrow & \swarrow & \\ \widehat{HFK}(L_+, i(L_+)) & & \end{array}$$

depending on whether the two strands of  $L_+$  at the crossing belong to the same component or not. In both cases, the exact triangle is isomorphic to

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \widehat{HFK}(L_0, i(L_0)) \\ \uparrow & \swarrow & \\ \mathbb{Z} & & \end{array}$$

By [4], for any oriented link  $L$   $\widehat{HFK}(L, i(L)) \otimes \mathbb{Q}$  is nontrivial. In particular this holds for  $L_0$ . One then easily sees that

$$\widehat{HFK}(L_0, i(L_0)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The case when  $P = (a_1 a_2 a_3)^t a_1 \sim a_1 (a_1 a_2 a_3)^t$  can be reduced to the previous one by Lemma 2.7. □

**Proof of Theorem 2.6.** Our theorem now follows from Lemma 2.7, Propositions 2.8, 2.10 and 2.11. □

**Remark 2.12.** With more care, one can get some information of the Maslov grading. For example, in Proposition 2.8, the topmost term lies at Maslov grading  $l(P) + \frac{|L|-1}{2}$ .

**Remark 2.13.** During the course of this work, we noted the paper [11], in which Stoimenow studied the skein polynomial of closed 3-braids, also using Xu’s theorem. Our result here should be compared with Stoimenow’s work.

### 3. Proof of the Main Theorem

In order to check a given Seifert surface is a fibre of a fibration, we use Gabai’s method of disk decomposition [2]. Let us briefly recall this method first.

A *sutured manifold* is a pair  $(M, \gamma)$ , where  $M$  is a compact oriented 3-manifold with boundary, and the *suture*  $\gamma$  is an embedded closed (not necessarily connected) curve in  $\partial M$ , such that  $\gamma$  divides  $\partial M$  into two subsurfaces  $R_+(\gamma), R_-(\gamma)$ . A sutured manifold  $(M, \gamma)$  is a *product sutured manifold* (or a *product*) if the pair is homeomorphic to  $(S \times [0, 1], \partial S \times \frac{1}{2})$  for some compact surface  $S$ .

A *product disk*  $D$  is a properly embedded disk in  $M$ , such that  $\partial D$  intersects  $\gamma$  exactly twice. Given a product disk  $D \subset M$ , one can cut  $M$  open along  $D$ , to get a new manifold  $M' = M - D \times (-1, 1)$ . For the curve  $\gamma \cap \partial M'$ , one can connect the ends  $\gamma \cap (\partial D \times \{\pm 1\})$  by diameters on  $D \times \{\pm 1\}$ , to get a closed curve  $\gamma' \subset \partial M'$ . Thus, we get a new sutured manifold  $(M', \gamma')$ . This process is called a *product decomposition*, denoted by

$$(M, \gamma) \xrightarrow{D} (M', \gamma').$$

Gabai proved that if there is a product decomposition as above, then  $(M', \gamma')$  is a product sutured manifold if and only if  $(M, \gamma)$  is a product. Moreover, he observed that  $(M, \gamma)$  is product sutured manifold if and only if there exists a sequence of product decomposition

$$(M, \gamma) = (M_0, \gamma_0) \xrightarrow{D_1} (M_1, \gamma_1) \xrightarrow{D_2} (M_2, \gamma_2) \xrightarrow{D_3} \dots \xrightarrow{D_n} (M_n, \gamma_n),$$

such that  $(M_n, \gamma_n)$  is homeomorphic to  $(D^2 \times [0, 1], \partial D^2 \times \frac{1}{2})$ .

Suppose  $F \subset S^3$  is an embedded oriented surface with boundary, a neighborhood of  $F$  is  $F \times [-1, 1]$ . The *complementary sutured manifold* of  $F$  is the pair  $(S^3 - \text{int}(F \times [-1, 1]), \partial F \times 0)$ . Then  $F$  is a fibre of a fibration of  $S^3 - \partial F$  if and only if the complementary sutured manifold is a product. Gabai’s strategy to detect fibred links is to start with a Seifert surface of a link, then try to decompose the complementary sutured manifold along product disks. After a few product decompositions, the (complementary) sutured manifold may become simple enough to study directly.

Now we can proceed to the proof of our main theorem.

