Ph223b **Problem Set #3 (Parts IV.1 – IV.4)** February 27, 2006
(Due: March 13, 2006)

1. Zero sound in Fermi liquids

The zero sound is a high-frequency sound of a Fermi liquid occurring at $T \to 0$ where $\omega\tau >> 1$, τ being the collision time between excitations. This acoustic mode is a collective excitation mode of a Fermi liquid, which differs from the first sound, also known as a low-frequency sound ($\omega\tau << 1$) determined by the compressibility of a liquid. To understand zero sound and its physical characteristics, we investigate the propagation of low-energy excitations in a Fermi liquid and show that zero sound is associated with the poles thus derived.

(a) We begin by considering the time variation of the state described by the wave function

$$\Psi_0(t) = \frac{1}{\Omega}\sum_{\mathbf{k},\alpha} \psi_{I\mathbf{k}\alpha}(t)\,\psi^{\dagger}_{I(\mathbf{k}+\boldsymbol{\kappa})\alpha}(t)\,\Phi_0(t),$$

where $\psi_{I\mathbf{k}\alpha}$ and $\psi^{\dagger}_{I\mathbf{k}\alpha}$ are the operators with momentum $\mathbf{k}$ and spin index α in the interaction picture, Φ_0 is the wave function of the ground state in the interaction picture, and Ω is the sample volume. For $\kappa = (\boldsymbol{\kappa}, \omega)$ and $|\boldsymbol{\kappa}| << k_F$, show that the probability amplitude $\left\langle \Psi_0^*(t)\,\Psi_0(t')\right\rangle$ can be expressed in terms the Green's function G and the vertex contribution Γ as follows:

$$\left\langle \Psi_0^*(t)\,\Psi_0(t')\right\rangle = \frac{1}{\Omega^2}\sum_{\mathbf{k}_1,\mathbf{k}_2;\alpha,\beta}\left\langle \psi_{H\mathbf{k}_2\beta}(t)\,\psi^{\dagger}_{H(\mathbf{k}_2-\boldsymbol{\kappa})\beta}(t)\,\psi_{H\mathbf{k}_1\alpha}(t')\,\psi^{\dagger}_{H(\mathbf{k}_1+\boldsymbol{\kappa})\alpha}(t')\right\rangle$$
$$\equiv \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, ia^2\,\Upsilon(\boldsymbol{\kappa},\omega),$$

where the subscript H refers to the Heisenberg representation, and the function $\Upsilon(\boldsymbol{\kappa},\omega)$ is defined by

$$\Upsilon(\boldsymbol{\kappa},\omega) \equiv \left[\frac{2}{ia^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3} G(\mathbf{k})\,G(\mathbf{k}+\boldsymbol{\kappa}) - \frac{1}{a^2}\int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{d^3\mathbf{k}_2}{(2\pi)^3} G(\mathbf{k}_1)\,G(\mathbf{k}_1+\boldsymbol{\kappa})\,\Gamma_{\alpha\beta,\alpha\beta}(k_1,k_2;\kappa)\,G(\mathbf{k}_2)\,G(\mathbf{k}_2-\boldsymbol{\kappa})\right]$$

(b) Noting that the dominant contributions to $\Upsilon(\boldsymbol{\kappa},\omega)$ arise from integration near the point $\left(|\mathbf{k}| = k_F, \varepsilon = 0\right)$ as seen in Part IV.2, verify the following relations:

$$\Upsilon(\boldsymbol{\kappa},\omega) = \left(\frac{k_F^2}{(2\pi)^3 v}\right)\int d\Omega_1 \left(\frac{\mathbf{v}_1\bullet\boldsymbol{\kappa}}{\omega - \mathbf{v}_1\bullet\boldsymbol{\kappa}}\right)\left[2 + \left(\frac{k_F^2}{(2\pi)^3 v}\right)\int d\Omega_2 \left(\frac{\mathbf{v}_2\bullet\boldsymbol{\kappa}}{\omega - \mathbf{v}_2\bullet\boldsymbol{\kappa}}\right) a^2\Gamma_{\alpha\beta,\alpha\beta}(k_1,k_2;\kappa)\right]$$
$$\equiv \left(\frac{k_F^2}{(2\pi)^3 v}\right)\int d\Omega_1\, \Upsilon_{1\alpha\alpha}(\boldsymbol{\kappa},\omega),$$

