

# Quantum Algorithms using the Curvelet Transform

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# Quantum algorithms

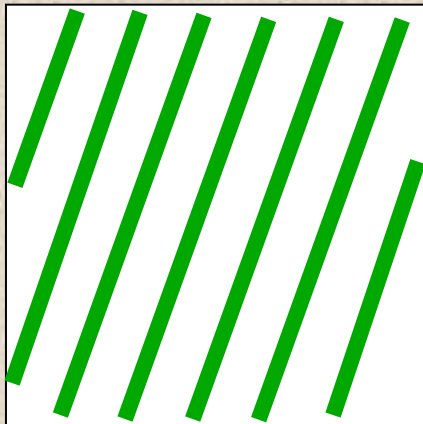
- Solving a hidden subgroup problem using the quantum Fourier transform
  - On the group  $\mathbf{Z}_M$  => Shor's algorithm
  - On the dihedral group => approximating the unique shortest vector in a lattice?
  - On the symmetric group => graph isomorphism?

What about other unitary transforms?

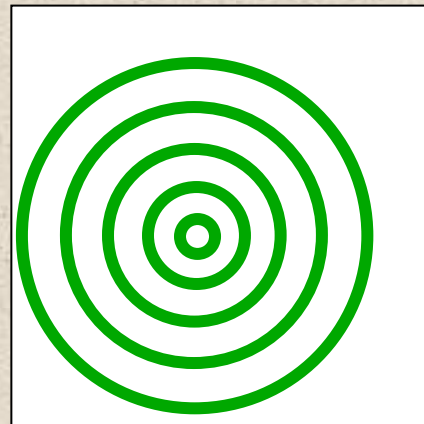
- The curvelet transform
  - A directional wavelet transform on  $\mathbf{R}^n$
  - Does this lead to interesting quantum algorithms?

# Beyond the hidden subgroup problem

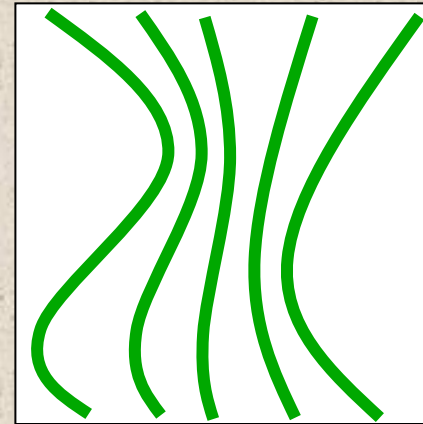
- Want to extract information about a function  $F$  that is promised to have a certain structure
  - Level sets of  $F$  are...



Flats in  $(\mathbf{Z}_M)^n$  :  
Fourier transform



Concentric spherical  
shells: curvelet  
transform



General smooth  
surfaces of dim.  $n-1$ :  
curvelet transform

# Summary of our results

- Finding the center of a ball in  $\mathbf{R}^n$  (approximately):

	Given:	Can succeed w/ probability:
Quantum:	1 quantum sample	$> \text{constant}$
Classical:	1 random sample	$< 2^{-\Omega(n)}$ (exp. small)

- Finding the center of a radial function on  $\mathbf{R}^n$  :
  - Given oracle access; assume function makes discrete steps; function can return values in any set

