PART IV. Gauge Theory

In Part IV we return to relativistic quantum field theory to address issues of gauge invariance, gauge fields, magnetic monopoles, Aharonov-Bohm effect, breaking of continuous symmetries and the resulting Nambu-Goldstone bosons, non-abelian gauge theory, and Anderson-Higgs mechanism of massive gauge bosons under gauge symmetry breaking. These topics associated with gauge theory will be useful for our consideration in Part VI of topological objects and topological field theory that are beyond the descriptions of Feynman diagrams. In particular, the Chern-Simons term in (2+1)-dimensions will be introduced in the context of gauge theory, which is of particular importance to the description of the fractional quantum Hall fluids discussed in Part VI.

IV.1 Gauge Invariance

We have discussed the subject of gauge invariance in the context of quantum electrodynamics (QED) in Part II, which we briefly review below before generalizing it to a broader context.

As discussed earlier, the Lagrangian for QED in the limit of zero photon mass ($\mu \to 0$) is

$$\mathcal{L} = \bar{\psi}\left[i\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right) - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{IV.1}$$

where ψ represents the fermion field operator, m is the fermion mass, $F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, and $\boldsymbol{A}$ denotes the electromagnetic vector potential. The Lagrangian $\mathcal{L}$ given above is invariant under the following gauge transformation:

$$\psi(x) \to e^{i\Lambda(x)}\psi(x) \, , \tag{IV.2}$$

$$A_{\mu}(x) \to A_{\mu}(x) + \frac{1}{ie}e^{-i\Lambda(x)}\partial_{\mu}e^{i\Lambda(x)} = A_{\mu}(x) + \frac{1}{e}\partial_{\mu}\Lambda(x) \, , \tag{IV.3}$$

and $F_{\mu\nu}(x)$ also remains invariant under the gauge transformation. We have also shown in Part II that the gauge invariance under the gauge transformation given above is diagrammatically equivalent to the following photon propagator:

$$iD_{\mu\nu} = \frac{i}{k^2}\left(\left(1-\xi\right)\frac{k_{\mu}k_{\nu}}{k^2} - \eta_{\mu\nu}\right), \tag{IV.4}$$

where ξ can be any number depending on our choice of the gauge. More generally, given any physical amplitude $M^{\mu}(\mathbf{k})$ with external electrons on shell for a process with a photon carrying momentum $\mathbf{k}$ coming out (or going into) a vertex labeled by the Lorentz index μ, it can be proven that the following *Ward* identity holds for QED, the simplest gauge theory:

$$k_{\mu}M^{\mu}(\mathbf{k}) = 0 \, . \tag{IV.5}$$

The Ward identity is a special case of the *Ward-Takahashi* identity illustrated in Fig. IV.1.1, the latter can be expressed in terms of the vertex contribution Γ^{μ} and the fermion propagators S below:

$$-ik_{\mu}\Gamma^{\mu}(\mathbf{p}+\mathbf{k},\mathbf{p}) = S^{-1}(\mathbf{p}+\mathbf{k}) - S^{-1}(\mathbf{p}) \, , \tag{IV.6}$$

where

$$S(\mathbf{p}) \equiv \frac{i}{\not{p} - m - \Sigma(\mathbf{p})}. \tag{IV.7}$$

In fact, EQ. (IV.6) can be expressed in a more general form if we define *M* as a correlation function in QED given in Fig. IV.1.2 so that

$$k_\mu M^\mu(\mathbf{k};\mathbf{p}_1\cdots\mathbf{p}_n;\mathbf{q}_1\cdots\mathbf{q}_n) = e\sum_i\left[M_0(\mathbf{p}_1\cdots\mathbf{p}_n;\mathbf{q}_1\cdots(\mathbf{q}_i-\mathbf{k})\cdots\mathbf{q}_n) - M_0(\mathbf{p}_1\cdots(\mathbf{p}_i+\mathbf{k})\cdots\mathbf{p}_n;\mathbf{q}_1\cdots\mathbf{q}_n)\right]. \tag{IV.8}$$

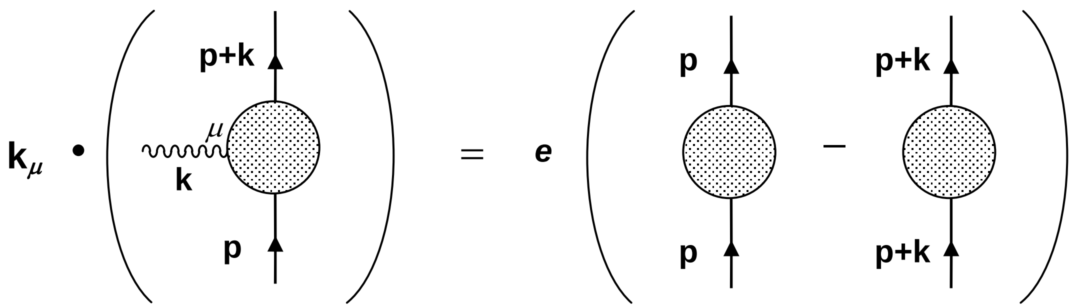

Fig. IV.1.1 Diagrammatic expression for the Ward-Takahashi identity given in EQ. (IV.6).

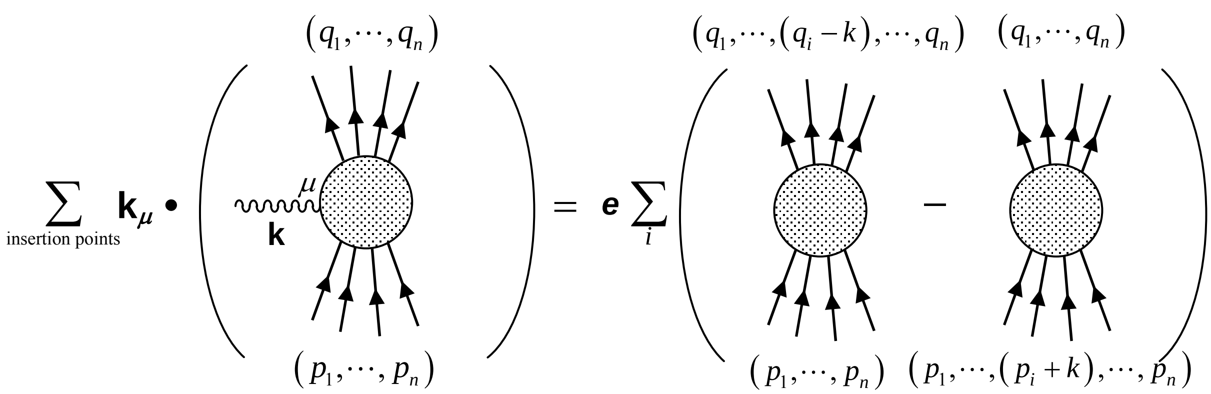

Fig. IV.1.2 Diagrammatic expression for the generalized Ward-Takahashi identity given in EQ. (IV.8).

Therefore, the Ward identity in EQ. (IV.5) is a special case of the Ward-Takahashi identity in the limit of $\mathbf{k} \to 0$. In literature the terms *Ward identity*, *current conservation*, and *gauge invariance* are often used interchangeably, because the Ward identity is the diagrammatic expression of the conservation of the electric current, which is in turn a consequence of gauge invariance. We further note that in analogy to the Ward-Takahashi identity in QED, similar identities known as the *Slavnov-Taylor identities* also hold for non-abelian gauge theories. While we do not get into rigorous proofs for these identities and refer you to references in the end of Part IV, we emphasize that the broad applicability of gauge invariance in quantum field theory has important implication on our later discussion of gauge transformation and gauge symmetry. In fact, the applicability of gauge invariance can be extended beyond Feynman diagrams to topological field theory, as we shall see later in the case Chern-Simons theory.

IV.2 Differential Forms, Magnetic Monopole, and Aharonov-Bohm Effect

The gauge transformation described in Part IV.1 and our later discussion of the non-abelian gauge theory can be more conveniently expressed in the language of *differential forms*. In this section we define the basic concepts of differential forms, and then apply differential forms to the gauge transformations associated with magnetic monopoles and the Aharonov-Bohm effect.

[Differential forms]

We first define the differentials under the coordinate transform $x \to x'$:

$$dx^{\mu} = \left(\partial x^{\mu} / \partial x'^{\nu}\right) dx'^{\nu} . \quad \text{(IV.9)}$$

The object $A \equiv A_{\mu} dx^{\mu}$ is called a 1-form and is given by:

$$A = A_{\mu}\left(\partial x^{\mu} / \partial x'^{\nu}\right) dx'^{\nu} \equiv A'_{\nu} dx'^{\nu} , \quad \text{(IV.10)}$$

which reproduces the standard transformation law of vectors under coordinate transformation. Similarly, the 2-form object F is given by:

$$F \equiv \left(1/2!\right) F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} , \quad \text{(IV.11)}$$

where

$$dx^{\mu} \wedge dx^{\nu} = -dx^{\nu} \wedge dx^{\mu} . \quad \text{(IV.12)}$$

In general, a *p-form* is defined as:

$$H \equiv \left(1/p!\right) H_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \cdots \wedge dx^{\mu_p} . \quad \text{(IV.13)}$$

Using EQ. (IV.13), we now define a differential operation d to act on any a p-form H as follows:

$$dH \equiv \left(1/p!\right) \partial_{\nu} H_{\mu_1\mu_2\cdots\mu_p} dx^{\nu} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \cdots \wedge dx^{\mu_p} . \quad \text{(IV.14)}$$

Given the definitions in EQs. (IV.10), (IV.11) and (IV.14), we see that if A_{μ} is the electromagnetic potential, then $F = dA$ is the field 2-form and $F_{\mu\nu}$ is indeed the electromagnetic field. Moreover, we find an important identity:

$$dd = 0, \quad \text{(IV.15)}$$

which is easily proven as follows. For an arbitrary p-form H, we find that

$$\begin{aligned} ddH &= \frac{1}{p!}\partial_{\lambda}\partial_{\nu} H_{\mu_1\mu_2\cdots\mu_p} dx^{\lambda} \wedge dx^{\nu} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \ldots \wedge dx^{\mu_p} \\ &= \frac{1}{2}\frac{1}{p!}\left[\partial_{\lambda}, \partial_{\nu}\right] H_{\mu_1\mu_2\cdots\mu_p} dx^{\lambda} \wedge dx^{\nu} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \ldots \wedge dx^{\mu_p} = 0 . \end{aligned} \quad \text{(IV.16)}$$

Using the differential forms given above, we define a p-form α as *closed* if $d\alpha = 0$. In addition, a p-form α is said to be *exact* if there is a (p–1)-form β such that $\alpha = d\beta$. From EQ. (IV.15), we see that an exact form is always closed. On the other hand, a closed form is not necessarily globally exact. An apparent example is the electromagnetic field F associated with the magnetic monopole, which is locally but not globally exact. On the other hand, in the absence of magnetic monopoles, the 2-form F becomes globally exact. We shall discuss below how gauge theory works leads to the Dirac quantization condition for magnetic charge.

