222D5220 **Problem Set #4 (Parts III.2, IV, V & VI)** September 6, 2007
(Due: September 24, 2007)

1. The gap equation of a three-dimensional BCS superconductor

We consider in the following the behavior of the superconducting gap Δ of a three-dimensional BCS superconductor under varying conditions.

(a) In the weak coupling limit where $\Delta(0) \equiv \Delta_0 \ll \omega_D$ and ω_D denotes the Debye frequency, show that the BCS gap equation in EQ. (III.157) becomes:

$$\ln\left(\frac{\Delta_0}{\Delta}\right) = 2\int_0^{\omega_D} \frac{d\xi}{\left(\xi^2+\Delta^2\right)^{1/2}} \frac{1}{\exp\left[\beta\left(\xi^2+\Delta^2\right)^{1/2}\right]+1}, \tag{1}$$

and that for $T \ll T_c$, EQ. (III.160) (which is reproduced below) can be verified:

$$\Delta(T) \approx \Delta_0 - \left(2\pi\Delta_0 k_B T\right)^{1/2} \exp\left(-\frac{\Delta_0}{k_B T}\right). \tag{2}$$

(b) Now consider another extreme condition near T_c, prove that from the gap equation in EQ. (III.156), the condition in EQ. (III.161) (which is reproduced below) is satisfied:

$$\Delta(T) \approx k_B T_c \pi \left(\frac{8}{7\zeta(3)}\right)^{1/2} \left(1-\frac{T}{T_c}\right)^{1/2} \approx 3.06 k_B T_c \left(1-\frac{T}{T_c}\right)^{1/2}. \tag{3}$$

(c) Next, we assume that the superconductor carries a uniform current so that the gap function takes the form $\Delta = |\Delta| e^{i2\mathbf{q}\cdot\mathbf{x}}$, where $|\mathbf{q}| \ll k_F$ and k_F is the Fermi momentum. Solve the corresponding Gorkov equations for the Green's functions $\mathcal{G}$ and $\mathcal{F}^\dagger$, and discuss how the supercurrent depends on $|\mathbf{q}|$.

(d) Continuing from part **(c)**, find the self-consistent equation for the gap function $|\Delta|$.

(e) Continuing from part **(d)**, show that in the limit of $T \to 0$ the gap function $|\Delta|$ is independent of $|\mathbf{q}|$ for $|\mathbf{q}| < q_c \approx \left(\Delta_0 / v_F\right)$, whereas near T_c, the following relation is satisfied:

$$\left[\frac{\Delta(T)}{k_B T_c}\right]^2 \approx \frac{8\pi^2}{7\zeta(3)}\left(1-\frac{T}{T_c}\right) - \frac{2}{3}\left(\frac{k_F}{m k_B T_c}\right)^2 q^2. \tag{4}$$

2. Gauge bosons and Higgs fields upon spontaneous symmetry breaking

(a) We have seen in Part IV.4 that a gauged $U(1)$ theory:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D\varphi^\dagger D\varphi + \mu^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2 \tag{5}$$

upon spontaneous symmetry breaking leads to a Lagrangian in EQ. (IV.101), which we reproduce below

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[(ve)^2 + 2(v\chi)e^2 + (\chi e)^2\right]A'^2_\mu + \frac{1}{2}(\partial\chi)^2 - \mu^2\chi^2 - \mu\sqrt{\lambda}\chi^3 - \frac{\lambda}{4}\chi^4 + \frac{\mu^4}{4\lambda}. \qquad (6)$$

where $v \equiv \sqrt{\mu^2/\lambda}$, $A'_\mu \equiv A_\mu - e^{-1}\partial_\mu\theta$, θ is defined by $\varphi = \rho e^{i\theta}$, and the fluctuation field χ is defined by $\rho = \left(1/\sqrt{2}\right)(v+\chi)$. Given EQ. (6) and define $M \equiv ve$, show that the gauge boson propagator is:

$$\frac{-i}{k^2 - M^2 + i\alpha}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right), \qquad \left(\alpha \to 0^+\right) \qquad (7)$$

and the χ propagator is:

$$\frac{i}{k^2 - 2\mu^2 + i\alpha}, \qquad \left(\alpha \to 0^+\right) \qquad (8)$$

(b) Find the Feynman rules for the interaction vertices of the aforementioned broken gauge theory.