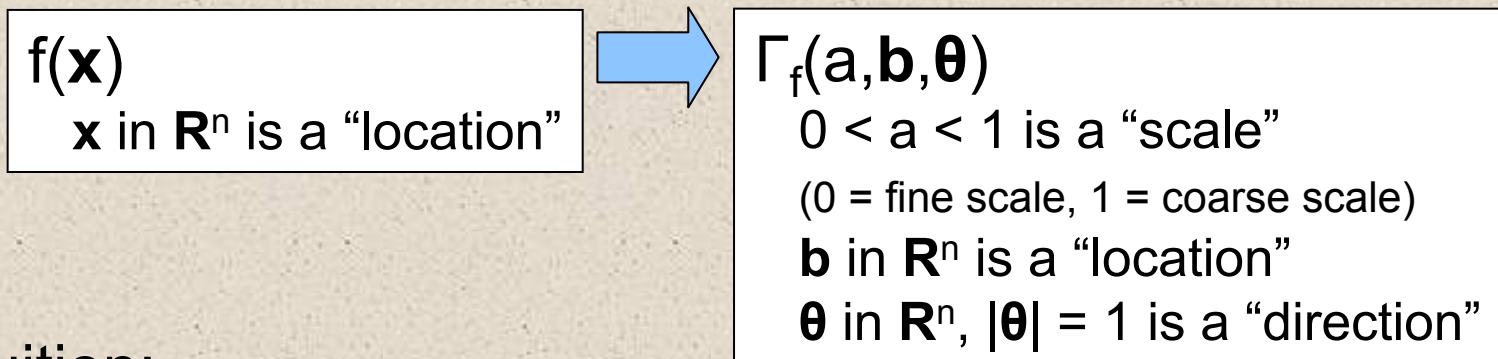
	To succeed w/ probability:	Need:
Quantum:	$> \text{constant}$	$< O(1)$ queries
Classical:	$> \text{constant}$	$> \Omega(n)$ queries

# Summary of our results

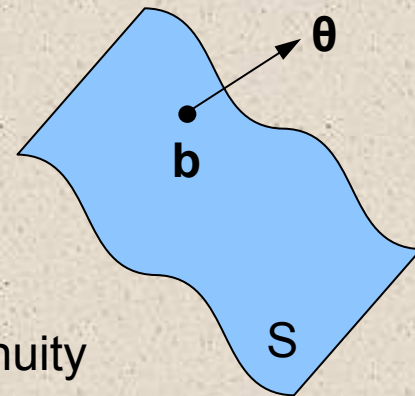
- Analysis of these algorithms:
  - We prove correctness of these algorithms in an idealized “continuous” model over  $\mathbf{R}^n$
  - Discrete analogues of these algorithms can be implemented efficiently (in quantum poly time)
  - We give heuristic arguments to suggest that the discrete algorithms are also correct

# What is the curvelet transform?

- A directional wavelet transform (Candes & Donoho, 1999, 2002)

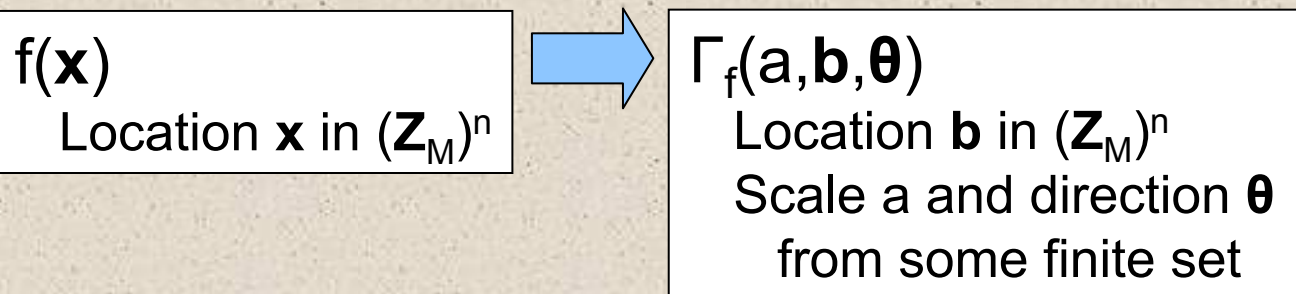
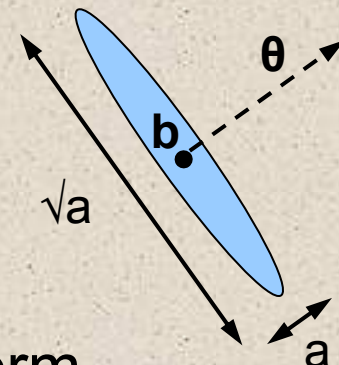


- Intuition:
  - Suppose  $f$  is discontinuous along a smooth surface  $S$  of dimension  $n-1$
  - Then  $\Gamma_f(a, \mathbf{b}, \boldsymbol{\theta})$  is “large” whenever:
    - $\mathbf{b}$  lies on  $S$  and  $\boldsymbol{\theta}$  is normal to  $S$  at  $\mathbf{b}$ 
      - A.k.a., the “wavefront set”
      - $a$  measures the “sharpness” of the discontinuity



