222D5220 **Problem Set #1** **(Parts I & II.1 – II.2)** August 10, 2007
(Due date: August 17, 2007)

1. Second-quantizing the interaction Hamiltonian of fermions

Consider the field operator ψ defined by $\psi(\mathbf{r}) = \sum_{\mathbf{k}} \varphi_{\mathbf{k}}(\mathbf{r})\, a_{\mathbf{k}}$, where $\varphi_{\mathbf{k}}(\mathbf{r})$ are a complete set of single particle states characterized by the quantum number **k**, **r** is the spatial coordinate, and $a_{\mathbf{k}}$ is the fermion annihilation operator.

Show that the second-quantized expression for the interaction Hamiltonian of a system of N interacting fermions, $\mathcal{H}' = \sum_{i,j\neq i}^{N} \mathcal{V}(\mathbf{r}_i, \mathbf{r}_j)/2$, is given by EQ. (I.28):

$$\mathcal{H}' = \frac{1}{2}\sum_{k,l,s,t} \langle kl|\mathcal{V}|st\rangle\, a_k^\dagger a_l^\dagger a_t a_s \equiv \frac{1}{2}\sum_{k,l,s,t} \left[\int d\mathbf{r}_1 d\mathbf{r}_2\, \varphi_k^*(\mathbf{r}_1)\varphi_l^*(\mathbf{r}_2)\,\mathcal{V}(\mathbf{r}_1,\mathbf{r}_2)\,\varphi_s(\mathbf{r}_1)\varphi_t(\mathbf{r}_2)\right] a_k^\dagger a_l^\dagger a_t a_s\,, \tag{P1.1}$$

where $\mathcal{V}(\mathbf{r}_1, \mathbf{r}_2)$ describes the interaction potential between fermions at $\mathbf{r}_1$ and $\mathbf{r}_2$. To simplify this problem without going over all possible combinations of the indices k, l, s, t, you may consider two special cases, one is for $k < l < s < t$, and the other for $k = s$ and $l > t$.

2. The interaction picture for a spin-1/2 particle in a time-dependent magnetic field

When we consider the interaction picture for various physical systems, it occurs at times that the best choice for $\mathcal{H}_0$ in the total Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$ is not necessarily the entire familiar or constant part. As an example, consider the following Hamiltonian for a spin-1/2 particle with gyromagnetic ratio γ in a time-dependent magnetic field $\boldsymbol{B}(t) = b_1\left[\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}\right] + B_0\hat{z}$ rotating about the z-axis with frequency ω so that

$$\begin{aligned}\mathcal{H}(t) &= -\gamma\left[b_1\cos(\omega t)\mathbf{S}_x + b_1\sin(\omega t)\mathbf{S}_y + B_0\mathbf{S}_z\right]\\ &= \frac{\hbar}{2}\left[-\gamma b_1\cos(\omega t)\boldsymbol{\sigma}_x - \gamma b_1\sin(\omega t)\boldsymbol{\sigma}_y + \omega_L\boldsymbol{\sigma}_z\right],\end{aligned} \tag{P1.2}$$

where $\boldsymbol{\sigma}_{x,y,z}$ represent the (2×2) Pauli matrices, and $\omega_L \equiv \gamma B_0$.

(a) A seemingly straightforward choice for $\mathcal{H}_0$ and $\mathcal{H}'$ in the above Hamiltonian is:

$$\mathcal{H}_0 = \frac{\hbar\omega_L}{2}\boldsymbol{\sigma}_z, \quad \mathcal{H}'(t) = -\frac{\hbar\gamma b_1}{2}\left[\cos(\omega t)\boldsymbol{\sigma}_x + \sin(\omega t)\boldsymbol{\sigma}_y\right] = -\frac{\hbar\gamma b_1}{2}\begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0\end{pmatrix}. \tag{P1.3}$$

With the $\mathcal{H}_0$ and $\mathcal{H}'$ given above, find the corresponding $\mathcal{H}_I'(t)$ in the interaction picture, and show that it does not even commute with itself at different times.

(b) The situation in Part (a) is certainly not desirable for considering the time evolution of the system. On the other hand, a different choice of $\mathcal{H}_0$ and $\mathcal{H}'$ can lead to much simplified solutions. Specifically, consider

$$\mathcal{H}_0 = \frac{\hbar\omega}{2}\boldsymbol{\sigma}_z, \quad \mathcal{H}'(t) = \frac{\hbar}{2}\left[(\omega_L - \omega)\boldsymbol{\sigma}_z - \gamma b_1\left(\cos(\omega t)\boldsymbol{\sigma}_x + \sin(\omega t)\boldsymbol{\sigma}_y\right)\right]. \tag{P1.4}$$

Find the corresponding $\mathcal{H}_I'(t)$ and use it to solve for $U(t,0)$ explicitly.