**Lemma 3.1.** *Suppose  $w$  is a shortest word for  $L$ ,  $w$  is not necessarily in Xu’s form. If the array  $\bar{a}_1 a_3 a_1 a_2$  appears in  $w$ , then we can replace the array by  $\bar{a}_1 a_2$ , thus get a new word  $w'$ , with closure  $L'$ . Then  $L$  is fibred with fibre  $B_w$ , if and only if  $L'$  is fibred with fibre  $B_{w'}$ .*

**Proof.** We draw the local picture of the closed braid near the array  $\bar{a}_1 a_3 a_1 a_2$  as in Fig. 2(a). We thicken  $B_w$  to  $B_w \times I$ , and consider its complementary sutured manifold. In Fig. 2(b), we draw the suture (as curves) on the boundary of the handlebody  $B_w \times I$ . There is an obvious product disk in the complementary sutured manifold, namely, the disk bounded by the dashdotted rectangle specified in Fig. 2(b).

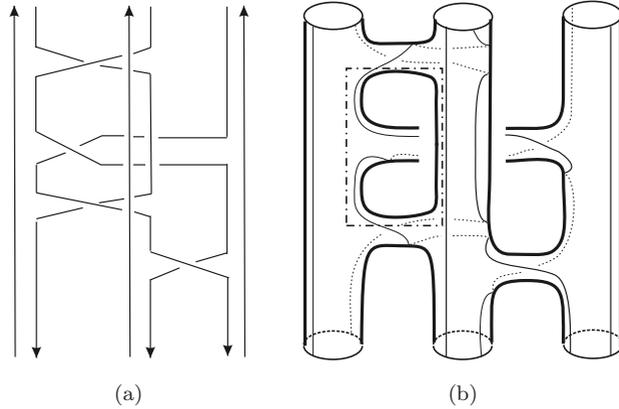


Fig. 2.

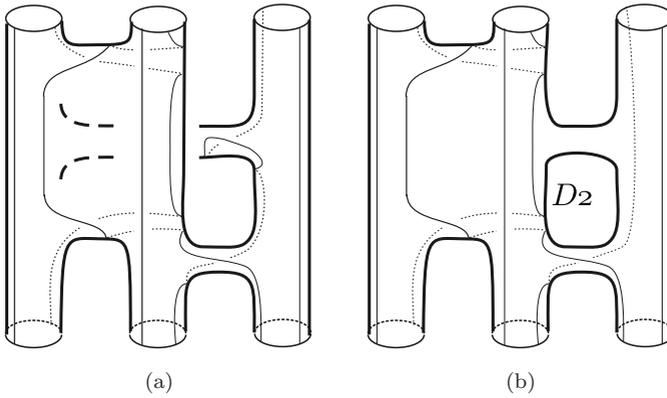


Fig. 3.

We decompose the complementary sutured manifold along the product disk, thus get Fig. 3(a). After an isotopy, we get Fig. 3(b), where the product disk  $D_2$  is clearer.

Now decompose the complementary sutured manifold in Fig. 3(b), thus get Fig. 4(a). After an isotopy, we get Fig. 4(b), which is just the local picture of a Bennequin surface near the array  $\bar{a}_1 a_2$ .

Our conclusion holds by [2, Lemma 2.2]. □

**Lemma 3.2.** *Suppose  $w$  is a shortest word for  $L$ ,  $w$  is not necessarily in Xu's form. If the array  $\bar{a}_1 a_2 a_3 a_1 a_2$  appears in  $w$ , then we can replace the array by  $\bar{a}_1 a_2$ , thus get a new word  $w'$ , with closure  $L'$ . Then  $L$  is fibred with fibre  $B_w$ , if and only if  $L'$  is fibred with fibre  $B_{w'}$ .*

**Proof.** We note that the algebraic relation

$$\cdots \bar{a}_1 a_2 \cdots = \cdots a_3 \bar{a}_1 \cdots$$

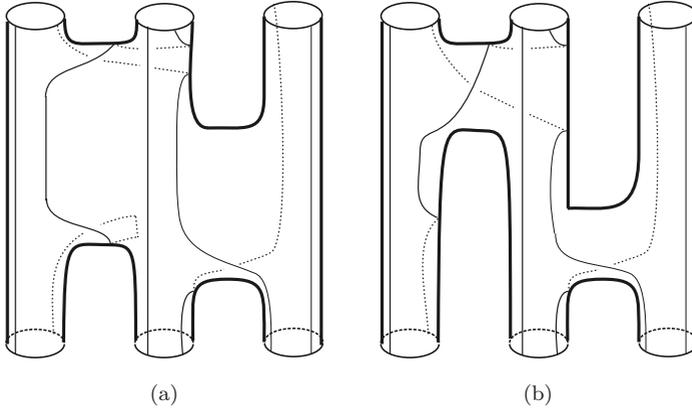


Fig. 4.

