Ph223a **Problem Set #2 (Part II.1 -- II.5)** October 12, 2005
(Due: October 26,2005)

1. The spherical harmonics $Y_{lm}(\theta,\phi)$ $(m=-l,-l+1,\cdots,l-1,l)$ with even (odd) l constitute a basis for the irreducible representation $D^{(l+)}$ ($D^{(l-)}$) of the rotation-inversion group $O(3)$. A rotation through an angle α about the z-axis is represented by the irreducible representation $D^{(l)}$ with the character

$$\chi^{(l)}(\alpha)=\frac{\sin\left[(2l+1)\alpha/2\right]}{\sin(\alpha/2)}.$$

(a) Consider the states of an atom split by a perturbation with the cubic symmetry O_h. Show that the irreducible representation of $O(3)$ with l up to 3 are decomposed into the following representations of O_h:

State $s(l=0)=\Gamma_1$,
State $p(l=1)=\Gamma_{15}$,
State $d(l=2)=\Gamma'_{25}+\Gamma_{12}$,
State $f(l=3)=\Gamma_{15}+\Gamma_{25}+\Gamma'_2$.

(b) Find the proper linear combinations of spherical harmonics up to $l=2$ that are basis functions for the irreducible representations of the point group O_h. [*Hint:* Use the character table of O_h, and apply the projection operators to spherical harmonics.]

2. In Part II.2 we have seen a few locally isomorphic relations between some of the $SU(N)$ and $SO(N)$ groups for relatively small N values. In reality, these relations cannot be generalized to arbitrarily large N, as evidenced by the dimensions of $SU(N)$ and $SO(N)$ groups being (N^2-1) and $N(N-1)/2$, respectively. Although the locally isomorphic relations can be inspected from the compatibility of the dimensions of $SU(N)$ and $SO(N)$, the rigorous proof for such relations must involve explicit consideration of their Lie algebra.

(a) Consider specific generators $M^{ij}=i[\Gamma^i,\Gamma^j]/4$ for the $SO(N)$ group, where Γ^i and Γ^j $(1\le i,j\le N)$ satisfy the Clifford algebra $\left\{\Gamma^i,\Gamma^j\right\}=\Gamma^i\Gamma^j+\Gamma^j\Gamma^i=2\delta^{ij}$. These generators are known to produce the spinor representations of the $SO(N)$ group. Show that the generators thus defined satisfy the commutation relation $\left[M^{ij},M^{kl}\right]=i\left(\delta^{ik}M^{jl}-\delta^{jk}M^{il}+\delta^{jl}M^{ik}-\delta^{il}M^{jk}\right)$.

(b) Prove that the following relations hold for the tensor representations in $SU(5)$ and $O(3)$:

$$10\otimes 10=5^*\oplus 45^*\oplus 55^* \qquad \text{for } SU(5),$$

and $$3\otimes 3\otimes 3=7\oplus 5\oplus 5\oplus 3\oplus 3\oplus 3\oplus 1 \qquad \text{for } O(3).$$

3. Verify that the two Casimir invariants $C_1\equiv P^\mu P_\mu$ and $C_2\equiv W_\mu W^\mu$ commute with all 10 generators in the Poincaré group. Here P_μ denote the translation generators and W_μ represent the Pauli-Lubanski pseudo-vectors:

$$P_\mu=i\frac{\partial}{\partial x^\mu},\qquad W_\mu=-\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma,$$

and $M^{\mu\nu}$ are the generators of the Lorentz group.

4. Consider 4 particles confined to a two-dimensional plane. As discussed in Part II.4, if these particles are indistinguishable (distinguishable), the corresponding statistics is described by the pure (colored) braid group B_N (P_N).

(a) Assuming that the 4 particles are indistinguishable, find all possible braiding configurations that are consistent with the irreducible representation characterized by a phase $e^{4i\theta}$ if there are only 4 exchanges among the 4 particles.

The braiding configurations in part (a) can be considered as a class within the irreducible representation characterized by the phase $e^{4i\theta}$. Given that there are infinite irreducible representations in the pure braid group, we expect that there are also infinite classes associated with each irreducible representation. To get a better idea for how these infinite classes come about, let us again consider the same irreducible representation in part (a) but now allow 6 exchanges, which can be accomplished if there is both counter-clockwise and clockwise braiding, and the corresponding braiding configurations belong to a different class from the situation in part (a). Of course we can also ask for the braiding configurations under more exchanges and so on, leading to an infinite number of possibilities.

(b) Find all the braiding configurations associated with the irreducible representation characterized by the phase $e^{4i\theta}$ if there are 6 exchanges among the particles. Explain why for this irreducible representation an odd number of exchanges are not possible.

(c) Now let's assume that there are two different "colors" associated with the 4 particles, with 2 particles associated with each color. How do the braiding configurations modified in this case relative to part (a) if we consider the irreducible representation of a phase $e^{4i\theta}$ in this colored braid group?

5. In this problem we consider the crystals fields of an iron (Fe) impurity in two different locations of a honeycomb lattice. Suppose that a Fe-impurity is introduced into a honeycomb lattice, which is a two-dimensional approximation for a graphite structure. In the first case the Fe-impurity is placed in a substitutional location, as shown in Fig. 1(a), and in the second case the impurity is placed in an interstitial location, as shown in Fig. 1(b).

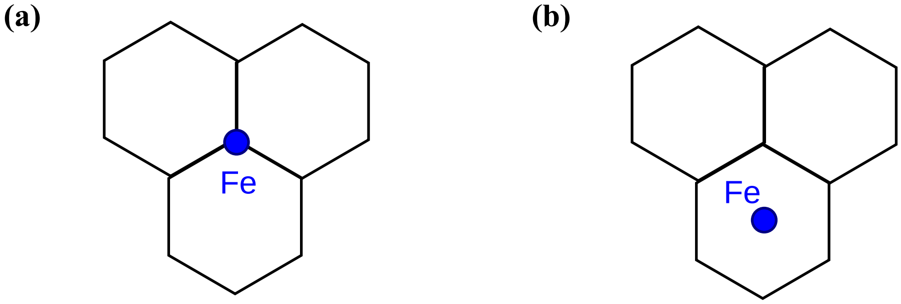

Fig. 1 (a) Substitutional and **(b)** interstitial locations of an iron impurity in a honeycomb lattice.

(a) Considering only the nearest neighbors of the Fe-impurity, find the difference in the crystal potential of the substitutional location and that of the interstitial location.

(b) For the interstitial case, express the crystal field in part (a) in terms of spherical harmonics for the lowest order terms with angular dependence.

(c) Using the proper point group symmetry and character table, give the crystal field splitting of the five-fold *d*-levels of the Fe-impurity in the crystal fields of part (a).