where Ω_i (i = 1,2) denote the solid angles of $(\mathbf{v}_i\bullet\boldsymbol{\kappa})$, and

$$(\omega - \mathbf{v}_1\bullet\boldsymbol{\kappa})\,\Upsilon_{1\alpha\gamma}(\boldsymbol{\kappa},\mathbf{n}_1,\omega) - \left(\frac{\mathbf{v}_1\bullet\boldsymbol{\kappa}\, k_F^2}{2(2\pi)^3 v}\right)\int d\Omega_2 \left[a^2\Gamma^{\omega}_{\gamma\eta,\alpha\xi}(\mathbf{n}_1,\mathbf{n}_2)\,\Upsilon_{1\eta\xi}(\boldsymbol{\kappa},\mathbf{n}_2,\omega)\right] = (\mathbf{v}_1\bullet\boldsymbol{\kappa})\,\delta_{\alpha\gamma},$$

with $\mathbf{n}_i$ denoting the normal vectors of $\mathbf{v}_i$.

(c) Using the results derived in **(b)**, show that the value of the probability amplitude $\left\langle \Psi_0^*(t)\,\Psi_0(t')\right\rangle$ as $t \to \infty$ is determined by the poles of the function $\Upsilon(\boldsymbol{\kappa},\omega)$ in the lower half-plane of the complex variable ω. These poles, which can be obtained by setting the right side of the last equation in **(b)** to 0, correspond to various branches of the zero-sound spectrum, and they are bosonic excitations of the Fermi liquid because they are associated with bilinear Fermion operators.

(d) To derive the zero-sound velocity c_0, we need to have an explicit form for the vertex contribution. Here we assume a vertex contribution due to two quasiparticle scattering by their Coulomb interaction $V(q)$ corrected to the lowest order in the polarization Π^0. Therefore, the pole of Γ in the plane of the complex variable ω is associated with the solution

$$1 = V(\mathbf{q})\,\Pi^{0R}\left(\mathbf{q},\,\Omega_{\mathbf{q}} - i\gamma_{\mathbf{q}}\right),$$

and the zero-sound velocity is given by $c_0 \equiv \Omega_{\mathbf{q}}/q$. Using the limiting values of Π^{0R} given in Part II.8 and assuming that $V(q)$ approaches a constant $V(0)$ as $q\to 0$, discuss why an undamped zero sound is only possible if $c_0 > v_F$, where v_F denotes the Fermi velocity. Also prove that explicit solutions to c_0 can be found in the weak and strong coupling limits (*i.e.*, for $0 < V(0) << (mk_F)^{-1}$ and $V(0) >> (mk_F)^{-1}$, respectively) as follows:

$$c_0 \approx v_F\left\{1 + 2\exp\left[-\frac{2\pi^2}{mk_F V(0)} - 2\right]\right\} \qquad \text{for } V(0) \ll \frac{1}{mk_F},$$

$$c_0 \approx v_F\left[\frac{mk_F V(0)}{3\pi^2}\right]^{1/2} \approx \left[\frac{k_F^3 V(0)}{3\pi^2 m}\right]^{1/2} \qquad \text{for } V(0) \gg \frac{1}{mk_F}.$$

In reality, we remark that the zero-sound velocity of the Fermi liquid for any coupling strength has been found to lie between the first-sound velocity $c_1 = v_F/\sqrt{3}$ and the Fermi velocity v_F.

