222D5220 **Problem Set #3 (Parts II.8 – II.10, III.1)** August 30, 2007
(Due: September 6, 2007)

1. Plasma oscillations in an electron gas

We have shown in Part II.9 that the linear response function of the density of an electron gas to an external perturbation is the retarded polarization propagator $\Pi^R(\mathbf{q},\omega)$, and that the poles of $\Pi^R(\mathbf{q},\omega)$ are associated with the excitation energies of collective modes in the interacting electron gas. In this problem we want to investigate a specific type of collective excitations of the degenerate electron gas known as the plasma oscillations. We shall restrict our consideration in the following to the ring diagrams.

(a) In the random phase approximation (RPA), the dielectric constant associated with the linear response of the density perturbation to an external impulsive potential $\varphi^{\text{ex}}(\mathbf{x},t)=e^{i\mathbf{q}\cdot\mathbf{x}}\varphi_0\,\delta(t)$ is given by $\varepsilon_r^R(\mathbf{q},\omega)=1-V(\mathbf{q})\Pi^{0R}(\mathbf{q},\omega)$, where $\Pi^{0R}(\mathbf{q},\omega)$ is the lowest order retarded polarization propagator. For $\omega=\Omega_p-i\gamma_p$ representing the poles so that $V(\mathbf{q})\Pi^{0R}(\mathbf{q},\Omega_p-i\gamma_p)=1$, verify that in the limit of very small damping of the collective modes $\gamma_p\ll\Omega_p$ the following conditions are satisfied:

$$V(\mathbf{q})\,\mathrm{Re}\left\{\Pi^{0R}(\mathbf{q},\Omega_p)\right\}=V(\mathbf{q})\,\mathrm{Re}\left\{\Pi^{0}(\mathbf{q},\Omega_p)\right\}=1,$$

$$\gamma_{\mathbf{q}}=\mathrm{Im}\left\{\Pi^{0R}(\mathbf{q},\Omega_p)\right\}\left[\left.\frac{\partial\,\mathrm{Re}\left\{\Pi^{0R}(\mathbf{q},\omega)\right\}}{\partial\omega}\right|_{\Omega_p}\right]^{-1}$$

$$=\mathrm{sgn}(\Omega_p)\,\mathrm{Im}\left\{\Pi^{0}(\mathbf{q},\Omega_p)\right\}\left[\left.\frac{\partial\,\mathrm{Re}\left\{\Pi^{0}(\mathbf{q},\omega)\right\}}{\partial\omega}\right|_{\Omega_p}\right]^{-1}.$$

(b) Using the results derived for $\Pi^0(\mathbf{q},\omega)$ in the limit of a fixed frequency ω and for $|\mathbf{q}|\equiv q\to 0$, show that for a three-dimensional spherical Fermi surface, the dielectric constant associated with the RPA is real and is given by

$$\lim_{q\to 0}\varepsilon_r^R(\mathbf{q},\omega)=1-\frac{\left(4\pi ne^2/m\right)}{\omega^2}\equiv 1-\left(\frac{\omega_p}{\omega}\right)^2.$$

Here n and m denotes the carrier density and mass, respectively, and ω_p is the plasma frequency.

(c) From the expression for $\mathrm{Re}\{\Pi^0(\mathbf{q},\omega)\}$ in EQ. (II.524), show that in the small q limit

$$\mathrm{Re}\left\{\Pi^{0R}(\mathbf{q},\omega)\right\}=\frac{k_F^3q^2}{3\pi^2m\omega^2}\left[1+\frac{3}{5}\left(\frac{k_Fq}{m\omega}\right)^2+\cdots\right],$$

where k_F is the Fermi momentum, and the dispersion relation for the plasma oscillation is given by

$$\Omega_q=\pm\omega_p\left[1+\frac{9}{10}\left(\frac{q^2\varepsilon_F}{6\pi ne^2}\right)+\cdots\right]\equiv\pm\omega_p\left[1+\frac{9}{10}\left(\frac{q}{q_{TF}}\right)^2+\cdots\right],$$

where q_{TF} is known as the Thomas-Fermi wave number. We remark that in the RPA, $\mathrm{Im}\{\Pi^0(\mathbf{q},\Omega_q)\}=0$ if $|\Omega_q|>(qk_F/m)+(q^2/2m)$, which implies that these plasma collective modes at long wavelengths are undamped in the lowest-order approximation. In reality, it can be shown that plasma oscillations are damped at all wavelengths if higher-order corrections are included.