(c) Consider an $SU(5)$ gauge theory with a Higgs field φ transforming as the 5-dimensional representation $\varphi^i, (i = 1, 2, \cdots, 5)$. Show that a vacuum expectation value of φ reduces the $SU(5)$ symmetry to $SU(4)$.

3. The one-band *t-J* model of the cuprates and the slave-boson formalism for pairing in the cuprates

We have described in Part V.2 the primary assumptions and key steps required to derive a microscopic model Hamiltonian for the strongly correlated cuprates from the Mott insulator limit where no double occupancies at the same site are allowed. Specifically, we may apply the Gutzwiller projection operators P_G to the one-band Hubbard Hamiltonian of the CuO_2 plane so that

$$\tilde{d}^\dagger_{i,\sigma} P_G \equiv \tilde{d}^\dagger_{i,\sigma}\left(1 - n_{i,-\sigma}\right), \qquad (V.52)$$

which is only non-trivial if the *i*-th site is not occupied. This procedure effectively reduces the Hilbert space associated with the original Hamiltonian of the Hubbard model to a no-double occupancy Hilbert space of a *t-J* Hamiltonian. Assuming hole doping into the CuO_2 plane so that there is strong *p-d* orbital hybridization and keeping to the lowest order in $(t^2/U) \equiv J$ in the nearly half-filling limit, we want to derive the effective one-band *t-J* Hamiltonian for the CuO_2 plane given by EQ. (V.51) from the one-band Hubbard model in EQ. (V.47). That is, we want to show that under the Gutzwiller projection, the one-band Hubbard model in EQ. (V.47) that consists of a kinetic term $\mathcal{H}_K$ and an interaction term $\mathcal{H}_U$:

$$\mathcal{H} = \sum_{\langle i,j\rangle,\sigma} t_{ij} d^\dagger_{i,\sigma} d_{j,\sigma} + U\sum_i n_{i,\uparrow} n_{j,\downarrow} \equiv \mathcal{H}_0 + \mathcal{H}_I \qquad (V.47)$$

can be projected out into the one-band *t-J* Hamiltonian given by EQ. (V.51) (which is reproduced below) in the half-filling limit:

$$\mathcal{H} = -t\sum_{\langle i,j\rangle}\Big[\left(1-n_{i,-\sigma}\right)\tilde{d}^\dagger_{i,\sigma}\tilde{d}_{j,\sigma}\left(1-n_{j,-\sigma}\right) + n_{i,-\sigma}\tilde{d}^\dagger_{i,\sigma}\tilde{d}_{j,\sigma}n_{j,-\sigma}$$
$$+\left(1-n_{i,-\sigma}\right)\tilde{d}^\dagger_{i,\sigma}\tilde{d}_{j,\sigma}n_{j,-\sigma} + n_{i,-\sigma}\tilde{d}^\dagger_{i,\sigma}\tilde{d}_{j,\sigma}\left(1-n_{j,-\sigma}\right)\Big] + \frac{U}{2}\sum_{i\sigma} n_{i,\sigma}n_{i,-\sigma}$$

$$= -t\sum_{\langle i,j\rangle}\left[\left(1-n_{i,-\sigma}\right)\tilde{d}^{\dagger}_{i,\sigma}\tilde{d}_{j,\sigma}\left(1-n_{j,-\sigma}\right)\right] + J\sum_{\langle i\,j\rangle}\left(\mathbf{S}_i\bullet\mathbf{S}_j - \frac{1}{4}n_{\tilde{d}\,i}n_{\tilde{d}\,j}\right) \text{ for no double occupancy.} \qquad \text{(V.51)}$$

