Ph223c **<u>Problem Set #1</u> (Parts IV.5 & V.1 – V.5)** April 10, 2006
(Due: April 26, 2006)

1. Luttinger liquid theory for spin-polarized fermions

In Part IV.5 we have derived the Luttinger liquid theory for one-dimensional conductor of length *L* by considering the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$, where the non-interacting Hamiltonian is expressed in terms of the free left- and right-moving waves of spin-polarized fermions as follows:

$$\mathcal{H}_0 = v_F \int_0^L dx\, \psi_1^\dagger(x)\left(-i\partial_x - k_F\right)\psi_1(x) + v_F \int_0^L dx\, \psi_2^\dagger(x)\left(i\partial_x - k_F\right)\psi_2(x) \equiv v_F \int_0^L dx\, \psi^\dagger(x)\,\mathbf{A}_0\,\psi(x) ,$$

with $$\mathbf{A}_0 \equiv \begin{pmatrix} -i\partial_x - k_F & 0 \\ 0 & i\partial_x - k_F \end{pmatrix}, \quad \psi(x) \equiv \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix},$$

and we have restricted the interacting Hamiltonian to the "head-on collisions" of right- and left-moving particles

$$\mathcal{H}_I = g\int_0^L dx\, \rho_1(x)\,\rho_2(x) , \qquad \rho_{i\sigma}(x) \equiv \psi_{i\sigma}^\dagger(x)\psi_{i\sigma}(x) , \ (i = 1, 2),$$

with the coupling coefficient $g_{12}(x-y)$ between the left- and right-moving waves approximated by an on-site interaction $g_{12}(x-y) = g\,\delta(x-y)$.

Now consider restoring the coupling terms $g_{11}(x-y)\,\rho_1(x)\,\rho_1(y)$ and $g_{22}(x-y)\,\rho_2(x)\,\rho_2(y)$ to the interaction Hamiltonian so that we have

$$\mathcal{H}_I = \int_0^L dx \int_0^L dy \left[g_{11}(x-y)\,\rho_1(x)\,\rho_1(y) + g_{12}(x-y)\,\rho_1(x)\,\rho_2(y) + g_{22}(x-y)\,\rho_2(x)\,\rho_2(y) \right] .$$

Using the new interaction Hamiltonian given above and following the same line of derivation in Part IV.5, find the corresponding expectation values of the Green's functions.

2. Lehmann spectral representation for bosons

The non-condensate Green's function of an interacting bose system in the matrix form can be expressed in terms of the following Lehmann spectral relation

$$\mathbf{G}'(p) = \Omega \lim_{\eta\to 0+} \sum_j \left[\frac{\langle \mathbf{O}|\Phi(0)|\, j,\mathbf{p}\rangle\langle j,\mathbf{p}|\Phi^\dagger(0)|\mathbf{O}\rangle}{\omega - \left(H_{j\mathbf{p}} - H_{00}\right) + i\eta} - \frac{\langle \mathbf{O}|\Phi^\dagger(0)|\, j,-\mathbf{p}\rangle\langle j,-\mathbf{p}|\Phi(0)|\mathbf{O}\rangle}{\omega + \left(H_{j,-\mathbf{p}} - H_{00}\right) - i\eta} \right],$$

where $p = (\omega, \mathbf{p})$ denotes the four momentum, the matrix form of the non-condensate field operator in the Heisenberg picture and the corresponding Green's function are given by

$$\Phi_{\hat{H}}(x) = \begin{pmatrix} \varphi_{\hat{H}}(x) \\ \varphi_{\hat{H}}^\dagger(x) \end{pmatrix}, \qquad \mathbf{G}'(p) = \begin{pmatrix} G'(p) & G'_{12}(p) \\ G'_{21}(p) & G'(-p) \end{pmatrix};$$

the operator $\hat{H}$ satisfies the relation

$$\hat{H} = E - \mu\hat{N} = \left(E_0 - \mu N_0 + \sum_j V_j \right) + \sum_{\mathbf{k}\neq 0} \left(\varepsilon_{\mathbf{k}}^0 - \mu\right) a_{\mathbf{k}}^\dagger a_{\mathbf{k}} ;$$

the states $\left| j,\mathbf{p}\right\rangle$ refer to those eigen-states with $(N \pm 1)$ particles if there are N particles in the interacting ground state $\left|\mathbf{O}\right\rangle$, and applying the momentum operator $\hat{P}$ and the operator $\hat{H}$ to $\left| j,\mathbf{p}\right\rangle$ yields the relations:

$$\hat{P}\left| j,\mathbf{p}\right\rangle = p\left| j,\mathbf{p}\right\rangle, \qquad \hat{H}\left| j,\mathbf{p}\right\rangle = H_{jp}\left| j,\mathbf{p}\right\rangle.$$

(a) Following similar lines of derivation given in Part III.7 for fermions, prove the above Lehmann spectral relation for interacting bosons.

(b) Show that the residues of $G'(p)$ are real, and that the residues of $G'_{12}(p)$ and $G'_{21}(p)$ are complex conjugate of each other.

3. A Bose condensate with a finite momentum

In Part V we have discussed the formalism for interacting bosons with a zero momentum condensate. Suppose that we generalize the condition for the condensate, so that the Bose condensation occurs in a state with momentum **q**, which is equivalent to a condensate in uniform motion with a velocity **v** = **q**/*m*, *m* being the mass of the boson. In this case, the condensate lines in the Feynman diagrams include a factor $e^{\pm i\mathbf{q}\bullet\mathbf{x}}$.

(a) Find the corresponding non-condensate Green's functions $G'(p)$, $G'_{12}(p)$ and $G'_{21}(p)$.

(b) Find the depletion of the condensate ($n - n_0$) for a condensate of momentum **q**. Show that the depletion at $T = 0$ does not differ from that of a zero-momentum condensate as long as the condensate velocity **v** is smaller than the Landau critical velocity.

4. Weakly interacting bosons at finite temperatures

Consider a weakly interacting Bose gas with a uniform condensate moving with a constant velocity **v**. The ensemble average of the boson field operator in this case becomes $\left\langle \psi(x)\right\rangle \equiv \Psi(x) = n_0^{1/2} e^{im\mathbf{v}\bullet\mathbf{x}}$.

(a) Find the corresponding chemical potential μ.

(b) Find the thermal Green's functions $g'(p)$ and $g'_{21}(p)$ for the $\Psi(x)$ given above.

(c) At finite temperatures, show that the density of the condensate $n_0(T)$ is given by

$$n_0(T) = n_0(0) - \frac{m}{12u}(k_B T)^2 \left[1 - \frac{|\mathbf{v}|^2}{u^2}\right]^{-1}.$$

(d) Verify that the momentum density to first order in **v** satisfies the following relation

$$\Omega^{-1}\mathbf{P} = \left[\rho - \rho_n(T)\right]\mathbf{v},$$

where ρ_n is the normal fluid mass density given by the Landau expression

$$\rho_n(T) = \frac{1}{6\pi^2}\int_0^\infty dk\, k^4 \left[-\frac{\partial f(\varepsilon_k)}{\partial \varepsilon_k}\right],$$

and $f(\varepsilon_k)$ denotes the quasiparticle distribution function.