2. Linear response, fluctuation-dissipation theorem & correlation functions

We have seen in Part II.9 that the general theory of linear response can be applied to the collective excitations of a system in response to external perturbation and also to the fluctuation-dissipation theorem. To investigate these issues further, we define a generalized retarded Green's function $\mathcal{G}^R$ and a generalized time-ordered Green's function $\mathcal{G}$ for operators A and B as follows:

$$\mathcal{G}^R(t) = -i\,\theta(t)\left\langle\left[A(t), B(0)\right]\right\rangle, \quad \mathcal{G}(t) = -i\left\langle T\left[A(t)B(0)\right]\right\rangle.$$

For a system with a Hamiltonian $\mathcal{H}$ and a complete set of eigen-states $\{|n\rangle\}$ such that $\mathcal{H}|n\rangle = \varepsilon_n|n\rangle$, the thermal average $\langle A(t)B(0)\rangle$ is given by

$$\left\langle A(t)B(0)\right\rangle = Z^{-1}\mathrm{Tr}\left\{e^{-\beta\mathcal{H}}e^{i\mathcal{H}t}Ae^{-i\mathcal{H}t}B\right\} = Z^{-1}\sum_n\left\langle n\right|e^{-\beta\mathcal{H}}e^{i\mathcal{H}t}Ae^{-i\mathcal{H}t}B\left|n\right\rangle,$$

where Z is the partition function.

(a) If we denote the Fourier transform of $\langle A(t)B(0)\rangle$ by $J_1(\omega)$, which is also known as the spectral density function associated with the time-ordered Green's function $\langle A(t)B(0)\rangle$, show that the Fourier transforms of the equivalent retarded Green's function $\mathcal{G}^R$ and the time-ordered Green's function $\mathcal{G}$ at $T = \beta^{-1}$ have the following forms (with $\eta = 0^+$):

$$\mathcal{G}^R(\omega) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\left(1-e^{-\beta\omega'}\right)\frac{J_1(\omega')}{\omega-\omega'+i\eta}, \text{ and } \mathcal{G}(\omega) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}J_1(\omega')\left[\frac{1}{\omega-\omega'+i\eta}-\frac{e^{-\beta\omega'}}{\omega-\omega'-i\eta}\right].$$

(b) If A and B are hermitian conjugates so that $J_1(\omega)$ is real, verify that

$$\mathrm{Im}\left\{\mathcal{G}^R(\omega)\right\} = -\frac{1}{2}\left(1-e^{-\beta\omega}\right)J_1(\omega) = \tanh\left(\frac{1}{2}\beta\omega\right)\mathrm{Im}\left\{\mathcal{G}(\omega)\right\},$$

$$\Rightarrow \quad J_1(\omega\neq 0) = -\frac{2}{1-e^{-\beta\omega}}\mathrm{Im}\left\{\mathcal{G}^R(\omega)\right\}.$$

The last line is a form of the fluctuation-dissipation theorem. In other words, the spectral response of the system (*i.e.* fluctuations) to an external perturbation at $t > 0$ gives rise to dissipation manifested by the imaginary part of the retarded Green's function.

3. Magnetic susceptibility in the generalized Hartree-Fock approximation

We have introduced the transverse magnetic susceptibility χ^{-+} in Part II.9 as the linear response function of a spin system to an external magnetic field. Here we want to evaluate χ^{-+} explicitly under the generalized Hartree-Fock approximation, which involves summing over the ladder diagrams containing repeated interactions of electron and hole lines, as shown in Fig. II.9.1. Thus, for

$$\chi^{-+}(\mathbf{x}-\mathbf{x}', t-t') = i\theta(t-t')\left\langle\left[\sigma^-(\mathbf{x},t), \sigma^+(\mathbf{x}',t')\right]\right\rangle = \sum_{\mathbf{p},\mathbf{q}}e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\chi^{-+}(\mathbf{p},\mathbf{q};t-t')$$