[Magnetic Monopoles – Dirac quantization of magnetic charge]

Let us assume the existence of a magnetic monopole with a charge g, which gives rise to an electromagnetic field $F = (g/4\pi)\, d\,(\cos\theta)\, d\phi$, where θ and ϕ are associated with the spherical coordinates. Noting that the electromagnetic field is related to the gauge potential A by the relation $F = dA$, one might conclude that $A = (g/4\pi) \cos\theta\, d\phi$. However, such a gauge potential is not defined at either the north or the south poles. If we instead introduce $A_N = (g/4\pi)\,(\cos\theta - 1)\, d\phi$ for the gauge potential covering the entire sphere except the south pole and $A_S = (g/4\pi)\,(\cos\theta + 1)\, d\phi$ for that covering the entire sphere except the north pole, then the two gauge potentials differ through a proper gauge transformation by

$$A_S - A_N = 2\frac{g}{4\pi} d\phi . \tag{IV.17}$$

Recall that the gauge transformation is given by

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{ie} e^{-i\Lambda(x)} \partial_\mu e^{i\Lambda(x)} = A_\mu(x) + \frac{1}{e}\partial_\mu \Lambda(x),$$
$$\Leftrightarrow \qquad A \rightarrow A + \frac{1}{ie} e^{-i\Lambda} d e^{i\Lambda} \tag{IV.18}$$

Suppose that A_N and A_S are related by the gauge transformation in EQ. (IV.18), we obtain $e^{i\Lambda}$ by comparing EQs. (IV.17) with (IV.18):

$$e^{i\Lambda} = e^{i(eg\phi)/(2\pi)} . \tag{IV.19}$$

However, in EQ. (VIII.19) $\phi = 0$ and $\phi = 2\pi$ correspond to the same point. Hence, we find

$$e^{i(eg0)/(2\pi)} = e^{i(eg2\pi)/(2\pi)} \rightarrow \quad e^{ieg} = 1 , \tag{IV.20}$$

which leads to the Dirac quantization condition for magnetic monopoles:

$$g = \frac{2\pi}{e} n , \tag{IV.21}$$

where n denotes an integer. Therefore, EQ. (IV.21) indicates that a magnetic monopole is quantized in units of $(2\pi/e)$, which is sometimes referred to as Dirac's quantization of the magnetic charge g.

[The Aharonov-Bohm effect]

The importance of gauge potential was well manifested by the Aharonov-Bohm effect, which states that a particle moving around a closed path will pick up a phase that represents the magnetic flux enclosed by the path, as schematically illustrated in Fig. IV.2.1, even if the particle does not pass through the area containing the magnetic flux. Specifically, we consider a magnetic field B confined to a region Ω and describe the non-relativistic Schrödinger's equation for a charged particle in the corresponding electromagnetic field:

$$\left[-\frac{1}{2m} \left(\nabla - ie\mathbf{A} \right)^2 + e\phi \right] \psi = E\psi , \tag{IV.22}$$

where $\boldsymbol{A}$ is the vector potential, ϕ is the electrostatic potential, and E is the energy of the particle. In Feynman's path integral formalism, the probability for an electron to propagate from point a to point b will involve the interference between contributions coming from path $P1$ and path $P2$ of the following form

$$\left(e^{ie\int_{P1} d\boldsymbol{x}\cdot\boldsymbol{A}}\right)\left(e^{ie\int_{P2} d\boldsymbol{x}\cdot\boldsymbol{A}}\right)^* = \left(e^{ie\oint d\boldsymbol{x}\cdot\boldsymbol{A}}\right) = \left(e^{ie\int d\boldsymbol{S}\cdot\boldsymbol{B}}\right) \equiv e^{ie\Phi}, \qquad \text{(IV.23)}$$

where Φ is the total flux enclosed by the closed curve $(P1 - P2)$. Therefore, the electron can feel the effect of the magnetic field even if it does not move into the region with a finite magnetic field.

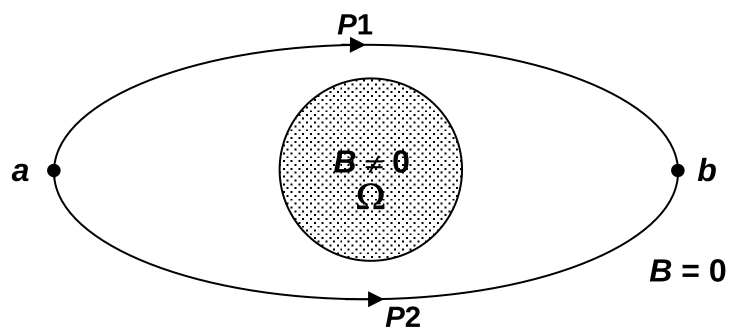

Fig. IV.2.1 Schematics of the Aharonov-Bohm effect: a particle moving from point a to point b experiences the interference of path P1 and path P2 due to the magnetic flux enclosed in Ω.

IV.3. Symmetry and Symmetry Breaking

In general symmetries of systems and related properties can be described in terms of group theory, and they can be classified in terms of the space-time and internal symmetries. In this section we introduce the concept of symmetry breaking and the physical consequences of symmetry breaking.

The physical universe and states of matter can be generally described in terms of symmetry and symmetry breaking. Indeed, a central theme of modern physics is the study of how symmetries of the Lagrangian can be broken. Let's take space for example, for almost all practical purposes we can consider space is homogeneous and isotropic. Even in the interior of a neutron star, the distortion of space due to gravitation can usually be treated as a small perturbation, which is locally negligible. Therefore, the laws that govern particles such as electrons and nuclei, or individual nucleons and mesons in nuclear many-body physics, have a very high degree of symmetry, corresponding to the homogeneity and isotropy of space. In most cases, they also obey parity and time-reversal symmetries. We can even speculate that before the cosmological Big Bang, the universe in a form of the elemental fireball probably contained the highest degree of symmetry that is not manifest to us.

In the case of matter, at very high temperatures the molecules dissociate and the atoms ionized, forming a plasma state (such as in the interior of the Sun) which is a locally homogeneous and isotropic gaseous mixture. Such a state is clearly compatible with the homogeneous and isotropic symmetry of space. On the other hand, cold matter is an entirely different story. As temperature cools down, two fundamental types of symmetry breaking can take place within matter. One is associated with nonlinear driven instabilities (such as a gravitational instability broke up the original homogeneous gas after the Big Bang, first into the ancestors of galactic clusters) where the local equations of state and the local properties remain more of less symmetric. The other is the phase transition (such as gas to liquid and liquid to solid phase transformation upon cooling) that involves a complete change in the microscopic properties of the matter. The phase transition case generally exhibits great regularity and stability. In addition, it has been proposed that laser represents a third category of symmetry breaking, in which there is a local, microscopic broken symmetry driven by an external pumping mechanism rather than by thermodynamic equilibrium.

It is worth noting, however, that broken symmetry is not the only way in which cold matter changes its behavior qualitatively. There are other alternatives, although rare in nature, in which a continuous phase transition to a qualitatively different behavior occurs without change of symmetry. Such phase transitions without explicit symmetry breaking in general cannot be described by the Landau theory of phase transitions, and are often associated with topological degrees of freedom. For phase transitions involving explicit symmetry breaking, low-energy excitations known as the Nambu-Goldstone bosons (or simply the "Goldstone modes" in condensed matter physics jargon) will appear. In contrast, for continuous phase transitions without explicit symmetry breaking, the corresponding low-energy excitations generally involve gauge bosons, which are usually not physical observables. In this section we investigate basic concepts of symmetry breaking and the effect of quantum fluctuations on symmetry breaking. The Yang-Mills non-abelian gauge theory, gauge bosons, and the Anderson-Higgs mechanism will be introduced in next section.

[Spontaneous symmetry breaking and Nambu-Goldstone bosons]

We begin with the familiar φ^4-theory for scalar bosons:

$$L = \frac{1}{2}\left[\left(\partial\vec{\varphi}\right)^2 - \mu^2\vec{\varphi}^2\right] - \frac{\lambda}{4}\left(\vec{\varphi}^2\right)^2, \tag{IV.24}$$

where we have defined $\vec{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_N)$. The Lagrangian in EQ. (IV.24) exhibits an *O*(*N*) symmetry under which $\vec{\varphi}$ transforms as an *N*-component vector. If we start adding terms that do not respect the *O*(*N*) symmetry to EQ. (IV.24), such as $\varphi_1^{\,2}$, $\varphi_1^{\,4}$ and $\varphi_1^{\,2}\vec{\varphi}^2$, we reduce the *O*(*N*) symmetry to *O*(*N*–1) symmetry under which $(\varphi_2, \varphi_3, \cdots, \varphi_N)$ rotates as an (*N*–1)-component vector. We can continue the similar process by breaking the *O*(*N*) symmetry down to *O*(*N*–*M*) with the addition of lower symmetry terms for any $M \le N$. We may also break the reflection symmetry $\varphi_a \to -\varphi_a$ (for any *a*) by adding such terms as $\varphi_a^{\,3}$. However, this type of breaking symmetry "by hand" is certainly not very interesting.

