Ph225a **Problem Set #6 (Chapters III.1. & III.2.)** December 1, 2004
(due at Noon on December 10, 2004)

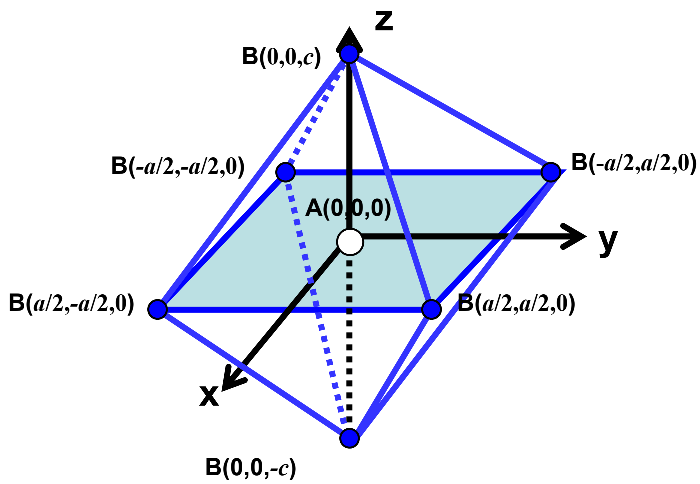
1. In the following we define R as the elements of the group $\mathcal{G}(R)$, h the order of the group, $D^{(\alpha)}$ an irreducible representation of $\mathcal{G}(R)$ with dimension l_α, $\chi^{(\alpha)}(R)$ the character of $D^{(\alpha)}$, and the operators $\hat{O}_R$ that transform functions $f(\vec{r})$ according to the relation $\hat{O}_R f(R\vec{r}) = f(\vec{r})$.

   (a) Using the character table of the cubic group O_h and the projection operators
   $$\hat{P}^{(\alpha)} = \frac{l_\alpha}{h}\sum_R \chi^{(\alpha)}(R)^* \hat{O}_R,$$
   show that the functions $(x^2 - y^2)$ and $(2z^2 - x^2 - y^2)$ are a set of basis functions for the irreducible representation Γ_{12}, and that the functions $z(x^2 - y^2)$, $y(z^2 - x^2)$ and $x(y^2 - z^2)$ are a set of basis functions for the irreducible representation Γ_{25}.

   (b) Find all the (2×2) matrices for the representation Γ_{12} by using the basis functions $(x^2 - y^2)$ and $(2z^2 - x^2 - y^2)$.

   (c) Find all the (3×3) matrices for the representation Γ'_{25} by using the basis functions xy, yz and zx.

2. Consider a molecule AB_6 where the A atom lies in the center of an octahedron, as shown in the figure, with the coordinates of the atoms specified, and a and c being positive constants.



   (a) Find the symmetry elements and their classes for the molecule, assuming that $c \neq a/\sqrt{2}$.

   (b) Identify the appropriate character table. Use the basis functions in the character table to construct two sets of (2×2) matrices which are irreducible representation of the symmetry group.

   (c) What additional symmetry operations can result from the condition $c = a/\sqrt{2}$? Find the corresponding symmetry group and the character table.

3. (a) Prove that $SO(4)$ is locally isomorphic to $SO(3)\otimes SO(3)$ by considering the algebra of the combinations $\frac{1}{2}(J^{ij} \pm \frac{1}{2}\,\varepsilon_{ijkl}\,J^{kl})$, where J^{ij} and ε_{ijkl} are the generators and the anti-symmetric constant tensor of $SO(4)$, respectively. {*Hint*: By defining $J_\pm^1 \equiv \left(J^{23} \pm J^{14}\right)/2$, $J_\pm^2 \equiv \left(J^{31} \pm J^{24}\right)/2$, and $J_\pm^3 \equiv \left(J^{12} \pm J^{34}\right)/2$, you may show that $\left[J_+^i, J_+^j\right] = i\varepsilon^{ijk} J_+^k$, $\left[J_-^i, J_-^j\right] = i\varepsilon^{ijk} J_-^k$, and $\left[J_+^i, J_-^j\right] = 0$.}

   (b) Prove that the tensor representation 5 and 5* in $SU(5)$ satisfy the following relations:
   $$5 \otimes 5 = 10 \oplus 15 \quad \text{and} \quad 5 \otimes 5^* = 1 \oplus 24.$$
   What does the product representation $5 \otimes 5$ in $SU(5)$ decompose into in the subgroup $SU(3)$?