CS 138, Homework 4

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May 15, 2009

1 14.1

Let x, y be an optimal solution for the fractional primal/dual problem. Let $\mathcal{C}$ be the collection of the sets picked by the modified algorithm. Since it contains the set cover obtained by Algorithm 14.1, $\mathcal{C}$ is a valid set cover. By the primal complementary slackness conditions, for any $S \in \mathcal{S}$, $x_S > 0 \Rightarrow \sum_{e:e\in S} y_e = c(S)$. Therefore, $c(\mathcal{C}) = \sum_{S:x_S>0} c(S) = \sum_{S:x_S>0} \sum_{e:e\in S} y_e \leq f \sum_{e:e\in S} y_e = f \cdot OPT$.

2 14.2

Since $\mathbf{E}[c(\mathcal{C})] = OPT_f$, by Markov's inequality, $\mathbf{Pr}[c(\mathcal{C}) \geq 10OPT] \leq 1/10$.

For each element $a \in U$, let I_a be the indicator random variable of the event where a is not covered by $\mathcal{C}$. $\mathbf{E}[I_a] = \mathbf{Pr}[a \text{ is not covered by } \mathcal{C}] \leq 1/e$ and Let $I = \sum_{a\in U} I_a$. By linearity of expectation,

$$\mathbf{E}[I] \leq \frac{n}{e}.$$

By Markov's inequality,

$$\mathbf{Pr}[I \geq n/2] \leq \frac{\mathbf{E}[I]}{n/2} \leq \frac{2}{e}$$

Therefore, with the probability at least $1 - 1/10 - 2/e \geq 1/10$, $\mathcal{C}$ covers at least half the elemtns at a cost of at most $10OPT$.

3 14.5

By Theorem 14.5, any extreme point solution (x_v) for the set of inequalites in LP (14.2) is half-integral. Find an extreme point solution (x_v). Given a coloring of G, without loss of generality, let red be the color such that $\sum_{(v\in RED)\wedge(x_v=1/2)} c(v) \geq \frac{1}{k} \sum_{x_v=1/2} c(v)$. Let $\sum_{x_v=1/2} c(v) = \alpha \cdot OPT_f$ some $0 \leq \alpha \leq 1$. Pick all vertices that are set to one, and also pick all non-red vertices that are set to half. Since there are no two adjacent red vertices, all edges are covered by the vertices we picked. The total cost is $\sum_{x_v=1} c(v) + \sum_{(v\notin RED)\wedge(x_v=1/2)} c(v) \leq (1-\alpha)OPT_f + 2\alpha(1-1/k)OPT_f \leq 1 + \alpha(1-2/k)OPT_f \leq (2 - \frac{2}{k})OPT$.

4 15.3

Consider a regular hypergraph, $G = (V, E)$, where

$$\begin{aligned} V &= \{v_1, ..., v_n\} \\ E &= \left\{(v_{i_1}, ..., v_{i_k}) : 1 \leq i_1 < ... < i_k \leq n\right\}. \end{aligned}$$

Let $U = E$, $\mathcal{S} = V$. For $e \in U$ and $v \in \mathcal{S}$, let $e \in V$ if and only if e is incident on v. Finally, let $c(v) = 1$ for all $v \in (S)$ to make an instance of the set cover problem.

For any feasible 0/1 solution, at least $n-k+1$ vertices must be picked, otherwise there is an edge incident on only unpicked vertices and it is not covered. Therefore, $OPT \geq n-k+1$. However, by letting $x_v = 1/k$ for all $v \in V$, we get a feasible fractional solution with the total cost n/k, implying $OPT_f \leq n/k$. Therefore, the integrality gap is at least $k(n-k+1)/n$, which converges to k as $n \to \infty$.