Suppose we simplify EQ. (IV.24) further by reducing it to a one-dimensional problem and by making the substitutions $\vec{\varphi} \to q, (-\mu^2) \to \mu^2$, and $\mu^2 > 0$, we have $V(q) = -(\mu^2 q^2/2) + (\lambda q^4/4)$, known as a double well potential as shown in Fig. IV.3.1, with two minima at $q = \pm v$ where $v \equiv (\mu^2/\lambda)^{1/2}$. In quantum mechanics, there is a tunneling barrier $[V(0) - V(q = \pm v)]$ between the two minima and the probability for a particle to be in one or the other must be equal so as to respect the reflection symmetry $q \to -q$ of our Lagrangian $L = (\dot{q}^2 + \mu^2 q^2)/2 - (\lambda q^4/4)$. In the ground state, this one-dimensional double-well problem leads to an even wave function $\psi(q) = \psi(-q)$. On the other hand, as we attempt to extend this problem to quantum field theory in the case of $N = 1$ with φ replacing *q*, although the potential $V(\varphi) = -(\mu^2\varphi^2/2) + (\lambda\varphi^4/4)$ looks just like *V*(*q*), the tunneling barrier is now given by $[V(0) - V(\varphi = \pm v)]\left(\int d^D x\right)$ with *D* being the spatial dimension of the system, which is a very large barrier so that tunneling is essentially shut down. Consequently, the ground state wave function must choose between $\varphi = +v$ or $\varphi = -v$, implying that the reflection symmetry $\varphi \to -\varphi$ is spontaneously broken even though we did not insert any symmetry breaking terms into the Lagrangian!

Now if we choose the ground state at $\varphi_0 = +v$ (*i.e.* the field φ acquires a vacuum expectation value +*v*) and allow fluctuations around φ_0 such that $\varphi = v + \varphi'$, we find that our Lagrangian (with *N* = 1) becomes:

$$
\begin{aligned}
L &= \frac{1}{2}\left[(\partial\varphi)^2 + \mu^2\varphi^2\right] - \frac{\lambda}{4}\varphi^4 = \frac{1}{2}\left[(\partial\varphi')^2 + \mu^2 v^2\left(1 + \frac{\varphi'}{v}\right)^2\right] - \frac{\lambda v^4}{4}\left(1 + \frac{\varphi'}{v}\right)^4 \\
&= \frac{1}{2}\left[(\partial\varphi')^2 + \mu^2 v^2\left(1 + 2\frac{\varphi'}{v} + \left(\frac{\varphi'}{v}\right)^2\right)\right] - \frac{\lambda v^4}{4}\left(1 + 4\frac{\varphi'}{v} + 6\left(\frac{\varphi'}{v}\right)^2\right) + O\left(\varphi'^3\right) \\
&= \frac{1}{2}(\partial\varphi')^2 + \frac{\mu^4}{4\lambda} - \frac{\mu^4}{\lambda}\left(\frac{\varphi'}{v}\right)^2 + O\left(\varphi'^3\right) = \frac{1}{2}(\partial\varphi')^2 + \frac{\mu^4}{4\lambda} - \frac{\left(\sqrt{2}\mu\right)^2}{2}\varphi'^2 + O\left(\varphi'^3\right).
\end{aligned}
\qquad \text{(IV.25)}
$$

Therefore the effective particle mass produced by the field φ' fluctuating around $\varphi_0 = +v$ is $\sqrt{2}\mu$, which is in fact heavier than the mass of φ without spontaneous symmetry breaking. [Can you use simple physics arguments to explain what may have given rise to this mass enhancement?]

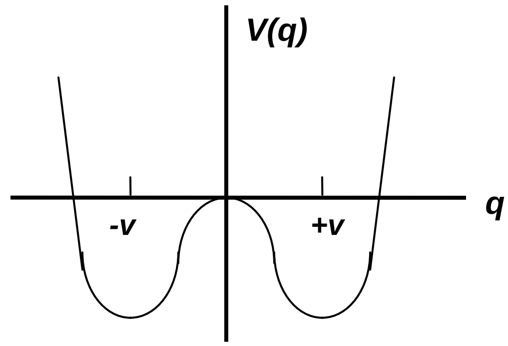

Fig. IV.3.1 The double-well potential $V(q)$ with two minima at $q = \pm v = \pm(\mu^2/\lambda)^{1/2}$.

The situation becomes dramatically different if we consider $N \geq 2$. For simplicity, let's consider the case $N = 2$ so that we are dealing with a Lagrangian with $O(2)$ symmetry. Now the shape of the potential is like a punted wine bottle or a Mexican hat, as shown in Fig. IV.3.2. In other words, there are an infinite number of $\vec{\varphi}$ values with $\vec{\varphi}^2 = \left(\mu^2/\lambda\right)$ that yield the minimum potential. The infinite number of possible directions associated with the ground state $\vec{\varphi}$ can be understood as the result of a continuous $O(2)$ symmetry. We expect that the physics derived from the Lagrangian should not be dependent on the choice of the $\vec{\varphi}$ direction, and for convenience, we choose $\vec{\varphi}$ to point along the 1 direction, so that $\varphi_1 = v = +\sqrt{\mu^2/\lambda}$ and $\varphi_2 = 0$. Next, we consider fluctuations around the choice of the field such that $\varphi_1 = v + \varphi_1'$ and $\varphi_2 = \varphi_2'$. The Lagrangian becomes:

$$L = \frac{1}{2}\left[\left(\partial\vec{\varphi}\right)^2 + \mu^2\vec{\varphi}^2\right] - \frac{\lambda}{4}\left(\vec{\varphi}^2\right)^2, \qquad \text{(IV.26)}$$

$$= \frac{1}{2}\left[\left(\partial\varphi_1'\right)^2 + \left(\partial\varphi_2'\right)^2\right] + \frac{\mu^2}{2}\left[v^2\left(1+\frac{\varphi_1'}{v}\right)^2 + \varphi_2'^2\right] - \frac{\lambda}{4}\left[v^2\left(1+\frac{\varphi_1'}{v}\right)^2 + \varphi_2'^2\right]^2$$

$$= \frac{1}{2}\left[\left(\partial\varphi_1'\right)^2 + \left(\partial\varphi_2'\right)^2\right] + \frac{\mu^4}{4\lambda} - \mu^2\left(\varphi_1'\right)^2 + O\left(\varphi^3\right). \qquad \text{(IV.27)}$$

Hence, we find that the field φ_1' acquires an effective mass $\sqrt{2}\mu$ just as the $N = 1$ case. However, the φ_2' field becomes massless because of the absence of a $\varphi_2{}^2$ term in EQ. (IV.27). This massless φ_2 field is in fact easily understood because it corresponds to fluctuations along the angular direction of the potential, and the latter apparently cost no energy. This situation associated with $N = 2$ is fundamentally different from the case for $N = 1$, the latter having a discrete symmetry (*i.e.* the reflection symmetry) only.

Another way of looking at the Lagrangian in EQ. (IV.27) under $O(2)$ symmetry is to construct a complex field φ from φ_1 and φ_2, such that $\varphi = \left(1/\sqrt{2}\right)\left(\varphi_1 + i\varphi_2\right)$ and $\varphi^\dagger\varphi = \left(\varphi_1{}^2 + \varphi_2{}^2\right)/2$. In this case the Lagrangian becomes:

$$L = \partial\varphi^\dagger\partial\varphi + \mu^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2, \qquad \text{(IV.28)}$$

which is invariant under the $U(1)$ transformation $\varphi \rightarrow e^{i\alpha}\varphi$. In other words, the groups $O(2)$ and $U(1)$ are locally isomorphic. Noting that EQ. (IV.28) is consistent with the $U(1)$ symmetry, we can rewrite the complex field in terms of an amplitude ρ and a phase θ, such that $\varphi(x)=\rho(x)e^{i\theta(x)}$. Hence, EQ. (IV.28) is rewritten into:

$$L=\rho^2\left(\partial\theta\right)^2+\left(\partial\rho\right)^2+\mu^2\rho^2-\lambda\rho^4\,. \tag{IV.29}$$

Under spontaneous symmetry breaking, we have amplitude fluctuations given by $\rho = v + \chi$ and $v=+\sqrt{\mu^2/\lambda}$ so that EQ. (IV.29) becomes:

$$L=\left(v^2+\sqrt{\frac{2\mu^2}{\lambda}}\chi+\chi^2\right)\left(\partial\theta\right)^2+\left[\left(\partial\chi\right)^2-2\mu^2\chi^2-\lambda\chi^4\right]. \tag{IV.30}$$

It is clear from EQ. (IV.30) that the phase $\theta(x)$ is the massless field.

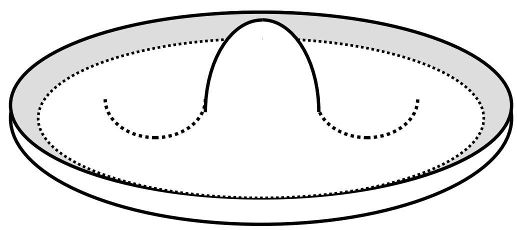

Fig. IV.3.2 The potential $V(\varphi)$ for $N = 2$, with an infinite number of minima at $\varphi = (\mu^2/\lambda)^{1/2}$.