also gives a local isotopy of the Bennequin surfaces. We have  $\bar{a}_1 a_2 a_3 a_1 a_2 = a_3 \bar{a}_1 a_3 a_1 a_2$ . By Lemma 3.1, we can replace  $a_3 \bar{a}_1 a_3 a_1 a_2$  by  $a_3 \bar{a}_1 a_2$ . Now  $a_3 \bar{a}_1 a_2 = \bar{a}_1 a_2^2$ , we get our conclusion by Lemma 2.7.  $\square$

**Lemma 3.3.** *Suppose  $L$  is an oriented fibred link with fibre  $F$ . If  $F'$  is a Seifert surface for  $L$  such that  $\chi(F') = \chi(F)$ , then  $F'$  is isotopic to  $F$  in the complement of  $L$ .*

**Proof.** Since  $F, F'$  are both Seifert surfaces for the oriented link  $L$ ,  $[F] = [F'] \in H_2(S^3, L)$ . Since  $F$  is a fibre of a fibration, it is the unique (up to isotopy) Thurston norm minimizing surface in its homology class, so our conclusion holds.  $\square$

**Proposition 3.4.** *Suppose  $w = NP$  is a shortest word in Xu’s form for  $L$ .  $l(N), l(P) > 0$ . Then  $L$  is fibred with fibre  $B_w$ .*

**Proof.** Without loss of generality, one can assume  $\bar{N}, P$  are strictly increasing, and the last letter in  $N$  is  $\bar{a}_1$ . By Lemmas 3.1 and 3.2, we can replace  $P$  by one of the following words:  $a_3, a_3 a_1, a_2, a_2 a_3, a_2 a_3 a_1$ . Then we consider  $\bar{P} \bar{N}$ . By cyclically permuted versions of Lemmas 3.1 and 3.2, we can replace  $\bar{N}$  by a word with length  $\leq 3$ .

Now there are only finitely many cases for  $NP$  we need to consider. (We note that  $NP$  should be cyclically reduced, this restriction also reduces the cases.) We enumerate these cases as follows:

$$\begin{aligned} & \bar{a}_1 a_3, \quad \bar{a}_1 a_2, \quad \bar{a}_1 a_2 a_3, \\ & \bar{a}_2 \bar{a}_1 a_3, \quad \bar{a}_2 \bar{a}_1 a_3 a_1, \quad \bar{a}_2 \bar{a}_1 a_2 a_3, \quad \bar{a}_2 \bar{a}_1 a_2 a_3 a_1 \\ & \bar{a}_3 \bar{a}_2 \bar{a}_1 a_3 a_1, \quad \bar{a}_3 \bar{a}_2 \bar{a}_1 a_2, \quad \bar{a}_3 \bar{a}_2 \bar{a}_1 a_2 a_3 a_1. \end{aligned}$$

For  $\bar{a}_1a_3$  or  $\bar{a}_1a_2$ , the closed braid is the unknot, and  $B_w$  is a disk. For  $\bar{a}_1a_2a_3$  or  $\bar{a}_2\bar{a}_1a_3$ , the closed braid is a Hopf link, and  $\chi(B_w) = 0$ , so  $B_w$  is an annulus hence a fibre of the complement of the Hopf link.

For  $w = \bar{a}_2\bar{a}_1a_3a_1$ , it is equal to

$$a_1\bar{a}_3\bar{a}_1\bar{a}_1a_3a_1 = a_1\bar{a}_2^2a_1.$$

The closure of the latter word is fibred by Lemma 2.7, the Euler characteristic of its fibre is  $-1$ , which is equal to  $\chi(B_w)$ . So  $B_w$  is also a fibre by Lemma 3.3. The case of  $\bar{a}_2\bar{a}_1a_2a_3$  is the same, since its inverse is  $\bar{a}_3\bar{a}_2a_1a_2$ , which is equivalent to  $\bar{a}_2\bar{a}_1a_3a_1$  by permuting the subscripts cyclically.

