Ph223a **Problem Set #1 (Part I)** September 28, 2005

1. Second-quantizing the interaction Hamiltonian of fermions

Consider the field operator ψ defined by $\psi(\mathbf{r}) = \sum_{\mathbf{k}} \varphi_{\mathbf{k}}(\mathbf{r}) c_{\mathbf{k}}$, where $\varphi_{\mathbf{k}}(\mathbf{r})$ are a complete set of single particle states characterized by the quantum number **k**, **r** is the spatial coordinate, and $c_{\mathbf{k}}$ is the fermion annihilation operator.

Show that the second-quantized expression for the interaction Hamiltonian of a system of N interacting fermions, $\mathcal{H}' = \sum_{i,j\neq i}^{N} \mathcal{V}(\mathbf{r}_i, \mathbf{r}_j)/2$, is given by EQ. (I.28):

$$\mathcal{H}' = \frac{1}{2}\sum_{k,l,s,t} \langle kl|\mathcal{V}|st\rangle\, a_k^\dagger a_l^\dagger a_t a_s \equiv \frac{1}{2}\sum_{k,l,s,t}\left[\int d\mathbf{r}_1 d\mathbf{r}_2\, \varphi_k^*(\mathbf{r}_1)\varphi_l^*(\mathbf{r}_2)\mathcal{V}(\mathbf{r}_1,\mathbf{r}_2)\varphi_s(\mathbf{r}_1)\varphi_t(\mathbf{r}_2)\right] a_k^\dagger a_l^\dagger a_t a_s\,,$$

where $\mathcal{V}(\mathbf{r}_1, \mathbf{r}_2)$ describes the interaction potential between fermions at $\mathbf{r}_1$ and $\mathbf{r}_2$.

2. Second-quantization of the Zeeman interaction

As an example to second-quantize a Hamiltonian that involves spin operations, we consider the Zeeman interaction, which is defined as the interaction of an electron spin with an external magnetic field $\boldsymbol{H}$. Assuming a spatial variation in the applied magnetic field so that $\boldsymbol{H} = H_0 \cos(\mathbf{q}\bullet\mathbf{r})\hat{z}$ where $\hat{z}$ denotes the unit vector along the z-direction, show that the interaction Hamiltonian $\mathcal{H}' = \mu_B \sigma_z H_0 \cos(\mathbf{q}\bullet\mathbf{r})$ can be second-quantized into the following expression:

$$\mathcal{H}' = \frac{1}{2}\mu_B H_0 \sum_{\mathbf{k}}\left(c_{\mathbf{k+q},\uparrow}^\dagger c_{\mathbf{k},\uparrow} - c_{\mathbf{k+q},\downarrow}^\dagger c_{\mathbf{k},\downarrow} + c_{\mathbf{k-q},\uparrow}^\dagger c_{\mathbf{k},\uparrow} - c_{\mathbf{k-q},\downarrow}^\dagger c_{\mathbf{k},\downarrow}\right),$$

where μ_B is the Bohr magneton, σ_z is the Pauli matrix:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and ↑ and ↓ denote the spin states parallel and anti-parallel to the applied magnetic field, respectively.

3. The interaction picture for a spin-1/2 particle in a time-dependent magnetic field

When we consider the interaction picture for various physical systems, it occurs at times that the best choice for $\mathcal{H}_0$ in the total Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$ is not necessarily the entire familiar or constant part. As an example, consider the following Hamiltonian for a spin-1/2 particle with gyromagnetic ratio γ in a time-dependent magnetic field $\boldsymbol{B}(t) = b_1\left[\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right] + B_0\hat{z}$ rotating about the z-axis with frequency ω so that

$$\begin{aligned}\mathcal{H}(t) &= -\gamma\left[b_1\cos(\omega t)\mathbf{S}_x + b_1\sin(\omega t)\mathbf{S}_y + B_0\mathbf{S}_z\right] \\ &= \frac{\hbar}{2}\left[-\gamma b_1\cos(\omega t)\boldsymbol{\sigma}_x - \gamma b_1\sin(\omega t)\boldsymbol{\sigma}_y + \omega_L\boldsymbol{\sigma}_z\right],\end{aligned}$$

where $\boldsymbol{\sigma}_{x,y,z}$ represent the (2 × 2) Pauli matrices, and $\omega_L \equiv \gamma B_0$.

(a) A seemingly straightforward choice for $\mathcal{H}_0$ and $\mathcal{H}'$ in the above Hamiltonian is:

$$\mathcal{H}_0 = \frac{\hbar\omega_L}{2}\boldsymbol{\sigma}_z, \quad \mathcal{H}'(t) = -\frac{\hbar\gamma b_1}{2}\left[\cos(\omega t)\boldsymbol{\sigma}_x + \sin(\omega t)\boldsymbol{\sigma}_y\right] = -\frac{\hbar\gamma b_1}{2}\begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix}.$$

With the $\mathcal{H}_0$ and $\mathcal{H}'$ given above, find the corresponding $\mathcal{H}_I'(t)$ in the interaction picture, and show that it does not even commute with itself at different times.

