<u>Problem Set #3</u> (Parts VII and VIII) March 1, 2010
(Due: March 17, 2010)

1. Bose-Einstein condensation temperature and the specific heat discontinuity

(**a**) Prove the statement in Part VII.2 that for a Bose system with a density of states of the form $\mathcal{N}(\varepsilon) = c_\alpha \varepsilon^{\alpha-1}$ where c_α and α are constants, there is a discontinuity in the specific heat ΔC given by

$$\Delta C = -\alpha^2 \frac{\zeta(\alpha)}{\zeta(\alpha-1)} N k_B$$

at the Bose-Einstein condensation (BEC) temperature T_c for $\alpha > 2$. Here $\zeta(\alpha)$ denotes the zeta function, k_B is the Boltzmann constant, and N is the total number of bosons in the system. Also show that the discontinuity disappears for $\alpha < 2$ by using the following identity

$$N - c_\alpha \Gamma(\alpha)\zeta(\alpha)(k_B T)^\alpha = c_\alpha \int_0^\infty d\varepsilon\, \varepsilon^{\alpha-1}\left[\frac{1}{e^{(\varepsilon-\mu)/k_BT}-1} - \frac{1}{e^{\varepsilon/k_BT}-1}\right]$$

at temperatures just above T_c. [*Hint*: You may simplify the above expression by using the approximation $(e^x - 1)^{-1} \approx x^{-1}$ for small x.]

(b) Next, consider a system of N identical bosons of mass m in an isotropic potential given by a specific interaction potential $V(\mathbf{r}) = \upsilon|\mathbf{r}|^\beta$, where υ and β are positive constants. Find the BEC transition temperature of the system and the temperature dependence of the depletion of the condensate.

2. Properties of the ground state for trapped bosons

The purpose of this problem is for you to apply two types of approximations that we have developed in the class notes to evaluate some properties of the ground state of trapped bosons.

(**a**) First we consider a system of identical N bosons of mass m interacting repulsively in an isotropic harmonic trap. Using the trial Gaussian wavefunction in EQ. (VII.94), find the kinetic energy per particle of a cloud of bosons in its ground state when (Na/a_{osc}) is large, where a denotes the scattering length and a_{osc} is the characteristic length of the isotropic harmonic potential.

(b) Next, let's consider the Thomas Fermi approximation where the kinetic energy is neglected. Assuming both spherically symmetric interaction and external trapped potentials (the latter not necessarily a harmonic potential), show that the interaction energy E_{int} and the potential energy E_{pot} satisfy the relation $(E_{int}/E_{pot}) = 2/3$. This result is in fact an expression of the virial theorem.

3. Magnetic critical fields of thin-film type-I superconductors

In this problem you are asked to apply Landau-Ginzburg theory to evaluate the magnetic critical fields of type-I superconductors. You'll find that the phase transition at the critical field of a type-I superconductor changes from first order to second order when the thickness of the sample becomes sufficiently small.

Consider a type-I superconducting slab of thickness $d < \xi(T)$ defined by the planes $z = \pm(d/2)$, where $\xi(T)$ is the Landau-Ginzburg coherence length. An external magnetic field $\mathbf{H} = H\,\hat{y}$ is applied parallel to the surface of the superconductor.

(a) If the film is sufficiently thin so that the superconducting order parameter ψ is approximately a constant within the superconductor, show that the application of the boundary condition $\mathbf{h}(z = \pm d/2) = H\,\hat{y}$ yields the following expression for the local field inside the superconductor $\mathbf{h}(|z| \le d/2)$:

$$h(z) = H\,\frac{\cosh(zF/\lambda)}{\cosh(\varepsilon F/\lambda)}, \qquad F \equiv \frac{|\psi|}{\psi_\infty}, \quad \varepsilon \equiv \frac{d}{\lambda}, \tag{1}$$

where λ is the magnetic penetration depth.

