Ph223a **<u>Problem Set #3</u> (Parts II.7 & III.1 – III.2)** October 26, 2005
(due on November 16, 2005)

1. Space group of three-dimensional graphite

Graphite is a well known semimetal with large electronic and structural anisotropies, and the two-dimensional graphite sheets are in fact the building blocks of such interesting structures as carbon nanotubes and bucky balls. Figures P3.1(a) and 1(b) illustrate the side- and top-views of the regular three-dimensional graphite structure with the so-called *A-B-A-B* stacking. The dark solid lines in Fig. 1(a) delineate the unit cell of graphite, and it is clear that there are 4 atoms per unit cell. The lattice constants are a_0 = 0.2456 nm for the two-dimensional graphite sheet and c_0 = 0.6696 nm for the c-axis spacing. The dark symbols denote the "*a*"-atoms of the *A* and *B* layers, and the open symbols denote the "*b*"-atoms of the *A* and *B* layers. The space group of graphite is D_{6h}^4, where D_{6h} denotes the point group symmetry, and the superscript 4 denotes the index of the space group. It is clear from the structure of graphite that D_{6h}^4 is a non-symmorphic space group.

(a) Construct the character table for the symmetry group D_{6h}^4.

(b) Find the lattice modes of graphite.

(c) Identify the Raman-active, infrared-active, and silent modes.

(d) Which lattice modes can be excited if the incident light is polarized along the c-axis of graphite?

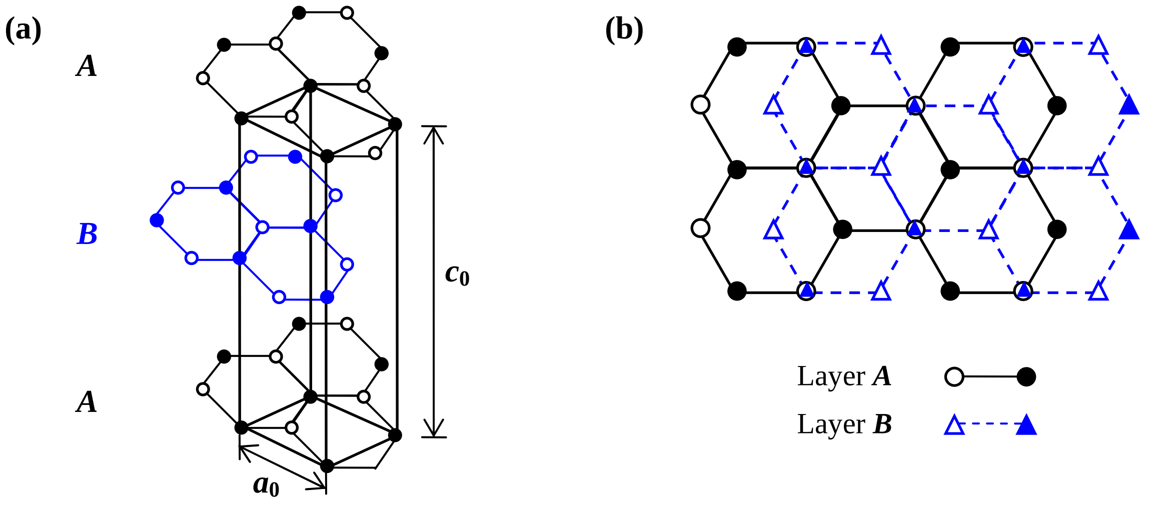

Fig. P3.1 Illustrations of the regular three-dimensional graphite structure with *A-B-A-B* stacking: **(a)** the side view, and **(b)** the top view. There are 4 atoms per unit cell, and the unit cell with lattice constants a_0 and c_0 is indicated by the thick solid lines.

2. The force law of a scalar field in (*D*+1)-dimensions

We have derived in Part III.2 the inverse-square force law for massless photons in a (3+1)-dimensional spacetime. Here we would like to investigate the force law in more generality.

(a) Find the force law mediated by a massless scalar field in (*D*+1)-dimensional spacetime.

(b) Find the long-range ($r \to \infty$) asymptotic behavior of the force law mediated by a massive scalar field in (*D*+1)-dimensional spacetime.

3. Noether's theorem for a system of *SO*(*N*) symmetry

As discussed in Part III.1, the Noether's theorem states that there is a conserved current associated with each generator of a continuous symmetry. In this problem we want to find the conserved currents of a specific system with continuous *SO*(*N*) symmetry and a Lagrangian density *L* given by the following:

$$L = \frac{1}{2}\left[\left(\partial \boldsymbol{\varphi}\right)^2 - m^2 \boldsymbol{\varphi}^2\right] - \frac{\lambda}{4}\left(\boldsymbol{\varphi}^2\right)^2 . \tag{P3.1}$$

Here m denotes the mass of the *N*-component scalar field $\boldsymbol{\varphi} = \left(\varphi_1, \cdots, \varphi_N\right)$ with φ_a $(a = 1, \cdots, N)$ transform like vectors, $\boldsymbol{\varphi}^2 \equiv \boldsymbol{\varphi} \bullet \boldsymbol{\varphi} = \varphi_a \varphi_a$ is invariant under *SO*(*N*) symmetry, and λ is a positive coupling constant. If we consider an infinitesimal change in the field so that

$$\delta\varphi_a(x) = \theta^A(x)\left(T^A\right)_{ab} \varphi_b(x) , \tag{P3.2}$$

where $\theta^A(x)$ denote parameters that depend on the 4-vector x, and T^A represent the $N(N-1)/2$ independent generators. In the special case of $\theta^A(x)$ = constant, we expect the action $S = \int d^4x\, L$ to be invariant under the infinitesimal transformation of the fields. In other words, $\delta S = 0$ for θ^A = constant. On the other hand, for arbitrary $\theta^A(x)$, δS does not necessarily vanish, and we expect that for arbitrary $\theta^A(x)$, δS takes on the following form

$$\delta S = \int d^4x \left[J^A(x)\right]^\mu \partial_\mu \theta^A(x) , \tag{P3.3}$$

so that $\delta S = 0$ for constant θ^A, and $\left[J^A(x)\right]^\mu$ denote conserved currents that are effectively the coefficients of $\partial_\mu \theta^A(x)$ according to EQ. (P3.3).

Using the Lagrangian given in EQ. (P3.1) and the transformation specified in EQ. (P3.2), verify the Noether's theorem and find an explicit expression for the conserved current $\left[J^A(x)\right]^\mu$.

4. Wick's theorem

Wick's theorem described in Part III.2 will become very important in our subsequent consideration of Feynman diagrams. The following two problems are examples for the application of Wick's theorem.

(a) Find the expectation value of $\langle x_i\, x_j\, x_k\, x_l\, x_m\, x_n\rangle$ using Wick's theorem, where $\langle x_i\, x_j\, x_k\, x_l\, x_m\, x_n\rangle$ is defined as:

$$\left\langle x_i x_j x_k x_l x_m x_n \right\rangle \equiv \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_N\, e^{-\frac{1}{2}x\bullet A\bullet x}\, x_i x_j x_k x_l x_m x_n}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_N\, e^{-\frac{1}{2}x\bullet A\bullet x}} , \tag{P3.4}$$

and A denotes a real symmetric $N \times N$ matrix. (Hint: your answer should contain 15 different terms.)

(b) Wick's theorem can also be applied to operators. As an example, consider the operator $O = aaa^\dagger a^\dagger$, where a and $a^\dagger$ are the annihilation and creation operators of a free boson system. Apply Wick's theorem and show that it does produce the correct result.