(b) The situation in Part (a) is certainly not desirable for considering the time evolution of the system. On the other hand, a different choice of $\mathcal{H}_0$ and $\mathcal{H}'$ can lead to much simplified solutions. Specifically, consider

$$\mathcal{H}_0 = \frac{\hbar\omega}{2}\boldsymbol{\sigma}_z, \quad \mathcal{H}'(t) = \frac{\hbar}{2}\left[(\omega_L - \omega)\boldsymbol{\sigma}_z - \gamma b_1\left(\cos(\omega t)\boldsymbol{\sigma}_x + \sin(\omega t)\boldsymbol{\sigma}_y\right)\right].$$

Find the corresponding $\mathcal{H}_I'(t)$ and use it to solve for $U(t,0)$ explicitly.

4. Low-energy excitations of a weakly interacting bosonic system

Consider a system of a large number of weakly interacting bosons with a repulsive interaction potential $V_{\mathbf{k}}$ that is a function of the wave vector **k**. The total number of particles N in the system is given by $N = N_0 + \sum_{\mathbf{k}\neq 0} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$, where $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$ denote the creation and annihilation operators for the bosons, respectively, and $N_0 = a_0^\dagger a_0$. If we limit the boson-boson interaction to the first order, which is justified if the interaction is sufficiently weak, the Hamiltonian $\mathcal{H}$ of the system may be expressed by the following:

$$\mathcal{H} = \frac{1}{2}N^2 V_0 + \sum_{\mathbf{k}\neq 0}\left[\left(\varepsilon_{\mathbf{k}} + NV_{\mathbf{k}}\right)\left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}}\right) + NV_{\mathbf{k}}\left(a_{\mathbf{k}} a_{-\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger\right)\right] \equiv \frac{1}{2}N^2 V_0 + \sum_{\mathbf{k}\neq 0}\mathcal{H}_{\mathbf{k}},$$

where $\varepsilon_{\mathbf{k}}$ is the eigen-energy of the unperturbed boson system in the absence of the interaction potential $V_{\mathbf{k}}$.

(a) Explain the physical significance of each term in the above Hamiltonian of weakly interacting bosons.

(b) To understand the low-energy excitations of the interacting bosons, we need to diagonalize $\mathcal{H}$ and find the energy dispersion relation of the system. The strategy is to introduce new boson operators $\alpha_{\mathbf{k}}^\dagger$ and $\alpha_{\mathbf{k}}$ that are linear combinations of $a^\dagger$ and a:

$$\alpha_{\mathbf{k}} = u_{\mathbf{k}} a_{\mathbf{k}} - v_{\mathbf{k}} a_{-\mathbf{k}}^\dagger, \qquad \alpha_{\mathbf{k}}^\dagger = u_{\mathbf{k}} a_{\mathbf{k}}^\dagger - v_{\mathbf{k}} a_{-\mathbf{k}};$$

so that $\alpha_{\mathbf{k}}^\dagger$ and $\alpha_{\mathbf{k}}$ satisfy the following commutation relations:

$$\left[\alpha_{\mathbf{k}}^\dagger, \mathcal{H}_{\mathbf{k}}\right] = -\lambda_{\mathbf{k}}\alpha_{\mathbf{k}}^\dagger; \qquad \left[\alpha_{\mathbf{k}}, \mathcal{H}_{\mathbf{k}}\right] = \lambda_{\mathbf{k}}\alpha_{\mathbf{k}}; \qquad \left[\alpha_{\mathbf{k}}, \alpha_{\mathbf{k}'}^\dagger\right] = \delta_{\mathbf{k}\mathbf{k}'}.$$

Using the above relations, show that $\mathcal{H}_{\mathbf{k}}$ can be diagonalized if

$$u_{\mathbf{k}} = \cosh\chi_{\mathbf{k}}, \quad v_{\mathbf{k}} = \sinh\chi_{\mathbf{k}}, \quad \text{and } \tanh\left(2\chi_{\mathbf{k}}\right) = -\frac{NV_{\mathbf{k}}}{\varepsilon_{\mathbf{k}} + NV_{\mathbf{k}}}.$$

(c) The expectation value of the low-energy excitations with wave vector **k** can be expressed in terms of the number operator $a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ and the ground state Φ_0:

$$\left\langle a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \right\rangle_0 \equiv \left\langle \Phi_0 \middle| a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \middle| \Phi_0 \right\rangle,$$

and the ground state satisfies the condition $\alpha_{\mathbf{k}} \Phi_0 = 0$. Using the boson operators introduced in part (b), show that the expectation value of the low-energy excitations is given by

$$\left\langle a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \right\rangle_0 = v_{\mathbf{k}}^2 = \frac{1}{2}\left[\cosh\left(2\chi_{\mathbf{k}}\right) - 1\right].$$

(d) Assuming $V_{\mathbf{k}}$ = constant, sketch the expectation value $\left\langle a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \right\rangle_0$ as a function of k, the magnitude of the wave vector, and explain the physical significance of a diverging expectation value as $k \to 0$.