$$= \sum_{\mathbf{p},\mathbf{q}}e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}i\theta(t-t')\left\langle\left[a^\dagger_{\mathbf{p}+\mathbf{q}\downarrow}(t-t')\,a_{\mathbf{p}\uparrow}(t-t'), \sigma^+(0,0)\right]\right\rangle,$$

where $\sigma^+(\mathbf{x}) = \sum_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}\sum_{\mathbf{p}}a^\dagger_{\mathbf{p}+\mathbf{q}\uparrow}a_{\mathbf{p}\downarrow}$ and $\sigma^-(\mathbf{x}) = \sum_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}\sum_{\mathbf{p}}a^\dagger_{\mathbf{p}+\mathbf{q}\downarrow}a_{\mathbf{p}\uparrow}$, the equation of motion for $\chi^{-+}(\mathbf{p},\mathbf{q};t)$ under a Hamiltonian $\mathcal{H}$ becomes:

$$i\frac{\partial}{\partial t}\chi^{-+}(\mathbf{p},\mathbf{q};t)=-\delta(t)\left\langle\left[a^{\dagger}_{\mathbf{p+q}\downarrow}a_{\mathbf{p}\uparrow},\sigma^{+}(0,0)\right]\right\rangle+i\theta(t)\left\langle\left[\left[a^{\dagger}_{\mathbf{p+q}\downarrow}(t)\,a_{\mathbf{p}\uparrow}(t),\mathcal{H}\right],\sigma^{+}(0,0)\right]\right\rangle.$$

(a) Now let's consider a model Hamiltonian (known as the Hubbard model)

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I=\left(\sum_{\mathbf{p}\sigma}\omega_{\mathbf{p}}a^{\dagger}_{\mathbf{p}\sigma}a_{\mathbf{p}\sigma}\right)+\left(\frac{U}{N}\sum_{\mathbf{pp'q}}a^{\dagger}_{\mathbf{p+q}\uparrow}a_{\mathbf{p}\uparrow}a^{\dagger}_{\mathbf{p'-q}\downarrow}a_{\mathbf{p'}\downarrow}\right),$$

where U represents an on-site repulsion potential. Under the generalized Hartree-Fock approximation where one sums over all Wick contractions and using the expectation value

$$\left\langle a^{\dagger}_{\mathbf{p}\alpha}a_{\mathbf{p'}\beta}\right\rangle=\delta_{\mathbf{pp'}}\delta_{\alpha\beta}\,f_{\mathbf{p}\alpha},$$

show that the following relation holds:

$$\left[a^{\dagger}_{\mathbf{p+q}\downarrow}(t)\,a_{\mathbf{p}\uparrow}(t),\mathcal{H}\right]=-\left(\omega_{\mathbf{p+q}}-\omega_{\mathbf{p}}\right)a^{\dagger}_{\mathbf{p+q}\downarrow}a_{\mathbf{p}\uparrow}+\frac{U}{N}\sum_{\mathbf{p'}}\left\{\left(f_{\mathbf{p}\uparrow}-f_{\mathbf{p+q}\downarrow}\right)a^{\dagger}_{\mathbf{p+p'+q}\downarrow}a_{\mathbf{p+p'}\uparrow}+\left(f_{\mathbf{p'}\downarrow}-f_{\mathbf{p'}\uparrow}\right)a^{\dagger}_{\mathbf{p+q}\downarrow}a_{\mathbf{p}\uparrow}\right\}.$$

(b) From the result in part (a) and the definitions

$$\chi(\omega)=\int_{-\infty}^{\infty}dt\,\chi(t)\,e^{i\omega t},\qquad \chi(\mathbf{q})=\sum_{\mathbf{p}}\chi(\mathbf{p},\mathbf{q}),\qquad \tilde{\omega}_{\mathbf{p}\sigma}\equiv\omega_{\mathbf{p}}-\frac{U}{N}\sum_{\mathbf{p'}}f_{\mathbf{p'},-\sigma}$$

verify that the Fourier transform of the transverse magnetic susceptibility satisfies the following relation:

$$\frac{1}{N}\chi^{-+}(\mathbf{q},\omega)=\frac{\Gamma^{-+}(\mathbf{q},\omega)}{1-U\Gamma^{-+}(\mathbf{q},\omega)},\ \text{where}\ \ \Gamma^{-+}(\mathbf{q},\omega)=\frac{1}{N}\sum_{\mathbf{p}}\frac{f_{\mathbf{p}\uparrow}-f_{\mathbf{p+q}\downarrow}}{\omega-\left(\tilde{\omega}_{\mathbf{p}\downarrow}-\tilde{\omega}_{\mathbf{p+q}\uparrow}\right)+i\eta}.$$

