222D5220 **Problem Set #2 (Parts II.7 – II.8)** August 17, 2007
(Due: August 28, 2007)

1. Lehmann representation of the density fluctuation operator

The density fluctuation operator of a fermionic many-body system is a useful physical quantity for such consideration as the system response to external fields and the density-density correlation near phase transitions. Specifically, the density fluctuation operator is defined as

$$\tilde{n}(\mathbf{r}) \equiv \psi_\alpha^\dagger(\mathbf{r})\psi_\alpha(\mathbf{r}) - \frac{\langle \Psi_0 | \psi_\alpha^\dagger(\mathbf{r})\psi_\alpha(\mathbf{r}) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle},$$

where $\psi_\alpha(\mathbf{r})$ denotes the fermion field operator, α is the spin index, and $|\Psi_0\rangle$ is the exact ground state of the many-body system.

(a) Derive the Lehmann representation for $D(\mathbf{k},\omega)$, which is the Fourier transformation of the density-density correlation function $D(x,x')$ defined as

$$D(x,x') \equiv -i \frac{\langle \Psi_0 | T\left[\tilde{n}_H(x)\tilde{n}_H(x')\right] | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}.$$

Here T is the time-ordering operator, x is a four-vector, and $\tilde{n}_H(x)$ denotes the density fluctuation operator in the Heisenberg picture.

(b) Show that $D(\mathbf{k},\omega)$ has poles in the second and fourth quadrant of the complex ω-plane and construct the corresponding retarded and advanced functions similar to those associated with the Green's functions.

2. Application of Wick's theorem to particles and holes of a fermionic many-body system

The fermion operator $c_\mathbf{k}$ (ignoring spin for now) of a many-body system can be expressed in terms of the particle and hole operators as

$$c_\mathbf{k} = \theta(|\mathbf{k}| - k_F)\, a_\mathbf{k} + \theta(k_F - |\mathbf{k}|)\, b_{-\mathbf{k}}^\dagger,$$

where $\theta(k)$ is the step function in momentum space, k_F is the Fermi momentum, and a and b are the particle and hole operators, respectively. By applying Wick's theorem, prove the following relation:

$$\begin{aligned} c_{\mathbf{k}_1}^\dagger c_{\mathbf{k}_2}^\dagger c_{\mathbf{k}_4} c_{\mathbf{k}_3} &= N\left(c_{\mathbf{k}_1}^\dagger c_{\mathbf{k}_2}^\dagger c_{\mathbf{k}_4} c_{\mathbf{k}_3}\right) + \theta(k_F - |\mathbf{k}_2|)\left[\delta_{\mathbf{k}_2\mathbf{k}_4} N\left(c_{\mathbf{k}_1}^\dagger c_{\mathbf{k}_3}\right) - \delta_{\mathbf{k}_2\mathbf{k}_3} N\left(c_{\mathbf{k}_1}^\dagger c_{\mathbf{k}_4}\right)\right] \\ &+ \theta(k_F - |\mathbf{k}_1|)\left[\delta_{\mathbf{k}_1\mathbf{k}_3} N\left(c_{\mathbf{k}_2}^\dagger c_{\mathbf{k}_4}\right) - \delta_{\mathbf{k}_1\mathbf{k}_4} N\left(c_{\mathbf{k}_2}^\dagger c_{\mathbf{k}_3}\right)\right] + \theta(k_F - |\mathbf{k}_1|)\,\theta(k_F - |\mathbf{k}_2|)\left[\delta_{\mathbf{k}_1\mathbf{k}_3}\delta_{\mathbf{k}_2\mathbf{k}_4} - \delta_{\mathbf{k}_1\mathbf{k}_4}\delta_{\mathbf{k}_2\mathbf{k}_3}\right], \end{aligned}$$

where $N(\cdots)$ represents the normal-ordered product of operators within the parenthesis.