3. Gell-Mann & Low Theorem

In deriving the Gell-Mann & Low theorem, we have used the following identity

$$\left(\frac{-i}{\hbar}\right)^{n-1}\frac{1}{(n-1)!}g^n = i\hbar g\frac{\partial}{\partial g}\left(\frac{-i}{\hbar}\right)^{n}\frac{1}{n!}g^n \quad , \tag{P1.5}$$

where g represents the coupling constant associated with the Hamiltonian in the interaction picture $\mathcal{H}_I'(t)$, n denotes positive integers, and the total Hamiltonian $\mathcal{H}$ is given by $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$. Using EQ. (P1.5) and EQ. (I.106):

$$\left(\mathcal{H}_0 - E_0\right)\left|\Psi_\varepsilon\right\rangle = -\sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^{n-1}\frac{1}{n!}\int_{-\infty}^{0}dt_1\ldots\int_{-\infty}^{0}dt_n\, e^{\varepsilon(t_1+\ldots+t_n)}\left(\sum_{\nu=1}^{n}\frac{\partial}{\partial t_\nu}\right)\hat{T}\left[\mathcal{H}_I'(t_1)\ldots\mathcal{H}_I'(t_n)\right]\left|\Phi_0\right\rangle,$$

prove that the following formalism in EQ. (I.107) indeed holds:

$$\left(\mathcal{H}_0 - E_0\right)\left|\Psi_\varepsilon\right\rangle = -\mathcal{H}'\left|\Psi_\varepsilon\right\rangle + i\hbar\varepsilon g\frac{\partial}{\partial g}\left|\Psi_\varepsilon\right\rangle.$$

Here E_0 denotes the eigen-energy of $\left|\Phi_0\right\rangle = \left|\Psi_I(-\infty)\right\rangle$ so that $\mathcal{H}_0\left|\Phi_0\right\rangle = E_0\left|\Phi_0\right\rangle$, and $\left|\Psi_\varepsilon\right\rangle$ is the state vector in the interaction picture defined by $\left|\Psi_\varepsilon\right\rangle \equiv \left|\Psi_{I\varepsilon}(0)\right\rangle$.

4. Noether's theorem for a system of *SO*(*N*) symmetry

As discussed in Part II.1, the Noether's theorem states that there is a conserved current associated with each generator of a continuous symmetry. In this problem we want to find the conserved currents of a specific system with continuous *SO*(*N*) symmetry and a Lagrangian density *L* given by the following:

$$L = \frac{1}{2}\left[\left(\partial\boldsymbol{\varphi}\right)^2 - m^2\boldsymbol{\varphi}^2\right] - \frac{\lambda}{4}\left(\boldsymbol{\varphi}^2\right)^2. \tag{P1.6}$$

Here m denotes the mass of the N-component scalar field $\boldsymbol{\varphi} = (\varphi_1,\cdots,\varphi_N)$ with φ_a $(a = 1,\cdots,N)$ transform like vectors, $\boldsymbol{\varphi}^2 \equiv \boldsymbol{\varphi}\bullet\boldsymbol{\varphi} = \varphi_a\varphi_a$ is invariant under *SO*(*N*) symmetry, and λ is a positive coupling constant. If we consider an infinitesimal change in the field so that

$$\delta\varphi_a(x) = \theta^A(x)\left(T^A\right)_{ab}\varphi_b(x), \tag{P1.7}$$

where $\theta^A(x)$ denote parameters that depend on the 4-vector x, and T^A represent the $N(N-1)/2$ independent generators. In the special case of $\theta^A(x)$ = constant, we expect the action $S = \int d^4x\,L$ to be invariant under the infinitesimal transformation of the fields. In other words, $\delta S = 0$ for θ^A = constant. On the other hand, for arbitrary $\theta^A(x)$, δS does not necessarily vanish, and we expect that for arbitrary $\theta^A(x)$, δS takes on the following form

$$\delta S = \int d^4x\left[J^A(x)\right]^\mu\partial_\mu\theta^A(x), \tag{P1.8}$$

so that $\delta S = 0$ for constant θ^A, and $\left[J^A(x)\right]^\mu$ denote conserved currents that are effectively the coefficients of $\partial_\mu \theta^A(x)$ according to EQ. (P1.8).

Using the Lagrangian given in EQ. (P1.6) and the transformation specified in EQ. (P1.7), verify the Noether's theorem and find an explicit expression for the conserved current $\left[J^A(x)\right]^\mu$.

5. Wick's theorem

Wick's theorem described in Part II.2 will become very important in our subsequent consideration of Feynman diagrams. The following two problems are examples for the application of Wick's theorem.

(a) Find the expectation value of $\langle x_i\, x_j\, x_k\, x_l\, x_m\, x_n\rangle$ using Wick's theorem, where $\langle x_i\, x_j\, x_k\, x_l\, x_m\, x_n\rangle$ is defined as:

$$\left\langle x_i x_j x_k x_l x_m x_n \right\rangle \equiv \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_N \; e^{-\frac{1}{2}x \bullet A \bullet x} x_i x_j x_k x_l x_m x_n}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_N \; e^{-\frac{1}{2}x \bullet A \bullet x}}, \qquad \text{(P1.9)}$$

and A denotes a real symmetric $N \times N$ matrix. (Hint: your answer should contain 15 different terms.)

(b) Wick's theorem can also be applied to operators. As an example, consider the operator $O = aaa^\dagger a^\dagger$, where a and $a^\dagger$ are the annihilation and creation operators of a free boson system. Apply Wick's theorem and show that it does produce the correct result.