Now we are ready to describe the Goldstone's theorem, which states that *whenever a continuous symmetry is spontaneously broken, massless fields known as Nambu-Goldstone bosons emerge*. Before we provide the proof for the Goldstone's theorem that applies to all systems of continuous symmetry, we first generalize our discussion of the specific Lagrangian in EQ. (IV.26) to $O(N)$ symmetry, and show that there are $(N - 1)$ Nambu-Goldstone bosons upon spontaneous symmetry breaking. Without losing generality, we can assume a vacuum expectation value for the $\boldsymbol{\varphi}$-field (among an infinite number of possible choices) is acquired at $\boldsymbol{\varphi}_0=\left(0,0,\cdots,+v\right)$ where $v = \pm(\mu^2/\lambda)^{1/2}$. The field $\boldsymbol{\varphi}$ fluctuating around $\boldsymbol{\varphi}_0$ is given by $\boldsymbol{\varphi}=\left(\varphi_1',\varphi_2',\cdots,v+\varphi_N'\right)$, so that the Lagrangian up to $O(\varphi^3)$ becomes:

$$\begin{aligned}L&=\frac{1}{2}\sum_{i=1}^{N}\left(\partial\varphi_i'\right)^2+\frac{\mu^2}{2}\left[v^2+2v\varphi_N'+\boldsymbol{\varphi}'^2\right]-\frac{\lambda}{4}\left[v^4+4v^2\varphi_N'^{\,2}+4v^3\varphi_N'+2v^2\boldsymbol{\varphi}'^2\right]+O\left(\varphi^3\right)\\&=\frac{1}{2}\sum_{i=1}^{N}\left(\partial\varphi_i'\right)^2+\frac{\mu^4}{4\lambda}-\mu^2\varphi_N'^{\,2}+O\left(\varphi^3\right).\end{aligned} \tag{IV.31}$$

Consequently, all $(N - 1)$ fields $\varphi_1',\varphi_2',\cdots,\varphi_{N-1}'$ are massless, implying $(N - 1)$ Nambu-Goldstone bosons for the continuous $O(N)$ symmetry upon spontaneous symmetry breaking with the establishment of a specific vacuum expectation value.

To prove the Goldstone's theorem, we recall that there is a conserved charge Q associated with every continuous symmetry so that the Q commutes with the Hamiltonian $\mathcal{H}$ of the continuous symmetry system:

$$[\mathcal{H}, Q] = 0. \tag{IV.32}$$

We also recall that in quantum field theory, Q is associated with local currents and defined as:

$$Q = \int d^D\mathbf{x}\, J^0(\mathbf{x},t), \tag{IV.33}$$

where D denotes the spatial dimension and the conservation of charge implies that Q can be evaluated at any time. If we denote the ground state (or vacuum) as $|0\rangle$ and shift the Hamiltonian properly so that the condition $\mathcal{H}|0\rangle = 0$ is satisfied, we find that EQ. (IV.32) yields $Q|0\rangle = 0$ and $e^{i\theta Q}|0\rangle = 0$. In the event of spontaneous breaking of the continuous symmetry associated with Q, the ground state is no longer invariant under the symmetry transformation so that $Q|0\rangle \neq 0$. In this case, the energy of the symmetry broken state $Q|0\rangle$ can be evaluated by applying $\mathcal{H}$ to $Q|0\rangle$:

$$\mathcal{H}Q|0\rangle = \mathcal{H}Q|0\rangle - Q\mathcal{H}|0\rangle = \left[\mathcal{H}, Q\right]|0\rangle = 0. \tag{IV.34}$$

In other words, the energy of $Q|0\rangle$ is zero. If we consider a finite-momentum state $|s\rangle$:

$$|s\rangle = \int d^D\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} J^0(\mathbf{x},t)|0\rangle, \tag{IV.35}$$

we find that $|s\rangle$ has a spatial momentum $\mathbf{k}$, because we can act on $|s\rangle$ with the momentum operator P^j and use the fact that $P^j|0\rangle = 0$ to obtain $P^j|s\rangle = \int d^D\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\left[P^j, J^0(\mathbf{x},t)\right]|0\rangle = \int d^D\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\left(-i\partial^j\right)|0\rangle = k^j|s\rangle$. As the momentum $\mathbf{k}$ of the state $|s\rangle$ goes to zero, we find that $\lim_{\mathbf{k}\to 0}|s\rangle \to Q|0\rangle$ and its energy vanishes. Hence, $|s\rangle$ describes a massless particle, and its occurrence under spontaneous symmetry breaking of the continuous symmetry generated by Q exudes generality according to the above proof of the Goldstone's theorem.

Our proof of the Goldstone's theorem indicates that the number of Nambu-Goldstone bosons is equal to the number of conserved charges associated with a given Lagrangian that do not leave the ground state $|0\rangle$ invariant upon spontaneous symmetry breaking. For each such charge Q^α, we can construct a zero energy state $Q^\alpha|0\rangle$. In general, if the Lagrangian is invariant under the symmetry operations of group $\mathcal{G}$ with $n(\mathcal{G})$ generators, and if the ground state is only invariant under the symmetry operations of a subgroup $\mathcal{S}$ with $n(\mathcal{S})$ generators, then there are $[n(\mathcal{G}) - n(\mathcal{S})]$ Nambu-Goldstone bosons.

The occurrence of Nambu-Goldstone bosons is not limited to particle physics. In fact, it was first realized in condensed matter physics. One classic example is in the case of ferromagnetism. If the magnetic exchange interaction in a ferromagnet is completely isotropic, the spontaneous magnetization **M** of the ferromagnet upon cooling below its Curie temperature can in principle point along any direction in space if no external magnetic field is applied, because the Hamiltonian obeys the $SO(3) \sim SU(2)$ symmetry. However, in reality **M** can pick up a specific direction due to many reasons, including the presence of a very small external field, crystalline/geometric anisotropy, and grain boundary pinning effects. Once the direction of **M** is fixed, the continuous $SO(3)$ symmetry is broken, and a gapless spin wave with a local magnetization $\mathbf{M}(\mathbf{x})$ varying slowly from point to point in space-time is generated, which corresponds to the massless Nambu-Goldstone boson in the ferromagnet. Another classic example of spontaneous symmetry breaking in condensed matter physics is superconductivity. As discussed in Part III, the occurrence of superconductivity requires both the formation of bosonic Cooper pairs and condensation of the pairs, and the latter leads to the acquisition of macroscopic phase coherence in a superconductor and therefore spontaneous $U(1)$ symmetry breaking below the superconducting transition temperature. Similar to the occurrence of spin waves as the manifestation of the Goldstone mode in ferromagnets, "supercurrents" (*i.e.* electrical currents that move without dissipating energy) can be induced in a superconductor as the corresponding Nambu-Goldstone boson under spontaneous $U(1)$ symmetry breaking.

[Quantum fluctuations and spontaneous symmetry breaking]

So far our discussion of spontaneous symmetry breaking has been essentially classical. A natural question to pose at this point is the effect of quantum fluctuations on spontaneous symmetry breaking. Heuristically, we return to the scalar field theory involving spontaneous symmetry breaking of $O(N)$. Suppose that a vacuum expectation value is acquired in φ_N such that the fields $\varphi_1, \varphi_2, \cdots, \varphi_{N-1}$ are all rendered massless. We want to know whether quantum fluctuations in any one of these massless fields would wonder away from its ground state values. This issue can be addressed by considering the mean squared fluctuations of any one of the massless fields, say, φ_i ($1 \le i \le N-1$):

$$\left\langle \left[\varphi_i(0)\right]^2 \right\rangle = \frac{1}{Z}\int D\varphi\, e^{iS(\varphi)} \left[\varphi_i(0)\right]^2 = \lim_{x\to 0}\frac{1}{Z}\int D\varphi\, e^{iS(\varphi)}\left[\varphi_i(x)\varphi_i(0)\right]$$
$$= \lim_{x\to 0}\frac{1}{Z}\int \frac{d^d k}{(2\pi)^d}\left[\frac{e^{ikx}}{k^2+\mu^2-i\varepsilon}\right]_{\mu=0} = \lim_{x\to 0}\frac{1}{Z}\int \frac{d^d k}{(2\pi)^d}\left(\frac{e^{ikx}}{k^2}\right), \qquad \text{(IV.36)}$$

where we have used the fact that $\left[\varphi_i(x)\varphi_i(0)\right]$ is essentially associated with the massless boson propagator. Upon careful inspection of EQ. (IV.36), we note that for $d \ge 2$ an ultraviolet cutoff Λ must be introduced to prevent ultraviolet divergence. This cutoff can be easily justified if we consider a realistic physical system such as a ferromagnet, where the cutoff can be related to inverse of the lattice constant. On the other hand, for $d \le 2$, EQ. (IV.36) implies an infrared divergence for small k. In other words, our notion of spontaneous breaking of a continuous symmetry is not valid for $d \le 2$ because quantum fluctuations would drive the massless fields away from the ground state. The assertion that spontaneous breaking of a continuous symmetry is impossible in $d = 2$ is known as *the Coleman-Mermin-Wagner theorem*. So what is the physics implication of the Coleman-Mermin-Wagner theorem on two-dimensional ferromagnetic or superconducting thin films? [*N.B.*! We are concerned with condensed matter physics systems of finite transition temperatures, and we note that the Euclidean quantum field theory in d-dimensional space-time is equivalent to finite-temperature quantum statistics in d-dimensional space.] In the absence of impurities or other competing orders, topological defects known as vortex and anti-vortex pairs appear in two-dimensional ferromagnetic and superconducting or superfluid thin films (*i.e.* films with thicknesses smaller than the ferromagnetic or superconducting coherence length) near the bulk phase transition temperatures. These topological defects effectively disrupt the long-range spontaneous symmetry breaking and reduce the effective phase transition temperatures from the bulk values to the Kosterlitz-Thouless transition temperature, thus retaining the "respect" for the Coleman-Mermin-Wagner theorem. On the other hand, if impurities or competing orders (*i.e.*, low-temperature phases with eigen-energies comparable to that of superconductivity/ferromagnetism in a superconductor/ferromagnet) are present, they will tend to stabilize in these two-dimensional films (or structures with negligible planar coupling) and give rise to strong spatial inhomogeneity. The presence of strong spatial inhomogeneity also disrupts the occurrence of a global $U(1)$ continuous symmetry breaking in these systems, despite the retention of localized superconducting order. Indeed, such two-dimensional inhomogeneous phases have been empirically observed in some extremely two-dimensional perovskite cuprate superconductors $Bi_2Sr_2CaCu_2O_{8+x}$ and also in the "colossal magnetoresistance" (known as CMR) perovskite manganites.

