

Ph223c

Problem Set #3 (Chapters VII & VIII)

 May 18, 2006
 (due on June 2, 2006)

1. Gauge bosons and Higgs fields upon spontaneous symmetry breaking

 (a) We have seen in Part VII.4 that a gauged $U(1)$ theory:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D\phi^\dagger D\phi + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \quad (\text{P3.1.1})$$

upon spontaneous symmetry breaking leads to a Lagrangian in EQ. (VII.101), which we reproduce below

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[(v e)'^2 + 2(v\chi)e^2 + (\chi e)^2\right]A_\mu'^2 + \frac{1}{2}(\partial\chi)^2 - \mu^2\chi^2 - \mu\sqrt{\lambda}\chi^3 - \frac{\lambda}{4}\chi^4 + \frac{\mu^4}{4\lambda}. \quad (\text{P3.1.2})$$

 where $v \equiv \sqrt{\mu^2/\lambda}$, $A'_\mu \equiv A_\mu - e^{-1}\partial_\mu\theta$, θ is defined by $\phi = \rho e^{i\theta}$, and the fluctuation field χ is defined by $\rho = (1/\sqrt{2})(v + \chi)$. Given EQ. (P.3.1.2) and define $M \equiv v e$, show that the gauge boson propagator is:

$$\frac{-i}{k^2 - M^2 + i\alpha} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{M^2} \right), \quad (\alpha \rightarrow 0^+) \quad (\text{P3.1.2})$$

 and the χ propagator is:

$$\frac{i}{k^2 - 2\mu^2 + i\alpha}, \quad (\alpha \rightarrow 0^+) \quad (\text{P3.1.3})$$

(b) Find the Feynman rules for the interaction vertices of the aforementioned broken gauge theory.

 (c) Consider an $SU(5)$ gauge theory with a Higgs field ϕ transforming as the 5-dimensional representation $\phi^i, (i = 1, 2, \dots, 5)$. Show that a vacuum expectation value of ϕ reduces the $SU(5)$ symmetry to $SU(4)$.

2. The Chern-Simons term in (D+1)-dimensional space-time

 (a) Consider the mass dimensions of the Chern-Simons term and the Maxwell term in $(D+1)$ -dimensional space-time. Prove that at long distances the Maxwell term is irrelevant relative to the Chern-Simons term for $D = 2$, becomes comparable to the Chern-Simons term for $D = 3$, and dominates over the Chern-Simons term if $D = 4$.

 (b) In $(2+1)$ -dimensional space-time, calculate the propagator for a $U(1)$ -gauge field with the Chern-Simons term. Show that the Chern-Simons term gives rise to a topological mass and examine the behavior of the propagator in the long-wavelength limit.

 (c) For a Lagrangian L_0 with a conserved current J^μ in $(2+1)$ -dimensional space-time, we construct a Lagrangian L that involves the Chern-Simons term $\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$:

$$\mathcal{L} = \mathcal{L}_0 - \gamma \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + a_\mu J^\mu, \quad (\text{P3.2.1})$$

where a_μ denotes a gauge potential. Show that with the choice of the Lorentz gauge $\partial_\mu a^\mu = 0$, one can integrate out the gauge potential a in EQ. (P3.2.1) and obtain the following non-local Lagrangian:

$$\mathcal{L}_{\text{Hopf}} = \frac{1}{4\gamma} \left(J_\mu \frac{\varepsilon^{\mu\nu\lambda} \partial_\nu}{\partial^2} J_\lambda \right). \quad (\text{P3.2.2})$$

The Lagrangian $\mathcal{L}_{\text{Hopf}}$ in EQ. (P3.3.2) is known as *the Hopf term*, which, in the fractional quantum Hall fluids, is related to the quasiparticle interactions.

3. Dynamic and topological properties of the FQH states

(a) In Part VIII.3 we have primarily focused on the topological properties of the FQH states. Here we examine the dynamic properties of the Laughlin states by explicitly including the Maxwell term in the effective Chern-Simons theory:

$$\mathcal{L} = -\frac{m}{4\pi} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2g_1} E^2 - \frac{1}{2g_2} B^2, \quad (\text{P3.3.1})$$

where m is an odd integer, E and B represent the electric field and the magnetic field of the gauge field a_μ , respectively, and the spatial components of fields are given as follows:

$$E_i = \partial_0 a_i - \partial_i a_0 \quad \text{and} \quad B_i = \partial_1 a^2 - \partial_2 a^1. \quad (\text{P3.3.2})$$

Find the equation of motion for the collective fluctuations described by E and B .

(b) Now let's investigate the topological properties of the FQH states. As an example, consider a two-level hierarchical FQH state described by the K-matrix and charge vector \mathbf{q} :

$$\mathbf{K} \equiv \begin{pmatrix} p_1 & -1 \\ -1 & p_2 \end{pmatrix} \quad \text{and} \quad \mathbf{q}^T = (q_1, q_2) = (1, 0). \quad (\text{P3.3.3})$$

Show that the filling factor of the system can be expressed as follows:

$$\nu = \mathbf{q}^T \mathbf{K}^{-1} \mathbf{q}. \quad (\text{P3.3.4})$$

(c) A generic quasiparticle of the FQH state described in (b) can be labeled by (l_1, l_2) , which corresponds to a quasiparticle carrying l_1 units of the $a_{1\mu}$ charge and l_2 units of the $a_{2\mu}$ charge so that it is described by

$$(l_1 a_{1\mu} + l_2 a_{2\mu}) J^\mu, \quad (\text{P3.3.5})$$

where J^μ is a conserved current. Show that the statistics of the (l_1, l_2) quasiparticle is given by

$$\theta = \pi \mathbf{l}^T \mathbf{K}^{-1} \mathbf{l} = \frac{1}{p_1 p_2 - 1} (p_1 l_2^2 + p_2 l_1^2 + 2l_1 l_2), \quad (\text{P3.3.6})$$

(d) Prove that the electric charge of the (l_1, l_2) quasiparticle described in (c) is given by

$$Q_q = -e \mathbf{q}^T \mathbf{K}^{-1} \mathbf{l} = -e \left(\frac{p_2 l_1 + l_2}{p_1 p_2 - 1} \right). \quad (\text{P3.3.7})$$