where the spin operators $\mathbf{S}_i$ are given by $\mathbf{S}_i \equiv \sum_{\mu\nu}\frac{1}{2}d^{\dagger}_{i\mu}\boldsymbol{\sigma}_{\mu\nu}d_{i\nu}$ with $\boldsymbol{\sigma}_{\mu\nu}$ denoting the Pauli matrices, the charge operators are given by $n_{\tilde{d}\,i} \equiv \sum_{\sigma}\tilde{d}^{\dagger}_{i,\sigma}\tilde{d}_{i,\sigma}$, $\langle i, j\rangle$ refers to sum over nearest neighbors, $U \equiv \varepsilon^0_p - \varepsilon^0_d$, $t \simeq zt^2_{pd}\big/\left(\varepsilon^0_p - \varepsilon^r_d\right)$, z denotes the hole doping level and t_{pd} is the hopping coefficient in the two-band model, and the cross terms involving $\left(1-n_{i,-\sigma}\right)n_{j,-\sigma}$ and $n_{i,-\sigma}\left(1-n_{j,-\sigma}\right)$ can be removed from the second line of EQ. (V.51).

(a) Before making the Gutzwiller projection, let us consider a more general projection procedure for the one- band Hubbard model. For an arbitrary projection operator P_1 that satisfies the condition $P_1^2 = P_1$, if we define a second projection operator $P_2 \equiv 1 - P_1$, we find that $P_2^2 = P_2$ and $P_1P_2 = P_2P_1 = 0$. Next, the eigen-value problem $\mathcal{H}\psi = E\psi$ for the Hamiltonian $\mathcal{H}$ can be rewritten into the following:

$$\mathcal{H}\left(P_1+P_2\right)\psi = E\left(P_1+P_2\right)\psi\,, \qquad (9)$$

$$\Rightarrow \quad P_2\mathcal{H}\left(P_1+P_2\right)\psi = P_2E\left(P_1+P_2\right)\psi \quad \Rightarrow \quad P_2\psi = -\frac{1}{\left(P_2\mathcal{H}P_2 - E\right)}P_2\mathcal{H}P_1\psi\,. \qquad (10)$$

From $P_1\mathcal{H}\left(P_1+P_2\right)\psi = P_1E\left(P_1+P_2\right)\psi$ and the above expression for $P_2\psi$, show that we can define an effective Hamiltonian $\mathcal{H}_{\text{eff}}$ so that $\mathcal{H}_{\text{eff}}P_1\psi = EP_1\psi$, where

$$\mathcal{H}_{\text{eff}} \equiv P_1\mathcal{H}P_1 - P_1\mathcal{H}\frac{1}{\left(P_2\mathcal{H}P_2 - E\right)}P_2\mathcal{H}P_1\,. \qquad (11)$$

This effective Hamiltonian $\mathcal{H}_{\text{eff}}$ therefore reduces the Hilbert space of the original Hamiltonian $\mathcal{H}$ into that of the projection operator P_1.

(b) Now if we specify the projection operator P_1 in Part **(a)** as the Gutzwiller projection operator P_G where

$$P_G \equiv \prod_i\left(1-n_{i,\uparrow}n_{i,\downarrow}\right),$$

we find that
$$P_G\mathcal{H}_IP_G = 0 \quad \Rightarrow \quad P_G\mathcal{H}P_G = P_G\mathcal{H}_0P_G\,, \qquad (12)$$
$$P_G\mathcal{H}_IP_2 = 0 \quad \Rightarrow \quad P_G\mathcal{H}P_2 = P_G\mathcal{H}_0P_2\,, \qquad (13)$$
and
$$P_2\mathcal{H}_IP_G = 0 \quad \Rightarrow \quad P_2\mathcal{H}P_G = P_2\mathcal{H}_0P_G\,. \qquad (14)$$