We note that $\chi^{-+}(\mathbf{q},\omega)$ given above is the general expression for the frequency- and momentum-dependent magnetic susceptibility of an interacting electron gas, whereas the function $\Gamma^{-+}(\mathbf{q},\omega)$ is the same unperturbed particle-hole polarization propagator as appeared in the Coulomb interaction. It is worth noting that for $\omega = 0$ and $\mathbf{q} = 0$, $\Gamma^{-+}(\mathbf{q},\omega)$ reduces to the well known Pauli susceptibility, which is proportional to the density of states at the Fermi level.

4. Equation of motion of phonon field operators under electron-phonon interaction

Given the general relation $\partial O_H/\partial t=i\left[\mathcal{H},O_H\right]$ for the Heisenberg operators O_H, we consider in this problem the Heisenberg phonon field φ_H for the electron-phonon interaction Hamiltonian $\mathcal{H}$:

$$\mathcal{H}=\mathcal{H}_{\text{el}}+\mathcal{H}_{\text{ph}}+\gamma\int d^3\mathbf{x}\,\psi^{\dagger}_{H\alpha}(\mathbf{x})\psi_{H\alpha}(\mathbf{x})\,\varphi_H(\mathbf{x}),$$

where $\psi_{H\alpha}$ and $\psi^{\dagger}_{H\alpha}$ represent the electron Heisenberg operators, $\mathcal{H}_{\text{el}}$ and $\mathcal{H}_{\text{ph}}$ are the Hamiltonians for the electrons and phonons, respectively, and γ is the electron-phonon coupling coefficient.

(a) Prove that the Heisenberg phonon fields satisfy the following relations in the limit of an infinite Debye frequency ($\omega_D \to \infty$):

$$\left[\varphi_H(x),\varphi_H(x')\right]_{t=t'}=0\ ,\qquad \left[\varphi_H(x),\frac{\partial\varphi_H(x')}{\partial t'}\right]_{t=t'}=-i\nabla^2_{\mathbf{x}}\,\delta(\mathbf{x}-\mathbf{x}')\,.$$

(b) From the relations given in **(a)**, derive the following equation of motion for the phonon fields:

$$\left[\nabla^2-\frac{1}{u_0^2}\frac{\partial^2}{\partial t^2}\right]\varphi_H(x)=-\gamma\,\nabla^2_{\mathbf{x}}\left[\psi^\dagger_{H\alpha}(x)\psi_{H\alpha}(x)\right],$$

where u_0 is the speed of sound.

(c) Defining the exact phonon Green's function

$$iD(x-x')=\langle 0|T\left[\varphi_H(x)\varphi_H(x')\right]|0\rangle\,,$$

where $|0\rangle$ denotes the exact Heisenberg ground state of the coupled electron-phonon system, show that the following relation is satisfied:

$$\left[\nabla^2_{\mathbf{x}}-\frac{1}{u_0^2}\frac{\partial^2}{\partial t^2}\right]iD(x-x')=-i\nabla^2_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')-\gamma\,\nabla^2_{\mathbf{x}}\langle 0|T\left[\psi^\dagger_{H\alpha}(x)\psi_{H\alpha}(x)\varphi_H(x')\right]|0\rangle\,.$$

(d) How are the relations given in **(a)** – **(c)** modified for a finite Debye frequency ω_D?

5. Magnetic critical fields of thin-film type-I superconductors

In this problem you are asked to apply Landau-Ginzburg theory to evaluate the magnetic critical fields of type-I superconductors. You'll find that the phase transition at the critical field of a type-I superconductor changes from first order to second order when the thickness of the sample becomes sufficiently small.

Consider a type-I superconducting slab of thickness $d<\xi(T)$ defined by the planes $z=\pm(d/2)$, where $\xi(T)$ is the Landau-Ginzburg coherence length. An external magnetic field $\mathbf{H}=H\,\hat{y}$ is applied parallel to the surface of the superconductor.