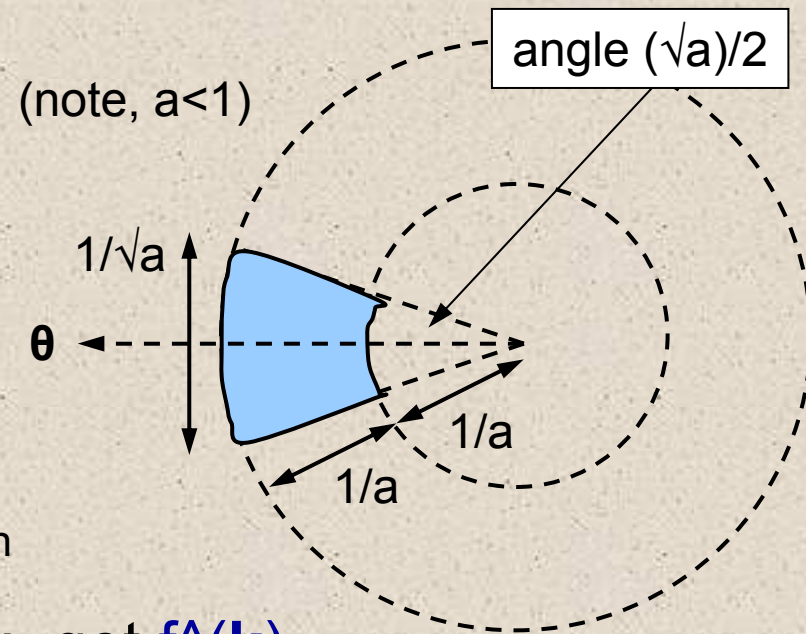
# What is the curvelet transform?

- Curvelet basis functions look like plane waves, but localized in small regions of space
  - High-frequency oscillations in the  $\theta$  direction, supported on a plate-like region centered at  $\mathbf{b}$  and orthogonal to  $\theta$
- Note, this is the *continuous* curvelet transform, there is also a finite discrete version



# The continuous curvelet transform

Window function  $\chi_{a\theta}$  is supported on a sector of frequency space:



- Given  $f(\mathbf{x})$ , a function on  $\mathbf{R}^n$
- Take the Fourier transform: get  $f^\wedge(\mathbf{k})$
- For each scale  $a$  and direction  $\theta$ , multiply by a smooth window function  $\chi_{a\theta}$ : get  $f^\wedge(\mathbf{k}) \chi_{a\theta}(\mathbf{k})$
- Take the inverse Fourier transform: get  $\Gamma_f(a, \mathbf{b}, \theta)$