Let's proceed with further investigation of the effect of quantum fluctuations. In our earlier discussion of spontaneous symmetry breaking, we search for the minima of the potential $V(\varphi)$ in order to find the vacuum expectation value of the ground state, which is essentially a semi-classical approximation. To incorporate quantum fluctuation effects, we want to find an effective potential $V_{\rm eff}(\varphi)$ whose minima provide the true vacuum states of the theory without any approximation. We shall show in the following that the effective potential, when considered to the first order of quantum fluctuations, consists of the classical potential energy $V(\varphi)$, which is associated with the background φ-field, plus a second term that corresponds to the vacuum energy density of a scalar field with a mass corrected for the interaction of the particle with the background φ. We begin with consideration of the φ^4-theory that includes proper counter terms (for renormalization purposes, which we do not address at present):

$$L = \frac{1}{2}\left[\left(\partial\varphi\right)^2 - \mu_p{}^2\varphi^2\right] - \frac{\lambda_p}{4!}\varphi^4 + A\left(\partial\varphi\right)^2 + B\varphi^2 + C\varphi^4 . \qquad \text{(IV.37)}$$

Similar to what we have seen in the case of Landau-Ginzburg theory, we note that for $\mu_p^2 > 0$ the action is minimized at $\varphi = 0$, and quantization of the small fluctuations around $\varphi = 0$ yields scalar particles scattering off each other. In contrast, for $\mu_p^2 < 0$, the action is minimized at some non-trivial vacuum expectation values $\varphi_{\rm min}$, resulting in spontaneous symmetry breaking, as discussed earlier. The following question naturally arises from the aforementioned findings: Do quantum fluctuations break the symmetry when $\mu_p = 0$? We shall examine this issue by calculating the connected generating functional $W(J)$ in the presence of the source $J(x)$, where $W(J)$ is defined through the path integral:

$$Z = \exp\left[iW(J)\right] = \int D\varphi \exp\left[i\int d^4x\left(L + J(x)\varphi(x)\right)\right] \equiv \int D\varphi \exp\left[i\left(S(\varphi) + J\varphi\right)\right]. \qquad \text{(IV.38)}$$

It should be emphasized that in EQ. (IV.38) we have effectively removed the "disconnected graphs" (*i.e.*, graphs without external legs) by normalizing the path integral $Z(J)$ relative to the contribution $Z(J{=}0)$ from the disconnected graphs. Thus, following our earlier discussions in Part III, $W(J)$ can be expanded in the functional Taylor series:

$$W(J) = \sum_n \frac{1}{n!}\int d^4x_1 \cdots d^4x_n\, G^{(n)}\left(x_1 \cdots x_n\right) J\left(x_1\right)\cdots J\left(x_n\right), \qquad \text{(IV.39)}$$

where the successive coefficients in the series are the connected Green's functions, with $G^{(n)}$ representing the sum of all connected Feynman diagrams with n external legs. Specifically, $G^{(n)}$ can be obtained by differentiating $W(J)$ n times relative to J.

Next we define the classical field φ_c:

$$\varphi_c(x) \equiv \frac{\delta W(J)}{\delta J(x)} = \frac{1}{Z}\int D\varphi \exp\left[iS(\varphi) + J\varphi\right]\varphi(x) = \left\langle 0\middle|\hat{\varphi}\middle|0\right\rangle, \qquad \text{(IV.40)}$$

and the effective action $\Gamma\left(\varphi_c\right)$:

$$\Gamma\left(\varphi_c\right) \equiv W(J) - \int d^4x\, J\left(x\right)\varphi_c\left(x\right). \qquad \text{(IV.41)}$$

[You may view the definition of the effective action $\Gamma\left(\varphi_c\right)$ as a functional that picks up the contributions from quantum fluctuations because it effectively "removes" the classical part of the contribution associated with the source.] It follows from EQ. (IV.41) that

$$\frac{\delta\Gamma\left(\varphi_c\right)}{\delta\varphi_c(x)} = -J(x). \qquad \text{(IV.42)}$$

We can also expand the effective action is a manner similar to that in EQ. (IV.39) for $W(J)$. That is,

$$\Gamma(\varphi_c) = \sum_n \frac{1}{n!}\int d^4x_1 \cdots d^4x_n \; \Gamma^{(n)}\left(x_1 \cdots x_n\right)\varphi_c\left(x_1\right)\cdots\varphi_c\left(x_n\right). \quad \text{(IV.43)}$$

The successive coefficients in the above series are the one-particle-irreducible Green's functions, with $\Gamma^{(n)}$ representing the sum of all one-particle-irreducible Feynman diagrams with *n* external legs. [*N.B.*: A one-particle-irreducible Feynman diagram is a connected diagram that cannot be disconnected by cutting a single internal line. By conventions, these diagrams are evaluated with no propagators on the external lines.] However, rather than making the expansion according to EQ. (IV.43) for the effective action, it is more convenient to expand $\Gamma\left(\varphi_c\right)$ in powers of momentum about the point where all external momenta vanish. In position space, such an expansion leads to the following form:

$$\Gamma(\varphi_c) = \int d^4x \left[-V_{eff}\left(\varphi_c\right) + \frac{1}{2}\left(\partial\varphi_c\right)^2 Z\left(\varphi_c\right) + \ldots \right], \quad \text{(IV.44)}$$

where … refers to terms with higher and higher powers of ∂. Comparing EQs. (IV.44) and (IV.43), we can see that the *n*th derivative of $V_{\text{eff}}(\varphi)$ is the sum of all one-particle-irreducible graphs with *n* vanishing external momenta. Also, $V_{\text{eff}}(\varphi_c)$ – an ordinary function rather than a functional, is called *the effective potential*. More specifically, we may take the Fourier transformation of Γ:

$$\Gamma^{(n)}(x_1,\cdots,x_n) = \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}(2\pi)^4\delta^4\left(k_1+\cdots+k_n\right)e^{i(k_1\cdot x_1+\cdots+k_n\cdot x_n)}\tilde{\Gamma}^{(n)}(k_1,\cdots,k_n), \quad \text{(IV.45)}$$

so that

$$\Gamma(\varphi_c) = \sum_n \frac{1}{n!}\int d^4x_1\cdots d^4x_n \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}e^{i(k_1\cdot x_1+\cdots+k_n\cdot x_n)}\left[\tilde{\Gamma}^{(n)}(0,\cdots,0)\varphi_c\left(x_1\right)\cdots\varphi_c\left(x_n\right)+\cdots\right]$$
$$= \int d^4x \sum_n \frac{1}{n!}\left[\tilde{\Gamma}^{(n)}(0,\cdots,0)\left(\varphi_c(x)\right)^n+\cdots\right]. \quad \text{(IV.46)}$$

Hence, the effective potential is related to Γ via the relation:

$$V_{eff}\left(\varphi_c\right) = -\sum_n \frac{1}{n!}\left[\tilde{\Gamma}^{(n)}(0,\cdots,0)\left(\varphi_c(x)\right)^n\right]. \quad \text{(IV.47)}$$

This expression implies that the effect of summing over the loop expansion produces a series of Feynman diagrams with zero momenta $\tilde{\Gamma}^{(n)}(0,\cdots,0)$ such that they act as the effective potential and are associated with a new effective action.

The above discussions have prepared us for the studies of the spontaneous symmetry breaking. Suppose that our Lagrangian density possesses an internal symmetry, which is broken if the quantum field φ develops a non-zero vacuum expectation value even when the source $J(x)$ vanishes. From EQs. (IV.40) and (IV.42), this occurs if

$$\frac{\delta\Gamma\left(\varphi_c\right)}{\delta\varphi_c(x)} = 0 \quad \text{(IV.48)}$$

for some non-zero values of φ_c. Moreover, since we are typically interested in cases where the vacuum expectation value is translational invariant, we can simplify the condition in EQ. (IV.48) into

$$\frac{dV_{eff}}{d\varphi_c} = 0 \tag{IV.49}$$

for some non-zero values of φ_c according to EQ. (IV.47). The value of φ_c for the minimum in $V_{\text{eff}}(\varphi)$ occurs is the expectation value $\langle \varphi_c \rangle$ of the new vacuum that incorporates effects of quantum fluctuations. Once the expectation value is known, we can obtain the effective mass $\mu(\varphi)$ of the meson (which now contains additional contributions due to the interaction between the meson and the background scalar field φ) by evaluating it at $\langle \varphi_c \rangle$ according to the following:

$$\mu^2(\varphi_c) = \left[\frac{d^2 V_{eff}}{d\varphi_c{}^2} \right]_{\langle \varphi_c \rangle} . \tag{IV.50}$$

Similarly, if we define the four-point function at $\langle \varphi_c \rangle$ as the coupling constant λ, we have

$$\lambda(\varphi_c) = \left[\frac{d^4 V_{eff}}{d\varphi_c{}^4} \right]_{\langle \varphi_c \rangle} . \tag{IV.51}$$