Using the above identities and taking the half-filling limit, show that in the lowest order of (E/U), the effective Hamiltonian becomes

$$\mathcal{H}_{\text{eff}} \simeq P_G\mathcal{H}_0P_G - \left(P_G\mathcal{H}_0P_2\mathcal{H}_0P_G/U\right). \qquad (15)$$

(c) Now consider the numerator in the second term of $\mathcal{H}_{\text{eff}}$ and show that it can be expressed as

$$
\begin{aligned}
P_G\mathcal{H}_0P_2\mathcal{H}_0P_G &= \sum_{i,j,i',j',s,s'} t_{ij}t_{i'j'}P_G d^\dagger_{i,s}d_{j,s}P_2 d^\dagger_{j',s'}d_{i',s'}P_G \\
&\approx \sum_{\langle i,j\rangle,s,s'} \left|t_{ij}\right|^2 P_G d^\dagger_{i,s}d_{j,s}n_{j\uparrow}n_{j\downarrow}d^\dagger_{j,s'}d_{i,s'}P_G \\
&= \sum_{\langle i,j\rangle,s,s'} \left|t_{ij}\right|^2 P_G d^\dagger_{i,s}d_{i,s'}d_{j,s}d^\dagger_{j,s'}n_jP_G \ , \qquad (16)
\end{aligned}
$$

where we have used the condition $j = j'$ because of the constraints on P_G and P_2 and have ignored terms with $i \neq i'$.

(d) Given the results in Part **(c)** and the identity for the multiplication rule of the Pauli matrix elements $\sigma^a_{\alpha\beta}\sigma^a_{\mu\nu} = 2\delta_{\alpha\nu}\delta_{\beta\mu} - \delta_{\alpha\beta}\delta_{\mu\nu}$ with a = 1, 2, 3, show that

$$
P_G\mathcal{H}_0P_2\mathcal{H}_0P_G = \sum_{\langle i,j\rangle,s,s'} \left|t_{ij}\right|^2 P_G\left(\frac{n_in_j}{2} - 2\mathbf{S}_i \bullet \mathbf{S}_j\right)P_G \ , \qquad (17)
$$

where we have used the relations

$$
d^\dagger_{i,s}d_{i,s'} = \frac{1}{2}\delta_{ss'}n_i + S^a\sigma^a_{ss'} \qquad \text{and} \qquad d_{i,s}d^\dagger_{i,s'} = \left(1 - \frac{1}{2}n_i\right)\delta_{ss'} - S^a\sigma^a_{ss'} \ . \qquad (18)
$$

Finally, verify the validity of EQ. (V.51) using the results derived from Part **(a)** to Part **(d)**.

(e) In the mean-field limit the exchange interaction term in the one-band *t-J* model can be rewritten by means of the slave-boson formalism into the following form for pseudo fermions $\tilde{d}$ and $\tilde{d}^\dagger$:

$$
\mathcal{H}_I = J\sum_{\langle ij\rangle}\left(\mathbf{S}_i \bullet \mathbf{S}_j - \frac{1}{4}n_{\tilde{d}i}n_{\tilde{d}j}\right) \equiv -\frac{J}{2}\sum_{\langle ij\rangle,\sigma}\left(\tilde{d}^\dagger_{i,\sigma}\tilde{d}^\dagger_{j,-\sigma}\tilde{d}_{i,-\sigma}\tilde{d}_{j,\sigma} + \tilde{d}^\dagger_{i,\sigma}\tilde{d}^\dagger_{j,-\sigma}\tilde{d}_{j,-\sigma}\tilde{d}_{i,\sigma}\right), \qquad \text{(V.56)}
$$

The relation given in EQ. (V.56) is essential for "finding attraction from repulsion" in the CuO_2 plane. That is, the repulsive antiferromagnetic exchange interaction can lead to an effective Cooper pairing due to a special "quantum choreography" of the pseudo fermions imposed by the slave-boson formalism. Prove that EQ. (V.56) is indeed correct.