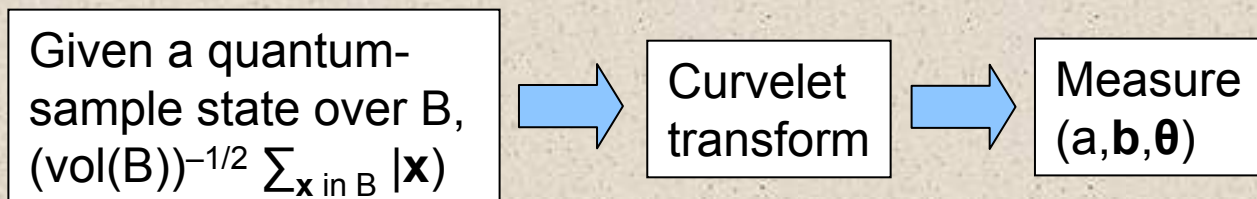
# A fast quantum curvelet transform

- Discrete curvelet transform
  - $\mathbf{b}$  in  $(\mathbf{Z}_M)^n$  ;  $\mathbf{a}, \boldsymbol{\theta}$  from some discrete set
  - Not unitary, but an isometric embedding:  $|\psi\rangle \rightarrow U(|\psi\rangle \text{ tensor } |0\rangle)$
- Quantum curvelet transform
  - Given input state + ancilla:  $\sum_{\mathbf{x}} f(\mathbf{x}) |\mathbf{x}\rangle |0, \mathbf{0}\rangle$
  - Apply QFT:  $\sum_{\mathbf{k}} f^\wedge(\mathbf{k}) |\mathbf{k}\rangle |0, \mathbf{0}\rangle$
  - Prepare superposition:  $\sum_{\mathbf{k}} f^\wedge(\mathbf{k}) |\mathbf{k}\rangle \sum_{\mathbf{a}, \boldsymbol{\theta}} \chi_{\mathbf{a}, \boldsymbol{\theta}}(\mathbf{k}) |\mathbf{a}, \boldsymbol{\theta}\rangle$
  - Apply QFT<sup>-1</sup>:  $\sum_{\mathbf{b}} \sum_{\mathbf{a}, \boldsymbol{\theta}} \Gamma_f(\mathbf{a}, \mathbf{b}, \boldsymbol{\theta}) |\mathbf{b}\rangle |\mathbf{a}, \boldsymbol{\theta}\rangle$
- Can be computed efficiently
  - Takes time  $\text{poly}(n, \log M)$
  - Provided that the  $\chi_{\mathbf{a}, \boldsymbol{\theta}}(\mathbf{k})$  are products of 1-D functions
    - E.g., using spherical coordinates

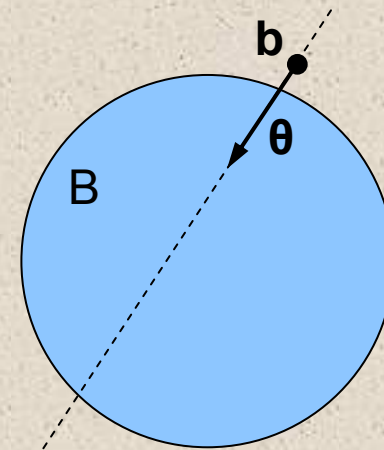
What can we do with this?

# Finding the center of a ball

- Let  $B$  be a ball in  $\mathbf{R}^n$



- Claim: with significant probability,
  - the scale  $a$  is small
  - the line  $\mathbf{b} + \lambda \boldsymbol{\theta}$  passes near the center of the ball
- To find the center:
  - Guess some point along the line, in the interval  $[\mathbf{b} - C\beta\boldsymbol{\theta}, \mathbf{b} + C\beta\boldsymbol{\theta}]$ , where  $C = \text{const.}$ ,  $\beta = \text{radius of the ball}$