(a) If the film is sufficiently thin so that the superconducting order parameter ψ is approximately a constant within the superconductor, show that the application of the boundary condition $\mathbf{h}(z=\pm d/2)=H\,\hat{y}$ yields the following expression for the local field inside the superconductor $\mathbf{h}(|z|\le d/2)$:

$$h(z)=H\,\frac{\cosh(zF/\lambda)}{\cosh(\varepsilon F/\lambda)}\,,\qquad F\equiv\frac{|\psi|}{\psi_\infty}\,,\quad \varepsilon\equiv\frac{d}{\lambda}\,, \tag{1}$$

where λ is the magnetic penetration depth.

(b) Having obtained the spatial dependence of the local field, we are ready to find an expression for the superconducting order parameter by averaging the kinetic energy of the supercurrent over the thickness of the film and then minimizing the corresponding Landau-Ginzburg free energy density of the superconducting film relative to F^2. Following the aforementioned procedure, show that the normalized order parameter F satisfies the following relation:

$$F^2 = 1 + \frac{m^* \left\langle v_S^2 \right\rangle}{2\alpha}, \qquad \left\langle v_S^2 \right\rangle \equiv \frac{1}{d}\int_{-d/2}^{d/2} dz \left(v_S^2 \right) = \frac{1}{2}\left[\frac{2e\lambda H}{m^* F \cosh\left(\varepsilon F/2\right)} \right]^2 \left[\frac{\sinh\left(\varepsilon F\right)}{\varepsilon F} - 1 \right] \tag{2}$$

(c) From EQs. (1) and (2), show that the order parameter F and the external magnetic field H are related as follows:

$$\left(\frac{H}{H_c} \right)^2 = 4F^2\left(1-F^2\right)\left\{ \frac{\cosh^2\left(\varepsilon F/2\right)}{\left[\sinh\left(\varepsilon F\right)/\left(\varepsilon F\right)\right]-1} \right\} \tag{3}$$

where H_c is the thermodynamic field.

(d) From EQ. (3), find $F(H)$ in two extreme cases $\varepsilon F \ll 1$ and $\varepsilon F \gg 1$.

(e) To derive the critical field H_T for the thin-film superconductor, we note that at the critical field the Gibbs free energy density of the superconductor, g_S, becomes equal to that of the normal state g_N, where

$$g_N = f_N - \left[H^2/\left(8\pi\right) \right], \tag{4}$$

and f_N is the Helmholtz free energy density in the normal state. Using the results derived thus far, show that the Gibbs free energy in the superconducting state satisfies the following relation:

$$\begin{aligned} g_S &= f_S - \frac{\langle h \rangle H}{4\pi} = \left[f_N - \frac{H_c^2}{8\pi}F^4 + \frac{\left\langle h^2 \right\rangle}{8\pi} \right] - \frac{\langle h \rangle H}{4\pi}, \\ &= f_N + \frac{H^2}{8\pi}\left[\frac{\sinh\left(\varepsilon F\right)+\varepsilon F}{\varepsilon F\left(1+\cosh\left(\varepsilon F\right)\right)} - \frac{4}{\varepsilon F}\tanh\left(\frac{\varepsilon F}{2}\right) \right] - \frac{H_c^2}{8\pi}F^4 . \end{aligned} \tag{5}$$

(f) Using the condition $g_S = g_N$ at $H = H_T$ and EQs. (3) and (5), show that the critical field H_T can be obtained by solving the following equation for the normalized order parameter $F(H)$:

$$Y_1\left(F\right) \equiv 1 + \frac{1}{6}\left(\frac{F^2}{1-F^2} \right) = \frac{1}{3}\frac{\varepsilon F\left[\cosh\left(\varepsilon F\right)-1\right]}{\sinh\left(\varepsilon F\right)-\varepsilon F} \equiv Y_2\left(F\right). \tag{6}$$

(g) Discuss why the phase transition at $H = H_T$ is first order if $\varepsilon > \sqrt{5}$ and is second order if $\varepsilon < \sqrt{5}$. (*Hint:* You may plot Y_1 and Y_2 in EQ. (6) as a function of F and also consider F as a function of ε.)