Next, we must calculate $W(J)$ in order to obtain all the aforementioned results explicitly. A common approach is to find the so-called "steepest descent point" φ_s in the path integral in EQ. (IV.38), namely, the solution to the following variation:

$$\left. \frac{\delta\left[S(\varphi) + \int d^4x\, J(x)\varphi(x) \right]}{\delta\varphi} \right|_{\varphi_s} = 0 , \tag{IV.52}$$

or more explicitly for $S(\varphi) = \int d^4x\, L(\varphi, \partial\varphi) = \int d^4x \left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi) \right] = -\int d^4x \left[\frac{1}{2}\varphi\partial^2\varphi + V(\varphi) \right]$:

$$\partial^2\varphi_s(x) + V'\left[\varphi_s(x)\right] = J(x) . \tag{IV.53}$$

Now if we express the field as $\varphi = \varphi_s + \tilde{\varphi}$, the path integral in EQ. (IV.38) can be rewritten into the following (with $\hbar$ restored):

$$\begin{aligned} Z = \exp\left[iW(J)\right] &= \int D\varphi \exp\left[(i/\hbar)\left(S(\varphi) + J\varphi\right) \right] \\ &\simeq \exp\left\{ (i/\hbar)\left[S(\varphi_s) + J\varphi_s \right] \right\} \int D\tilde{\varphi} \exp\left\{ (i/\hbar) \int d^4x \frac{1}{2}\left[(\partial\tilde{\varphi})^2 - V''(\varphi_s)\tilde{\varphi}^2 \right] \right\} \\ &\simeq \exp\left\{ (i/\hbar)\left[S(\varphi_s) + J\varphi_s \right] - \frac{1}{2}\operatorname{Tr}\log\left[\partial^2 + V''(\varphi_s) \right] \right\} . \end{aligned} \tag{IV.54}$$

Therefore we have determined the connected generating functional $W(J)$:

$$W(J) = \left[S(\varphi_s) + J\varphi_s \right] + \frac{i\hbar}{2}\operatorname{Tr}\left\{ \log\left[\partial^2 + V''(\varphi_s) \right] \right\} + O\left(\hbar^2\right) . \tag{IV.55}$$

In light of EQ. (IV.40) for the definition of the classical field φ_c, we vary EQ. (IV.55) relative to *J* and find that

$$\varphi_c = \frac{\delta W(J)}{\delta J} = \frac{\delta\left[S(\varphi_s) + J\varphi_s\right]}{\delta\varphi_s}\frac{\delta\varphi_s}{\delta J} + \varphi_s + O(\hbar) = \varphi_s + O(\hbar), \tag{IV.56}$$

where we have used the condition in EQ. (IV.52). Consequently, we find that the classical field is equal to the steepest descent value φ_s to the leading order in $\hbar$, and we are now in business to derive the effective action $\Gamma(\varphi_c)$ and various other important quantities discussed earlier.

Using EQs. (IV.41) and (IV.45), we obtain the effective action:

$$\Gamma(\varphi_c) = S(\varphi_c) + \frac{i\hbar}{2}\mathrm{Tr}\left\{\log\left[\partial^2 + V''(\varphi_c)\right]\right\} + O(\hbar^2). \tag{IV.57}$$

In general it is difficult to evaluate the trace for any arbitrary $\varphi_c(x)$, and if we are primarily interested in systems with translation invariant vacuum expectation values $\langle\varphi_c\rangle$, we can simplify the problem by assuming that φ_c is independent of *x*, $V''(\varphi_c)$ is a constant and is related to the effective mass of the meson as given in EQ. (IV.50), and $\left[\partial^2 + V''(\varphi_c)\right]$ is translation invariant. Thus,

$$\begin{aligned}\mathrm{Tr}\left\{\log\left[\partial^2 + V''(\varphi_c)\right]\right\} &= \int d^4x \left\langle x\right|\log\left[\partial^2 + V''(\varphi_c)\right]\left|x\right\rangle \\ &= \int d^4x \int\frac{d^4k}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}\left\langle x|k'\right\rangle\left\langle k'\right|\log\left[\partial^2 + V''(\varphi_c)\right]\left|k\right\rangle\left\langle k|x\right\rangle \\ &= \int d^4x\int\frac{d^4k}{(2\pi)^4}\log\left[-k^2 + V''(\varphi_c)\right].\end{aligned} \tag{IV.58}$$

From EQs. (IV.44), (IV.57) and (IV.58), we obtain the Coleman-Weinberg effective potential:

$$V_{eff}(\varphi_c) = V(\varphi_c) - \frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\log\left[\frac{k^2 - V''(\varphi_c)}{k^2}\right] + O(\hbar^2), \tag{IV.59}$$

where we have added a φ_c-independent term to make the argument of the logarithm dimensionless. In addition, using $V(\varphi_c) = \frac{1}{2}\mu_p^2\varphi_c^2 + \frac{1}{4!}\lambda\varphi_c^4$ and EQ. (IV.50), we obtain the effective mass for the meson:

$$\left[\mu(\varphi_c)\right]^2 = V''(\varphi_c) = \mu_p^2 + \frac{1}{2}\lambda\varphi_c^2. \tag{IV.60}$$

In other words, the mass squared of the meson $\mu^2(\varphi_c)$ is corrected by a term $\left(\lambda\varphi_c{}^2/2\right)$ due to the interaction of the meson with the background field φ. Moreover, the logarithmic term in EQ. (IV.59) is in fact associated with the vacuum energy density of a scalar field (such as a meson field) with the effective mass determined by EQ. (IV.60). [You may consult the derivation in Zee's book that leads to EQ. (3) in Chapter II.5.] This excess energy correction is in addition to the classical energy density $V(\varphi)$ contained in the background φ.

The integral in EQ. (IV.59) is quadratically divergent, and it in fact consists of an infinite number of Feynman diagrams. To see the relevant Feynman diagrams associated with the integral, we expand the logarithm in EQ. (IV.59) into an infinite series and ignore the classical energy density for now:

$$V_{eff}\left(\varphi_c\right) = +\sum_{n=1}^{\infty} \frac{i\hbar}{2n}\int \frac{d^4k}{(2\pi)^4}\left[\frac{V''(\varphi_c)}{k^2}\right]^n . \tag{IV.61}$$

For $V''(\varphi) = \left(\lambda\varphi^2/2\right)$, each diagram consists of a loop with n interaction vertices attached to it. Some of the representative diagrams are shown in Fig. IV.3.3. In fact, the extra term inserted in EQ. (IV.59) to yield dimensionless argument in the logarithm is not arbitrarily chosen; it ensures consistency with the sum of the infinite one-loop Feynman diagrams.

To handle the divergence in EQ. (IV.59), we introduce proper counter terms so that

$$V_{eff}\left(\varphi_c\right) = V\left(\varphi_c\right) + \frac{\hbar}{2}\int \frac{d^4k_E}{(2\pi)^4}\log\left[\frac{k_E^{\ 2} + V''(\varphi_c)}{k_E^{\ 2}}\right] + B\varphi_c^{\ 2} + C\varphi_c^{\ 4} + O\left(\hbar^2\right), \tag{IV.62}$$

where we have made the Wick rotation into a Euclidean integral. Using the following identity

$$\int d^4k\, F\left(k^2\right) = \pi^2\int_0^{\infty} dk^2\left[k^2 F\left(k^2\right)\right] \tag{IV.63}$$

and choosing a cutoff for the integration in EQ. (IV.62) up to $k_E^2 = \Lambda^2$, we obtain (dropping $\hbar$ for now)

$$V_{eff}\left(\varphi_c\right) = V\left(\varphi_c\right) + \left\{\frac{\Lambda^2}{32\pi^2}V''(\varphi_c) - \frac{\left[V''(\varphi_c)\right]^2}{64\pi^2}\log\left[\frac{e^{1/2}\Lambda^2}{V''(\varphi_c)}\right]\right\} + B\varphi_c^{\ 2} + C\varphi_c^{\ 4} . \tag{IV.64}$$

Indeed we have just enough counter terms in EQ. (IV.64) to cancel the divergence because $V(\varphi_c)$ is a quartic polynomial in φ_c.

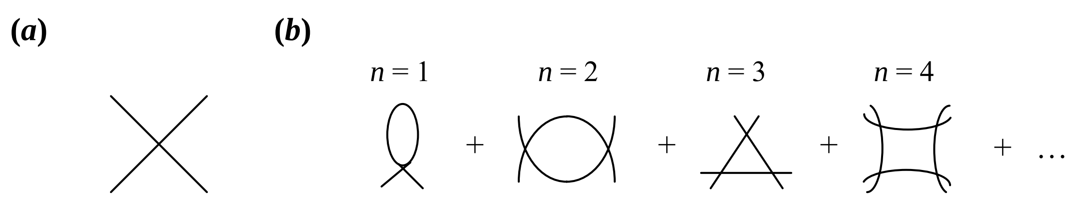

Fig. IV.3.3 The **(*a*)** no-loop and **(*b*)** one-loop approximation for the effective potential.

With EQ. (IV.64) in hand, we are ready to answer the question posed earlier. That is, does symmetry breaking occurs due to quantum fluctuations if $\mu_p = 0$ in EQ. (IV.37) so that $V(\varphi_c) = (1/4!)\lambda\varphi_c^{\ 4}$? Inserting $V(\varphi_c)$ into EQ. (IV.64), we obtain

$$V_{eff}\left(\varphi_c\right) = \left(\frac{\Lambda^2}{64\pi^2}\lambda + B\right)\varphi_c^{\ 2} + \left[\frac{1}{4!}\lambda + \frac{\lambda^2}{\left(16\pi\right)^2}\log\left(\frac{\lambda\varphi_c^{\ 2}}{2e^{1/2}\Lambda^2}\right) + C\right]\varphi_c^{\ 4} + O\left(\lambda^3\right). \tag{IV.65}$$

From EQ. (IV.65), we see that the Λ-dependent terms can be absorbed by choosing proper coefficients *B* and *C*.