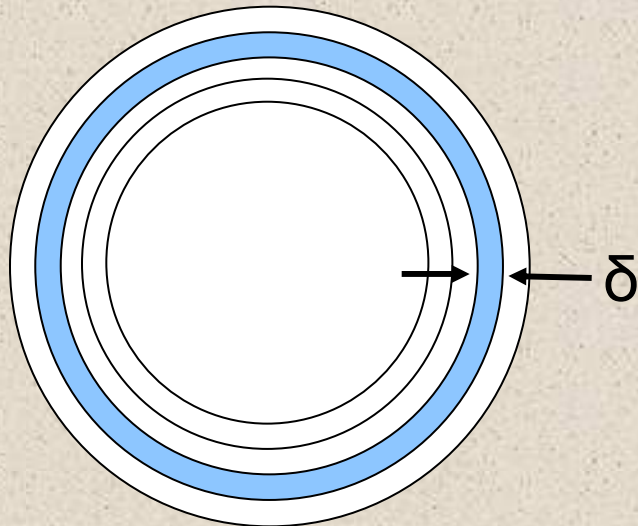


# Why is this interesting?

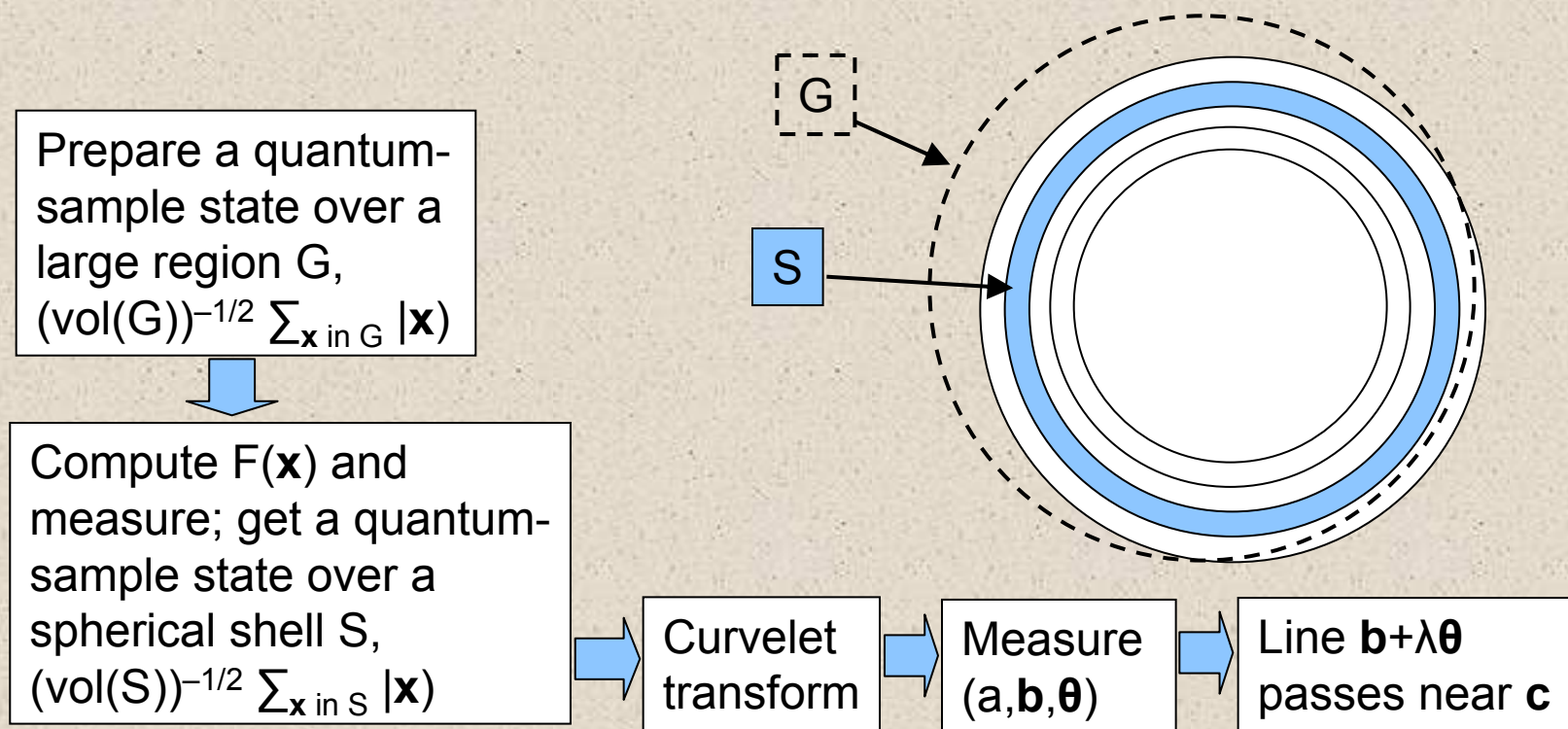
- Can get information from just **one** quantum-sample
  - “Single-shot quantum measurement”
  - For any constant  $\kappa < 1$ , the algorithm finds a point within distance  $\beta\kappa$  of the center, **with probability  $> \Omega(\kappa^3)$ , independent of  $n$**
  - Compare w/ classical sampling:
    - Pick a single random point in  $B$
    - We succeed **with probability  $\kappa^n$ , exponentially small in  $n$**
- Why? Because volume is concentrated near the surface of the ball
  - Bad for classical sampling, good for the curvelet transform!
  - But quantum advantage disappears if we have many samples
    - We can take several classical samples and average them

# Finding the center of a radial function

- Let  $F$  be a radial function on  $\mathbf{R}^n$ , centered at some unknown point  $\mathbf{c}$ 
  - We are given oracle access to  $F$
  - $F$  can return values in some arbitrary set
  - Assume that level sets of  $F$  are spherical shells
  - Let  $\delta$  = thickness of spherical shells
  - Let  $R$  = upper bound on distance from  $\mathbf{c}$  to the origin