For  $w = \bar{a}_2\bar{a}_1a_2a_3a_1$ , we have

$$w = \bar{a}_2a_3\bar{a}_1a_3a_1 = \bar{a}_2a_3^2\bar{a}_2a_1.$$

The latter can be reduced to  $\bar{a}_2a_3\bar{a}_2a_1$  by Lemma 2.7. Now

$$\bar{a}_2a_3\bar{a}_2a_1 = \bar{a}_2\bar{a}_1a_3a_1,$$

which is a case we have considered. So the closure of  $w$  is fibred, and  $\chi$  of its fibre is  $-2$ . Thus  $B_w$  is a fibre by Lemma 3.3. The case of  $\bar{a}_3\bar{a}_2\bar{a}_1a_3a_1$  is equivalent to this one by taking the inverse then permuting the subscripts.

For  $\bar{a}_3\bar{a}_2\bar{a}_1a_2$ , consider its inverse  $\bar{a}_2a_1a_2a_3$ . Permute the subscripts cyclically, we get  $\bar{a}_1a_3a_1a_2$ , then we can apply Lemma 3.1 to get our conclusion. The same strategy works for  $\bar{a}_3\bar{a}_2\bar{a}_1a_2a_3a_1$ . □

**Proof of Theorem 1.1.** By Propositions 2.8 and 3.4, we only need to consider the case that  $w = P$ ,  $P$  is strictly increasing. The case that  $l(P) \leq 1$  is easy. If  $l(P) = 3t + 2$ , ( $t \geq 0$ ) then it is conjugated to the form in Proposition 2.8. If  $l(P) = 3t + 3$  or  $l(P) = 3t + 4$  ( $t \geq 0$ ), then one can conjugate the word so that it is started with  $a_1$  and ended with  $a_3$ . Now  $a_2P$  is fibred by Proposition 2.8. □

#### 4. A Remark about Arbitrary Braids

Now we consider braids with arbitrary indices. Let  $A = (\sigma_1\sigma_2 \cdots \sigma_{n-1})^n \in B_n$ . Suppose  $w \in B_n$  is an  $n$ -braid, we say  $wA$  is obtained by adding a *full-twist* to  $w$ . Theorem 1.1 shows that, if a closed 3-braid is not fibred, then it differs from a fibred link by a full-twist. Our next proposition is a moderate generalization of this result to arbitrary braids.

**Proposition 4.1.** *Let  $w$  be an  $n$ -braid, then  $wA^k$  is a positive braid when the integer  $k$  is sufficiently large. Hence the closure of  $wA^k$  is fibred when  $k$  is large.*

**Proof.** It is well known that  $A$  lies in the center of  $B_n$ . (Actually  $A$  generates  $B_n$ , but we do not need this result.) Hence given a word  $w \in B_n$ , one can put  $A$  at any position in  $w$ , so as to get  $wA$ . Note that  $\sigma_1^{-1}A$  is a positive word, so after adding enough  $A$ 's, we can eliminate all the negative powers of  $\sigma_1$  in  $w$ .

Now suppose  $\sigma_2^{-1}$  appears in  $w$ , consider  $\sigma_2^{-1}A$ . We have

$$\sigma_2^{-1}\sigma_1\sigma_2\cdots = \sigma_1\sigma_2\sigma_1^{-1}\cdots.$$

So putting another  $A$  after the  $\sigma_1^{-1}$  will eliminate this negative power.

If  $\sigma_3^{-1}$  appears in  $w$ , consider  $\sigma_3^{-1}A$ . Since  $\sigma_3$  commutes with  $\sigma_1$ , we have

$$\sigma_3^{-1}\sigma_1\sigma_2\sigma_3\cdots = \sigma_1\sigma_3^{-1}\sigma_2\sigma_3\cdots = \sigma_1\sigma_2\sigma_3\sigma_2^{-1}\cdots.$$

As before, after adding two more full-twists, we can eliminate  $\sigma_2^{-1}$ .

Go on with the above arguments, we can eliminate all the negative powers in  $w$  eventually.

The last statement in the proposition holds, since closed positive braids are fibred [13].  $\square$

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