We can also evaluate the coupling constant associated with the effective potential in EQ. (IV.65) using the definition in EQ. (IV.51) with a new vacuum expectation value $\langle \varphi_c \rangle$. The choice of an expectation value $\langle \varphi_c \rangle$ is equivalent to fixing the effective mass of the meson because $\mu^2(\langle \varphi_c \rangle) = V''(\langle \varphi_c \rangle) = \lambda \langle \varphi_c \rangle^2 / 2$. Thus, we find that:

$$\left[\frac{d\lambda(\varphi_c)}{d\varphi_c}\right]_{\langle \varphi_c \rangle} = \frac{d}{d\varphi_c}\left[\frac{d^4 V_{eff}}{d\varphi_c{}^4}\right]_{\langle \varphi_c \rangle} = \frac{3}{16\pi^2}\frac{\lambda^2}{\langle \varphi_c \rangle} + O(\lambda^3). \tag{IV.66}$$

For simplicity, we can drop the subscript *c*, define $\lambda(\langle \varphi \rangle) \equiv \lambda_{\langle \varphi \rangle}$ and rewrite EQ. (IV.66) into

$$\langle \varphi \rangle \left[\frac{d\lambda_{\langle \varphi \rangle}}{d\langle \varphi \rangle}\right] = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) = \frac{3}{16\pi^2}\left(\lambda_{\langle \varphi \rangle}\right)^2 + O\left(\lambda_{\langle \varphi \rangle}{}^3\right). \tag{IV.67}$$

What EQ. (IV.67) implies is that the effective coupling constant $\lambda_{\langle \varphi \rangle}$ that incorporates quantum fluctuations in fact depends on the vacuum expectation value $\langle \varphi_c \rangle$ and therefore on the effective mass $\mu(\langle \varphi_c \rangle)$. Following similar arguments and imposing the new normalization condition $Z(\langle \varphi_c \rangle) = 1$ as well as the condition $\lambda = \lambda_{\langle \varphi \rangle} \equiv \left(d^4 V_{eff} / d\varphi_c{}^4\right)_{\langle \varphi \rangle}$, it can be shown that

$$B = -\frac{\Lambda^2}{64\pi^2}\lambda, \qquad C = \frac{\lambda^2}{(16\pi)^2}\left[\log\left(\frac{2\Lambda^2}{\lambda \langle \varphi_c \rangle^2}\right) - \frac{11}{3}\right]. \tag{IV.68}$$

Finally, collecting all necessary terms from the above calculations we obtain:

$$V_{eff}(\varphi) = \frac{1}{4!}\lambda_{\langle \varphi \rangle}\varphi^4 + \frac{\lambda_{\langle \varphi \rangle}{}^2}{(16\pi)^2}\varphi^4\left[\log\left(\frac{\varphi^2}{\langle \varphi \rangle^2}\right) - \frac{25}{6}\right] + O\left(\lambda_{\langle \varphi \rangle}{}^3\right). \tag{IV.69}$$

As expected, we find that EQ. (IV.69) is independent of the cutoff Λ. However, the logarithmic term in EQ. (IV.69) tells us that the effective potential is negative and the magnitude diverges with small φ, indicating that the incorporation of quantum fluctuations in fact breaks the symmetry of the φ^4-theory.

In fact, we can extend our discussions on scalar bosonic fields such as the φ^4-theory to theories involving fermions. For instance, if we add to the bosonic field the Dirac Lagrangian plus a coupling term between the fermion and boson fields, $\bar{\psi}(i\partial\!\!\!/ - m - f\varphi)\psi$, we arrive at the following path integral:

$$Z = \int D\varphi\, D\bar{\psi}\, D\psi\, \exp\left\{i\int d^4x\left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi) + \bar{\psi}(i\partial\!\!\!/ - m - f\varphi)\psi\right]\right\}. \tag{IV.70}$$

Integrating over the fermion fields first, we obtain

$$Z = \int D\varphi\, \exp\left\{i\int d^4x\left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right] + \mathrm{Tr}\left[\log(i\partial\!\!\!/ - m - f\varphi)\right]\right\}. \tag{IV.71}$$

We can evaluate the trace of the fermion term in EQ. (IV.71) as follows:

$$\begin{aligned}\mathrm{Tr}\left[\log\left(i\partial\!\!\!/-m-f\varphi\right)\right]&=\int d^4x\left\langle x\right|\log\left(i\partial\!\!\!/-m-f\varphi\right)\left|x\right\rangle\\&=\int d^4x\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\left\langle x|p'\right\rangle\left\langle p'\right|\log\left(i\partial\!\!\!/-m-f\varphi\right)\left|p\right\rangle\left\langle p|x\right\rangle.\\&=\int d^4x\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\log\left(p\!\!\!/-m-f\varphi\right)\end{aligned}\tag{IV.72}$$

Therefore the fermion field contributes to the total effective potential $V_{\text{eff}}(\varphi)$ an extra term:

$$V_F(\varphi)\equiv+i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\log\left(\frac{p\!\!\!/-m-f\varphi}{p\!\!\!/}\right),\tag{IV.73}$$

where we have inserted a term in the denominator of the logarithm to make the argument dimensionless. The physical significance of this insertion is analogous to that in EQ. (IV.59) for scalar bosons. Physically the fermion contribution $V_F(\varphi)$ represents the vacuum energy of a fermion with an effective mass $m(\varphi)\equiv m+f\varphi$. In other words, the effective mass of the fermion is modified because of its interaction with the background scalar field φ.

We may further simplify EQ. (IV.73) by the following consideration:

$$\begin{aligned}&\mathrm{Tr}\log\left[p\!\!\!/-m(\varphi)\right]=\mathrm{Tr}\log\left\{\gamma^5\left[p\!\!\!/-m(\varphi)\right]\gamma^5\right\}=\mathrm{Tr}\log\left[-p\!\!\!/-m(\varphi)\right]\\&=\frac{1}{2}\mathrm{Tr}\left\{\log\left[p\!\!\!/-m(\varphi)\right]+\log\left[p\!\!\!/+m(\varphi)\right]\right\}+\frac{1}{2}\mathrm{Tr}\log(-1)=\frac{1}{2}\mathrm{Tr}\log\left[(-1)\left(p^2-m(\varphi)^2\right)\right].\end{aligned}\tag{IV.74}$$

Hence, the fermion field from EQs. (IV.73) and (IV.74) becomes (with $\hbar$ restored):

$$\begin{aligned}V_F(\varphi)&=+i\hbar\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\log\left(\frac{p\!\!\!/-m(\varphi)}{p\!\!\!/}\right)=+i\hbar\int\frac{d^4p}{(2\pi)^4}\frac{1}{2}\mathrm{Tr}\log\left(\frac{p^2-m(\varphi)^2}{p^2}\right)\\&=2i\hbar\int\frac{d^4p}{(2\pi)^4}\log\left(\frac{p^2-m(\varphi)^2}{p^2}\right).\end{aligned}\tag{IV.75}$$

It is interesting to compare the sign in EQ. (IV.75) for the fermion field with that in EQ. (IV.59) for the bosonic scalar field. This is consistent with our understanding that the correction terms due to quantum fluctuations are associated with the vacuum energies of the bosonic and fermionic particles.

Our discussions on the scalar boson and fermion fields have led to the conclusion that corrections associated with quantum fluctuations can lead to spontaneous symmetry breaking for systems that their semi-classical approximations would not have indicated such symmetry breaking. Moreover, Coleman and Weinberg [*Phys. Rev. D* **7**, 1888 (1973)] have shown that both scalar mesons and vector photons can acquire a finite effective mass as the result of radioactive corrections, and these studies of massless scalar and vector electrodynamics, when extended to non-Abelian gauge theories, also yield similar results. In the special case of photons, Coleman and Weinberg have considered the electrodynamics of a complex scalar field:

$$L = \left[\left(\partial^{\mu} + ieA^{\mu}\right)\varphi^{\dagger}\right]\left[\left(\partial_{\mu} - ieA_{\mu}\right)\varphi\right] + \mu^2\varphi^{\dagger}\varphi - \lambda\left(\varphi^{\dagger}\varphi\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{IV.76}$$

The calculations can be simplified by employing the Landau gauge for the photon propagator:

$$D_{\mu\nu} = -i\left[\frac{\eta_{\mu\nu} - \left(k_{\mu}k_{\nu}/k^2\right)}{k^2 + i\varepsilon}\right], \tag{IV.77}$$

and it is found that the Lagrangian in EQ. (IV.76) acquires a correction term of the following form to the effective potential $V_{\text{eff}}(\varphi)$:

$$V_A(\varphi) \sim \int \frac{d^4k}{(2\pi)^4}\log\left[\frac{k^2 - \left(e^2\varphi^{\dagger}\varphi\right)}{k^2}\right] \equiv \int \frac{d^4k}{(2\pi)^4}\log\left[\frac{k^2 - M(\varphi)^2}{k^2}\right]. \tag{IV.78}$$

Similar to EQs. (IV.59) and (IV.75) for the cases of mesons and fermions, the effective mass acquired by photons is the result of interaction of photons with background scalar φ-field, whereas the energy correction in EQ. (IV.78) is associated with the vacuum energy of quantum fluctuations.

The symmetry breaking that we have discussed thus far is associated with explicit symmetries and is generally related to physical observables. In the next section we consider non-abelian gauge theory and discuss symmetry breaking associated with "hidden" gauge symmetries. As we shall see in Part VI, the concept of non-abelian gauge symmetry has important implications on topological field theory.