4. The Chern-Simons term in (D+1)-dimensional space-time

(a) Consider the mass dimensions of the Chern-Simons term and the Maxwell term in (D+1)-dimensional space-time. Prove that at long distances the Maxwell term is irrelevant relative to the Chern-Simons term for D = 2, becomes comparable to the Chern-Simons term for D = 3, and dominates over the Chern-Simons term if D = 4.

(b) In (2+1)-dimensional space-time, calculate the propagator for a U(1)-gauge field with the Chern-Simons term. Show that the Chern-Simons term gives rise to a topological mass and examine the behavior of the propagator in the long-wavelength limit.

(c) For a Lagrangian L_0 with a conserved current J^μ in (2+1)-dimensional space-time, we construct a Lagrangian L that involves the Chern-Simons term $\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$:

$$\mathcal{L} = \mathcal{L}_0 - \gamma \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + a_\mu J^\mu , \tag{19}$$

where a_μ denotes a gauge potential. Show that with the choice of the Lorentz gauge $\partial_\mu a^\mu = 0$, one can integrate out the gauge potential a in EQ. (19) and obtain the following non-local Lagrangian:

$$\mathcal{L}_{\text{Hopf}} = \frac{1}{4\gamma}\left(J_\mu \frac{\varepsilon^{\mu\nu\lambda}\partial_\nu}{\partial^2} J_\lambda \right). \tag{20}$$

The Lagrangian $\mathcal{L}_{\text{Hopf}}$ in EQ. (20) is known as *the Hopf term*, which, in the fractional quantum Hall fluids, is related to the quasiparticle interactions.

5. Dynamic and topological properties of the FQH states*

(a) In Part VI.4 we have primarily focused on the topological properties of the FQH states. Here we examine the dynamic properties of the Laughlin states by explicitly including the Maxwell term in the effective Chern-Simons theory:

$$\mathcal{L} = -\frac{m}{4\pi} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2g_1} E^2 - \frac{1}{2g_2} B^2 , \tag{21}$$

where m is an odd integer, E and B represent the electric field and the magnetic field of the gauge field a_μ, respectively, and the spatial components of fields are given as follows:

$$E_i = \partial_0 a_i - \partial_i a_0 \qquad \text{and} \qquad B_3 = \partial_1 a^2 - \partial_2 a^1 . \tag{22}$$

Find the equation of motion for the collective fluctuations described by E and B.

(b) Now let's investigate the topological properties of the FQH states. As an example, consider a two-level hierarchical FQH state described by the K-matrix and charge vector $\mathbf{q}$:

$$\mathbf{K} \equiv \begin{pmatrix} p_1 & -1 \\ -1 & p_2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{q}^{\mathrm{T}} = (q_1, q_2) = (1, 0) . \tag{23}$$

Show that the filling factor of the system can be expressed as follows:

$$\nu = \mathbf{q}^{\mathrm{T}} \mathbf{K}^{-1} \mathbf{q} . \tag{24}$$

(c) A generic quasiparticle of the FQH state described in **(b)** can be labeled by (l_1, l_2), which corresponds to a quasiparticle carrying l_1 units of the $a_{1\mu}$ charge and l_2 units of the $a_{2\mu}$ charge so that it is described by

$$\left(l_1 a_{1\mu} + l_2 a_{2\mu} \right) J^\mu , \tag{25}$$

where J^μ is a conserved current. Show that the statistics of the (l_1, l_2) quasiparticle is given by

$$\theta = \pi\, \boldsymbol{l}^{\mathrm{T}} \mathbf{K}^{-1} \boldsymbol{l} = \frac{1}{p_1 p_2 - 1}\left(p_1 l_2^2 + p_2 l_1^2 + 2 l_1 l_2 \right), \tag{26}$$

(d) Prove that the electric charge of the (l_1, l_2) quasiparticle described in **(c)** is given by

$$Q_q = -e\, \mathbf{q}^{\mathrm{T}} \mathbf{K}^{-1} \boldsymbol{l} = -e\left(\frac{p_2 l_1 + l_2}{p_1 p_2 - 1} \right). \tag{27}$$

[*Optional problem]