Ph223a **<u>Problem Set #4</u> (Parts III.3 – III.6)** November 21, 2005
(Due: December 7, 2005)

1. γ-matrices

(a) Provide the details of why the γ-matrices must be at least (4 × 4) for (3+1)-dimensional spacetime.

(b) Prove that $\gamma^{\mu} \not{p} \not{q} \not{r} \gamma_{\mu} = -2 \not{r} \not{q} \not{p}$.

(c) Prove that $\mathrm{Tr}\left\{\gamma^5 \gamma^{\mu} \gamma^{\nu} \gamma^{\lambda} \gamma^{\sigma}\right\} = -4i\varepsilon^{\mu\nu\lambda\sigma}$, where $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

2. Dirac equation in (*D*+1)-dimensions (*D* = 1, 2, 3)

(a) Derive the Dirac equation for (1+1)- and (2+1)-dimensions, and show that the mass term in (2+1)-dimensions violates the parity and time-reversal symmetries.

(b) For (3+1)-dimensions, find the Dirac equation in the non-relativistic limit.

3. Spinor fields

In our canonical quantization of the Dirac field, we decompose $\Psi(x)$ into plane waves, with *u* spinors corresponding to positive energy particles (moving forward in time) and *v* spinors corresponding to negative energy particles (moving backward in time). In the rest frame we have the spinors given by

$$u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad v_1 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad v_2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

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Now consider a pure Lorentz boost with a momentum $\vec{p} = (0, 0, p_z)$. If the rest mass of each particle under consideration is *m*, find the Lorentz transformed *u* and *v* spinors by applying the corresponding Lorentz transformation matrix $S(\Lambda)$ to the independent spinor basis in the rest frame.

4. *S*-matrix and Coulomb scattering

Finding the cross section of an electron scattered by the Coulomb potential of another charged particle is not a problem only limited to particle physics; in condensed matter physics a standard approach to calculating the electrical resistivity of metals or semiconductors resulting from scattering of conduction electrons by charged impurities is to find the Coulomb scattering cross section computed essentially in the same way as that in particle physics, except that the Coulomb potential in the former contains a larger dielectric constant than the vacuum dielectric constant and also a modified spatial dependence (such as the Thomas-Fermi approximation), which account for the many-body screening effect in solids. In this problem you are asked to consider the Coulomb scattering of electrons in vacuum under two extreme conditions known as the Mott scattering and the Rutherford scattering.

(a) Consider the two-fermion Coulomb scattering process involving an electron and another spin-1/2 fermion *f* of charge (*Ze*) such that $e^-(\boldsymbol{p}_1)\, f(\boldsymbol{p}_2) \to e^-(\boldsymbol{p}_1')\, f(\boldsymbol{p}_2')$, where the initial momentum of electron (spin-1/2 fermion) is given by $\boldsymbol{p}_1$ ($\boldsymbol{p}_2$) and the final momentum electron (spin-1/2 fermion) is denoted by $\boldsymbol{p}'_1$ ($\boldsymbol{p}'_2$). Sketch the lowest order Feynman diagram for this QED process.

(b) Find the *S*-matrix for the process described in part (a) and the corresponding scattering matrix element between the initial and final states.

(c) Show that the spin-averaged scattering cross section for the electron in the limit of $m_f >> m_e$ (where m_f and m_e are the spin-1/2 fermion and electron masses, respectively) is given by the following *Mott formula*:

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2 Z^2}{4|\boldsymbol{p}|^2\,\beta^2\sin^4(\theta/2)}\left[1-\beta^2\sin^2\left(\frac{\theta}{2}\right)\right],$$

where θ is the scattering angle, $\boldsymbol{p}$ and β are respectively the momentum and velocity of the electron in the spin-1/2 fermion rest frame, and $\alpha \equiv e^2/(4\pi)\approx 1/137$ is the fine structure constant. It is straightforward to show that in the non-relativistic limit of the Mott scattering, we obtain the following celebrated Rutherford scattering formula:

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2 Z^2}{4\left(m_e v^2\right)^2\sin^4(\theta/2)}\,.$$