CS 138, Homework 5

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Assume that $G = (V, E)$ is k-colorable ($|V| = \{1, ..., n\}$). There is a map $c : V \mapsto \{1, ..., k\}$ such that $c(i) \neq c(j)$ for any $(i, j) \in E$. For $i = 1, ..., k$, let $\mathbf{w}_i$ be the n-dimensional vector which has 0 in the last $n - k$ positions, $-\sqrt{\frac{k-1}{k}}$ in the ith position, and $\frac{1}{\sqrt{k(k-1)}}$ in the remaining positions. For any vertex i, let $\mathbf{v}_i = \mathbf{w}_{c(i)}$.

For each vertex $i \in V$,

$$\begin{aligned}
& \mathbf{v}_i \cdot \mathbf{v}_i \\
= \ & \mathbf{w}_{c(i)} \cdot \mathbf{w}_{c(i)} \\
= \ & \sum_{j=1}^{k} ((\mathbf{w}_{c(i)})_j)^2 \\
= \ & (k-1)\frac{1}{k(k-1)} + \frac{k-1}{k} \\
= \ & 1
\end{aligned}$$

For each $(i, j) \in E$, since $c(i) \neq c(j)$,

$$\begin{aligned}
& \mathbf{v}_i \cdot \mathbf{v}_j \\
= \ & \mathbf{w}_{c(i)} \cdot \mathbf{w}_{c(j)} \\
= \ & \sum_{l=1}^{k} (\mathbf{w}_{c(i)})_l (\mathbf{w}_{c(j)})_l \\
= \ & (k-2)\frac{1}{k(k-1)} - \frac{2}{k} \\
= \ & -\frac{1}{k-1}
\end{aligned}$$

Therefore, if G is k-colorable, then this vector program has a feasible solution.