# Finding the center of a radial function



- To find the center  $\mathbf{c}$ :
  - Do this twice, to find two lines  $L$  and  $L'$  that pass near  $\mathbf{c}$
  - Then return the point on  $L$  that lies nearest to  $L'$

# Why is this interesting?

- Claim: this algorithm finds the center “exactly,” when the shells are sufficiently thin
  - For any  $\mu$ , we can find a point within distance  $\mu$  of the center, provided that  $\delta < O(\mu^2/Rn^2)$
- This only requires  **$O(1)$  oracle queries**, independent of the dimension  $n$ 
  - Algorithm succeeds with probability  $> \Omega(1)$ , independent of  $n$
  - Any classical algorithm requires  **$\Omega(n)$  queries**
- Can also succeed when  $\delta$  is larger...
  - Run the algorithm iteratively to zoom in on the center
  - This works when  $\delta < O(\mu/n^2L^3)$ , where  $L = \lg \lg(R/\mu)$
  - Iterate  $O(L \lg L)$  times, use  $O(L \lg L)$  queries
    - Why  $\lg \lg(R/\mu)$ ? We make faster progress on the early iterations...

# Related work

- Quantum algorithm for estimating the gradient of a function on  $\mathbf{R}^n$  (Jordan, 2004)
  - Works when the function is smooth
  - Uses Fourier transform + phase kickback
    - Can be interpreted as a curvelet transform at a single location
- Quantum wavelet transforms (Hoyer, 1997; Fijany & Williams, 1998)
  - Can be implemented efficiently; any applications?
- Quantum algorithms for finding “hidden nonlinear structures” (Childs, Schulman & Vazirani, 2007)
  - Shifted subsets – use the Fourier transform
  - Hidden polynomials – use curvelets?
- Classical algs. for image processing, simulating wave propagation
  - Resolving the “wavefront set” (Candes & Donoho, 2002, 2003)
  - Nice results are known for general functions on  $\mathbf{R}^2$

# Proof ideas

- Curvelet transform of a radial function
  - Wlog, assume the object is centered at the origin
  - The probability of observing a fine scale element corresponds to the amount of power at high frequencies
    - $\Pr[a \leq \zeta] \approx \int_{|\mathbf{k}| \geq 1/(\lambda \zeta)} |\hat{f}(\mathbf{k})|^2 d\mathbf{k}$ , we get better accuracy when  $a$  is small
    - High-frequency components are due to the discontinuity of  $f$
  - The direction  $\boldsymbol{\theta}$  is uniformly distributed, and the location  $\mathbf{b}$  has expected value  $\mathbf{0}$
  - We can upper-bound the variance of  $\mathbf{b}$  in the directions orthogonal to  $\boldsymbol{\theta}$ 
    - Use Plancherel's theorem to go from spatial to frequency domain
    - Integrate using  $n$ -dimensional spherical coordinates
    - Behavior of Bessel functions  $J_\nu(z)$  in the transition regime  $z \approx \nu$
    - Lots of fun...
  - Hence the line  $\mathbf{b} + \lambda \boldsymbol{\theta}$  passes near the origin

# Conclusions

- The quantum curvelet transform
  - Can be computed efficiently, for a “nice” family of curvelets
- What does it do?
  - Suppose  $f$  is discontinuous along a smooth surface  $S$
  - Then  $|\Gamma_f|^2$  is large near the “wavefront set”:
    - Points  $\mathbf{b}$  that lie on  $S$ , and directions  $\boldsymbol{\theta}$  that are normal to  $S$
- One application: finding the center in  $\mathbf{R}^n$ 
  - Can find the center of a ball (approximately), using 1 quantum-sample
  - Can find the center of a radial function (exactly), using  $O(1)$  oracle queries
  - Open problems: discretization, different families of curvelets

# The big picture

- How can one use the quantum curvelet transform?
- What is the right framework?
  - Recall the hidden subgroup problem and the Fourier transform
  - Exponential speed-up over classical computation?
- When are curvelets useful?
  - Where does one get quantum states with “wavefront” features?
    - From quantum walks?
  - Think about more complicated geometric objects
    - Ellipsoids, polytopes?
    - Surfaces over finite fields?
- “Wavelet” techniques...
  - Construct basis functions with desirable properties
  - Work over spatial and frequency domains simultaneously

# Any questions?



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