2. Equation of motion of phonon field operators under electron-phonon interaction

Given the general relation $\partial O_H/\partial t = i\left[\mathcal{H}, O_H\right]$ for the Heisenberg operators O_H, we consider in this problem the Heisenberg phonon field φ_H for the electron-phonon interaction Hamiltonian $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_{\text{el}} + \mathcal{H}_{\text{ph}} + \gamma\int d^3\mathbf{x}\,\psi_{H\alpha}^\dagger(\mathbf{x})\psi_{H\alpha}(\mathbf{x})\varphi_H(\mathbf{x}),$$

where $\psi_{H\alpha}$ and $\psi_{H\alpha}^\dagger$ represent the electron Heisenberg operators, $\mathcal{H}_{\text{el}}$ and $\mathcal{H}_{\text{ph}}$ are the Hamiltonians for the electrons and phonons, respectively, and γ is the electron-phonon coupling coefficient.

(a) Prove that the Heisenberg phonon fields satisfy the following relations in the limit of an infinite Debye frequency ($\omega_D \to \infty$):

$$\left[\varphi_H(x),\varphi_H(x')\right]_{t=t'} = 0\ , \qquad \left[\varphi_H(x),\frac{\partial\varphi_H(x')}{\partial t'}\right]_{t=t'} = -i\nabla_{\mathbf{x}}^2\,\delta(\mathbf{x}-\mathbf{x}').$$

(b) From the relations given in **(a)**, derive the following equation of motion for the phonon fields:

$$\left[\nabla^2 - \frac{1}{u_0^2}\frac{\partial^2}{\partial t^2}\right]\varphi_H(x) = -\gamma\,\nabla_{\mathbf{x}}^2\left[\psi_{H\alpha}^\dagger(x)\psi_{H\alpha}(x)\right],$$

where u_0 is the speed of sound.

(c) Defining the exact phonon Green's function

$$iD(x-x') = \langle 0|T\left[\varphi_H(x)\varphi_H(x')\right]|0\rangle,$$

where $|0\rangle$ denotes the exact Heisenberg ground state of the coupled electron-phonon system, show that the following relation is satisfied:

$$\left[\nabla_{\mathbf{x}}^2 - \frac{1}{u_0^2}\frac{\partial^2}{\partial t^2}\right] iD(x-x') = -i\nabla_{\mathbf{x}}^2 \delta(\mathbf{x}-\mathbf{x}')\delta(t-t') - \gamma\nabla_{\mathbf{x}}^2 \langle 0|T\left[\psi_{H\alpha}^{\dagger}(x)\psi_{H\alpha}(x)\varphi_H(x')\right]|0\rangle .$$

(d) How are the relations given in **(a)** – **(c)** modified for a finite Debye frequency ω_D?

3. The Kohn effect

In this problem we investigate how the presence of electron-phonon coupling renormalizes the phonon spectra and gives rise to spectral anomalies at $|\mathbf{k}| = 2k_F$. This phenomenon has been empirically verified in many metallic systems, and is known as the Kohn effect.

(a) Consider the lowest-order proper self-energy correction $\Pi^*(q)$ to the phonon propagator due to electron-phonon coupling. Compute the phonon propagator explicitly under this lowest-order correction.

(b) Using the result in **(a)**, derive the following relations for the renormalized phonon frequency $\Omega_{\mathbf{q}}$ and the inverse lifetime $\delta_{\mathbf{q}}$

$$\Omega_{\mathbf{q}}^2 = \omega_{\mathbf{q}}^2\left[1 - 2\mathcal{N}(0)\gamma^2 g\left(\frac{q}{k_F}\right)\right], \qquad \delta_{\mathbf{q}} = \frac{\pi m\omega_{\mathbf{q}}^2\gamma^2\mathcal{N}(0)}{2qk_F}\theta(2k_F - q),$$

where the function $g(x)$ has been given in Part III.8, $\mathcal{N}(0)$ is the density of states at the Fermi level, and corrections of order of $(1/v_F)$ may be neglected for v_F measured in units of the sound speed. We note that $\left(\partial\Omega_q/\partial q\right)_{q=2k_F} \to \infty$, which is known as the Kohn effect. Moreover, $2\delta_{\mathbf{q}}$ corresponds to the ultrasonic attenuation constant in a pure metal, which can also be determined empirically.