(b) Having obtained the spatial dependence of the local field, we are ready to find an expression for the superconducting order parameter by averaging the kinetic energy of the supercurrent over the thickness of the film and then minimizing the corresponding Landau-Ginzburg free energy density of the superconducting film relative to F^2. Following the aforementioned procedure, show that the normalized order parameter F satisfies the following relation:

$$F^2 = 1 + \frac{m^* \langle v_S^2 \rangle}{2\alpha}, \qquad \langle v_S^2 \rangle \equiv \frac{1}{d}\int_{-d/2}^{d/2} dz\,\left(v_S^2\right) = \frac{1}{2}\left[\frac{2e\lambda H}{m^* F \cosh(\varepsilon F/2)}\right]^2 \left[\frac{\sinh(\varepsilon F)}{\varepsilon F} - 1\right] \tag{2}$$

(c) From EQs. (1) and (2), show that the order parameter F and the external magnetic field H are related as follows:

$$\left(\frac{H}{H_c}\right)^2 = 4F^2\left(1 - F^2\right)\left\{\frac{\cosh^2(\varepsilon F/2)}{\left[\sinh(\varepsilon F)/(\varepsilon F)\right] - 1}\right\} \tag{3}$$

where H_c is the thermodynamic field.

(d) From EQ. (3), find $F(H)$ in two extreme cases $\varepsilon F \ll 1$ and $\varepsilon F \gg 1$.

(e) To derive the critical field H_T for the thin-film superconductor, we note that at the critical field the Gibbs free energy density of the superconductor, g_S, becomes equal to that of the normal state g_N, where

$$g_N = f_N - \left[H^2/(8\pi)\right], \tag{4}$$

and f_N is the Helmholtz free energy density in the normal state. Using the results derived thus far, show that the Gibbs free energy in the superconducting state satisfies the following relation:

$$\begin{aligned} g_S &= f_S - \frac{\langle h \rangle H}{4\pi} = \left[f_N - \frac{H_c^2}{8\pi}F^4 + \frac{\langle h^2 \rangle}{8\pi}\right] - \frac{\langle h \rangle H}{4\pi}, \\ &= f_N + \frac{H^2}{8\pi}\left[\frac{\sinh(\varepsilon F) + \varepsilon F}{\varepsilon F\left(1 + \cosh(\varepsilon F)\right)} - \frac{4}{\varepsilon F}\tanh\left(\frac{\varepsilon F}{2}\right)\right] - \frac{H_c^2}{8\pi}F^4. \end{aligned} \tag{5}$$

(f) Using the condition $g_S = g_N$ at $H = H_T$ and EQs. (3) and (5), show that the critical field H_T can be obtained by solving the following equation for the normalized order parameter $F(H)$:

$$Y_1(F) \equiv 1 + \frac{1}{6}\left(\frac{F^2}{1-F^2}\right) = \frac{1}{3}\frac{\varepsilon F\left[\cosh(\varepsilon F)-1\right]}{\sinh(\varepsilon F)-\varepsilon F} \equiv Y_2(F). \tag{6}$$

(g) Discuss why the phase transition at $H = H_T$ is first order if $\varepsilon > \sqrt{5}$ and is second order if $\varepsilon < \sqrt{5}$. (*Hint:* You may plot Y_1 and Y_2 in EQ. (6) as a function of F and also consider F as a function of ε.)

4. The gap equation of a three-dimensional BCS superconductors

We consider in the following the behavior of the superconducting gap Δ of a three-dimensional BCS superconductor under varying conditions.

(a) In the weak coupling limit where $\Delta(0) \equiv \Delta_0 \ll \omega_D$ and ω_D denotes the Debye frequency, show that the BCS gap equation in EQ. (VIII.157) becomes:

$$\ln\left(\frac{\Delta_0}{\Delta}\right) = 2\int_0^{\omega_D} \frac{d\xi}{\left(\xi^2+\Delta^2\right)^{1/2}} \frac{1}{\exp\left[\beta\left(\xi^2+\Delta^2\right)^{1/2}\right]+1}, \tag{7}$$

and that for $T \ll T_c$, EQ. (VIII.160) (which is reproduced below) can be verified:

$$\Delta(T) \approx \Delta_0 - \left(2\pi\Delta_0 k_B T\right)^{1/2} \exp\left(-\frac{\Delta_0}{k_B T}\right). \tag{8}$$