3. The proper self-energy of a degenerate electron gas

(a) For a degenerate (*i.e.* high-density) electron gas with a Fermi momentum k_F, show that the proper self-energy $\Sigma^*(\mathbf{q})$ to first order in the Coulomb interaction is given by:

$$\Sigma^*_{(1)}(\mathbf{q}) = -\frac{e^2}{2\pi}\left(\frac{k_F^2 - q^2}{q}\ln\left|\frac{k_F + q}{k_F - q}\right| + 2k_F\right), \qquad \text{where } |\mathbf{q}| \equiv q.$$

(b) Using the proper self-energy given in (a), find and sketch the corresponding single-particle spectrum.

4. Spin-dependent interaction potential and Dyson's equation of a fermionic many-body system

Consider a uniform many-body system of spin-1/2 fermions with a spin-dependent pair interaction potential given by:

$$V(\mathbf{r}_1 - \mathbf{r}_2) = V_0(|\mathbf{r}_1 - \mathbf{r}_2|)\mathbf{1}(1)\mathbf{1}(2) + V_1(|\mathbf{r}_1 - \mathbf{r}_2|)\boldsymbol{\sigma}(1)\bullet\boldsymbol{\sigma}(2),$$

where **1** refers to the unit spin matrix and $\boldsymbol{\sigma}$ denotes the Pauli matrices.

(a) The exact interaction potential $U(q)_{\alpha\beta,\rho\tau}$ can be expressed in terms of the bare interaction potential $U_0(q)_{\alpha\beta,\rho\tau}$ and the proper polarization insertion $\Pi^*_{\mu\nu,\kappa\lambda}(q)$ by the following relation

$$U(q)_{\alpha\beta,\rho\tau} = U_0(q)_{\alpha\beta,\rho\tau} + U_0(q)_{\alpha\beta,\mu\nu}\,\Pi^*_{\mu\nu,\kappa\lambda}(q)U(q)_{\kappa\lambda,\rho\tau}.$$

Assuming that the proper polarization insertion $\Pi^*_{\mu\nu,\kappa\lambda}(q)$ can be approximated by $\Pi^0(q)\delta_{\nu\kappa}\delta_{\mu\lambda}/2$, prove that the interaction potential becomes

$$U(q)_{\alpha\beta,\rho\tau} = \frac{V_0(q)\delta_{\alpha\beta}\delta_{\rho\tau}}{1 - V_0(q)\Pi^0(q)} + \frac{V_1(q)\boldsymbol{\sigma}_{\alpha\beta}\bullet\boldsymbol{\sigma}_{\rho\tau}}{1 - V_1(q)\Pi^0(q)}.$$

(b) Find Dyson's equation for the polarization insertion Π in terms of the proper polarization insertion Π^* and U_0, and then solve this Dyson's equation to prove the following relation

$$\Pi_{\mu\nu,\kappa\lambda}(q) = \frac{1}{2}\Pi^0(q)\delta_{\nu\kappa}\delta_{\mu\lambda} + \frac{1}{2}\Pi^0(q)U(q)_{\nu\mu,\lambda\kappa}\frac{1}{2}\Pi^0(q).$$

5. The proper polarization of a spin-*S* fermion system

Consider a system of spin-*S* fermions interacting with a spin-independent static potential $V(\mathbf{q})$.

(a) By analyzing the Feynman diagrams, show that the proper polarization is given by

$$\Pi^*_{\alpha\beta,\lambda\mu}(\mathbf{q}) = \Pi^*_\sigma(\mathbf{q})(2S+1)^{-1}\delta_{\beta\lambda}\delta_{\alpha\mu} + \left[\Pi^*_{\alpha\alpha,\lambda\lambda}(\mathbf{q}) - \Pi^*_\sigma(\mathbf{q})\right](2S+1)^{-2}\delta_{\alpha\beta}\delta_{\lambda\mu}.$$

(b) Solve Dyson's equation for $\Pi_{\alpha\beta,\lambda\mu}(\mathbf{q})$, and compare the result with that obtained in Problem 4**(b)**.

(c) Show that the correlation function $D(\mathbf{q})$ is equal to $\Pi_{\alpha\alpha,\lambda\lambda}(\mathbf{q})$ and derive the expression

$$D(q) = \frac{\Pi^*(q)}{1 - V(\mathbf{q})\Pi^*(q)}.$$