(b) Now consider another extreme condition near T_c, prove that from the gap equation in EQ. (VIII.156), the condition in EQ. (VIII.161) (which is reproduced below) is satisfied:

$$\Delta(T) \approx k_B T_c \pi \left(\frac{8}{7\zeta(3)}\right)^{1/2}\left(1-\frac{T}{T_c}\right)^{1/2} \approx 3.06 k_B T_c \left(1-\frac{T}{T_c}\right)^{1/2}. \tag{9}$$

(c) Next, we assume that the superconductor carries a uniform current so that the gap function takes the form $\Delta = |\Delta| e^{i2\mathbf{q}\cdot\mathbf{x}}$, where $|\mathbf{q}| \ll k_F$ and k_F is the Fermi momentum. Solve the corresponding Gorkov equations for the Green functions $\mathcal{G}$ and $\mathcal{F}^\dagger$, and discuss how the supercurrent depends on $|\mathbf{q}|$.

(d) Continuing from part **(c)**, find the self-consistent equation for the gap function $|\Delta|$.

(e) Continuing from part **(d)**, show that in the limit of $T \to 0$ the gap function $|\Delta|$ is independent of $|\mathbf{q}|$ for $|\mathbf{q}| < q_c \approx (\Delta_0 / v_F)$, whereas near T_c, the following relation is satisfied:

$$\left[\frac{\Delta(T)}{k_B T_c}\right]^2 \approx \frac{8\pi^2}{7\zeta(3)}\left(1-\frac{T}{T_c}\right) - \frac{2}{3}\left(\frac{k_F}{m k_B T_c}\right)^2 q^2. \tag{10}$$

5. Tunneling spectroscopy of a superconductor with *p*-wave pairing symmetry

In Part VIII.3 we have primarily focused on singlet superconductors whose orbital pair wavefunctions are of even angular momentum quantum numbers l = 0, 2, … due to symmetry consideration, and the corresponding pairing symmetries are therefore *s*-wave, *d*-wave, etc. Similar symmetry consideration suggests that for triplet superconductors (such as certain heavy-fermion superconductors discussed in Part VIII.4) the orbital pair wavefunctions must be associated with odd angular momentum quantum numbers l = 1, 3, … so that the pairing symmetries are *p*-wave, *f*-wave, etc. In this problem, you are asked to derive the quasiparticle tunneling spectrum of certain triplet superconductors by means of numerical analysis.

(a) Using the generalized BTK formalism derived in Part VIII.3, find the quasiparticle tunneling spectrum (*i.e.*, the tunneling conductance σ_S versus quasiparticle energy E) of a superconductor with a pure *p*-wave pairing potential $\Delta(\mathbf{k}) = \Delta_p \cos\theta_{\mathbf{k}}$, provided that quasiparticles are tunneling along the nodal direction (i.e. the k_y-axis) of the superconductor. Here $\cos\theta_{\mathbf{k}} \equiv \left(k_x / |\mathbf{k}|\right)$, $|\mathbf{k}|^2 = k_x^2 + k_y^2$, and we have assumed a two-dimensional superconductor.

(b) Suppose that the time-reversal symmetry is broken in the *p*-wave superconductor by the presence of a small imaginary component in the pairing potential so that $\Delta(\mathbf{k}) = \Delta_p\left[\cos\theta_{\mathbf{k}} - i\delta \sin\theta_{\mathbf{k}}\right]$ where $\delta < 1$. Find the quasiparticle spectrum of this ($p_x - ip_y$)-wave superconductor for quasiparticles tunneling along the k_y-direction. It is worth noting that the quasiparticle wavefunctions associated with the vortex state of a ($p_x - ip_y$)-wave pairing superconductor in fact exhibit non-abelian statistics, and are therefore of interest in the possible construction of qubits for quantum computation.