IV.4 Non-Abelian Gauge Theory & the Anderson-Higgs Mechanism

The gauge invariance discussed in the context of quantum electrodynamics is associated with the vector potential of the electrodynamic field, so that the gauge symmetry is $U(1)$ and the gauge transformation is abelian. We may consider a more general situation for the following Lagrangian density under the $SU(N)$ gauge transformation:

$$L = \partial\varphi^\dagger \partial\varphi - V\left(\varphi^\dagger\varphi\right), \tag{IV.79}$$

where $\varphi(x) = \{\varphi_1(x), \varphi_2(x), \cdots, \varphi_N(x)\}$ is an N-component complex scalar field, and $V\left(\varphi^\dagger\varphi\right)$ denotes the potential of the system and is a polynomial of $\varphi^\dagger\varphi$. The scalar field transforms as $\varphi(x) \to U\varphi(x)$ with U being an element of $SU(N)$, so that $\varphi^\dagger \to \varphi^\dagger U^\dagger$ and $U^\dagger U = 1$. Clearly $\partial\varphi^\dagger\partial\varphi \to \partial\varphi^\dagger\partial\varphi$ and $\varphi^\dagger\varphi \to \varphi^\dagger\varphi$, so that the Lagrangian density in EQ. (IV.79) remains invariant.

However, what would happen to EQ. (IV.79) if we consider a new type of transformation that varies from place to place in space-time so that $U = U(x)$? This question was first posed by C. N. Yang and R. Mills in 1954, which led to the celebrated Yang-Mills non-abelian gauge theory.

[Non-abelian gauge field and gauge transformation]

Suppose that we define $U = U(x)$ and apply the transformation to EQ. (IV.79). Obviously $\varphi^\dagger\varphi$ remains invariant, whereas the kinetic term $\partial\varphi^\dagger\partial\varphi$ can no longer be invariant, because we have

$$\partial_\mu\varphi \to \partial_\mu(U\varphi) = U\partial_\mu\varphi + \left(\partial_\mu U\right)\varphi = U\left[\partial_\mu\varphi + \left(U^\dagger\partial_\mu U\right)\varphi\right], \tag{IV.80}$$

and the extra term $\left(U^\dagger\partial_\mu U\right)\varphi$ cannot be trivially removed. On the other hand, if we generalize the ordinary derivative ∂_μ to a covariant derivative D_μ by defining:

$$D_\mu\varphi(x) \equiv \left(\partial_\mu - iA_\mu\right)\varphi(x), \tag{IV.81}$$

where A_μ is a gauge potential, we can find a suitable gauge that yields $D_\mu\left[U(x)\varphi(x)\right] = U(x)D_\mu\varphi(x)$. The gauge potential that satisfies this condition yields the following gauge transformation

$$A_\mu \to UA_\mu U^\dagger - i\left(\partial_\mu U\right)U^\dagger = UA_\mu U^\dagger + iU\left(\partial_\mu U^\dagger\right), \tag{IV.82}$$

where we have used the fact that $U^\dagger U = 1$ so that $\left(\partial_\mu U\right)U^\dagger = -U\left(\partial_\mu U^\dagger\right)$. Thus, the gauge potential A_μ satisfying the relation in EQ. (IV.82) is referred to as the non-abelian gauge potential and EQ. (IV.82) is a non-abelian gauge transformation.

Based on EQ. (IV.82), we find that the gauge potential A_μ has the following properties:

1. The gauge potential A_μ must be an $(N \times N)$ matrix and $A_\mu^\dagger = A_\mu$ from EQ. (IV.82), so that it is consistent to take A_μ to be hermitian. As an example, in the case of $SU(2)$ symmetry, the unitary matrix is given by

$U = \exp\left(i\theta \bullet \tau/2\right)$ where $\theta \bullet \tau = \theta^a \tau^a$ with τ^a being the generators of the *SU*(2) group, which are the familiar Pauli matrices.

2. In the general case of *SU*(*N*) symmetry, the unitary matrix is given by $U = \exp\left(i\theta \bullet T\right)$, where T^a are the generators. Under infinitesimal non-abelian gauge transformation, we find $U \approx 1 + \left(i\theta \bullet T\right)$ so that EQ. (IV.82) is reduced to

$$A_\mu \to A_\mu + i\theta^a \left[T^a, A_\mu\right] + \partial_\mu \theta^a T^a . \tag{IV.83}$$

3. From EQ. (IV.83) we note that the trace of A_μ does not transform. Therefore, we can take A_μ to be traceless as well as hermitian, and we can always write $A_\mu = A_\mu^a T^a$ so that the matrix field A_μ can be decomposed into component fields A_μ^a.

4. Noting that the generators of *SU*(*N*) satisfy the relation $\left[T^a, T^b\right] = i f^{abc} T^c$, where the numbers f^{abc} are referred to as structure constants. Thus, EQ. (IV.83) leads to an expression for the component field A_μ^a:

$$A_\mu^a \to A_\mu^a - f^{abc}\theta^b A_\mu^c + \partial_\mu \theta^a . \tag{IV.84}$$

We further note that in the event of θ being independent of *x*, the component fields A_μ^a transform as the adjoint representation of the group.

5. In the special case of the *U*(1) symmetry, we have $U(x) = e^{i\theta(x)}$ and A_μ is just the abelian gauge potential so that EQ. (IV.82) is consistent with the usual abelian gauge transformation.

In general, the transformation *U* that depends on the space-time coordinates *x* is known as a *gauge transformation* or *local transformation*. A Lagrangian *L* is said to be gauge invariant if it is invariant under a gauge transformation. Thus, the following Lagrangian density is gauge invariant

$$L = \left(D_\mu \varphi\right)^\dagger \left(D_\mu \varphi\right) - V\left(\varphi^\dagger \varphi\right). \tag{IV.85}$$

To make connection of EQ. (IV.85) to such 2-forms as electromagnetic fields in Maxwell's equations, we first define a new gauge potential $\tilde{A}_\mu \equiv -iA_\mu$ so that $D_\mu \equiv \partial_\mu + \tilde{A}_\mu$. To construct 2-forms from the 1-form gauge potential, we expect linear combinations of such terms as $d\tilde{A}$ and $\tilde{A}^2$. Indeed it is found that for

$$\tilde{F} = d\tilde{A} + \tilde{A}^2 \equiv -iF , \tag{IV.86}$$

we have

$$d\tilde{A} + \tilde{A}^2 \to U\left(d\tilde{A} + \tilde{A}^2\right)U^\dagger \quad \Rightarrow \quad \tilde{F} \to U\tilde{F}U^\dagger , \tag{IV.87}$$

which can be verified by the following consideration. First, the term $\tilde{A}^2$ is given by

$$\tilde{A}^2 = \tilde{A}_\mu \tilde{A}_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}\left[\tilde{A}_\mu, \tilde{A}_\nu\right] dx^\mu \wedge dx^\nu , \tag{IV.88}$$

which does not vanish for non-abelian gauge potentials. Next, consider $\tilde{A}$ under the gauge transformation in EQ. (IV.82):

$$\tilde{A} \to U\tilde{A}U^{\dagger} + UdU^{\dagger} \quad , \tag{IV.89}$$

with U being a 0-form so that $dU^{\dagger} = \partial_{\mu} U^{\dagger} dx^{\mu}$. Applying d to EQ. (IV.89), we obtain

$$d\tilde{A} \to d\left(U\tilde{A}U^{\dagger} + UdU^{\dagger}\right) = dU\tilde{A}U^{\dagger} + Ud\tilde{A}U^{\dagger} - U\tilde{A}\,dU^{\dagger} + dU\,dU^{\dagger} . \tag{IV.90}$$

In addition, from EQ. (IV.89) we obtain $\tilde{A}^2$ as follows:

$$\tilde{A}^2 \to \left(U\tilde{A}U^{\dagger} + UdU^{\dagger}\right)\left(U\tilde{A}U^{\dagger} + UdU^{\dagger}\right) = U\tilde{A}^2U^{\dagger} + U\tilde{A}\,dU^{\dagger} - dU\tilde{A}U^{\dagger} - dU\,dU^{\dagger} , \tag{IV.91}$$

where we have used the identities $\left(\partial_{\mu}U\right)U^{\dagger} = -U\left(\partial_{\mu}U^{\dagger}\right)$ and $-dU\tilde{A}U^{\dagger} = U\tilde{A}\,dU^{\dagger}$. By combining EQs. (IV.90) and (IV.91), we obtain EQ. (IV.87) as asserted earlier.

If we rewrite EQ. (IV.86) explicitly in terms of the components, we have

$$\tilde{F} = \left(\partial_{\mu}\tilde{A}_{\nu} + \tilde{A}_{\mu}\tilde{A}_{\nu}\right)dx^{\mu}\wedge dx^{\nu} = \frac{1}{2}\left(\partial_{\mu}\tilde{A}_{\nu} - \partial_{\nu}\tilde{A}_{\mu} + \left[\tilde{A}_{\mu}, \tilde{A}_{\nu}\right]\right)dx^{\mu}\wedge dx^{\nu} \equiv \left(1/2!\right)\tilde{F}_{\mu\nu}\,dx^{\mu}\wedge dx^{\nu} , \tag{IV.92}$$

so that

$$\tilde{F}_{\mu\nu} = \left(\partial_{\mu}\tilde{A}_{\nu} - \partial_{\nu}\tilde{A}_{\mu} + \left[\tilde{A}_{\mu}, \tilde{A}_{\nu}\right]\right) \quad \Leftrightarrow \quad F_{\mu\nu} = \left(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - i\left[A_{\mu}, A_{\nu}\right]\right). \tag{IV.93}$$

Furthermore, if we express the gauge potential A_{μ} in terms of its component fields A_{μ}^{a} associated with the individual generators of the symmetry group, we may also write the 2-form $F_{\mu\nu}$ in terms of its component fields so that

$$F_{\mu\nu}^{a} = \left(\partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + f^{abc}A_{\mu}^{b}A_{\nu}^{c}\right). \tag{IV.94}$$

In a special case of $SU(2)$ symmetry, both F and A transform like vectors and $f^{abc} = \varepsilon^{abc}$, we restore EQ. (IV.94) to the familiar vector notation $\boldsymbol{F}_{\mu\nu} = \left(\partial_{\mu}\boldsymbol{A}_{\nu} - \partial_{\nu}\boldsymbol{A}_{\mu} + \boldsymbol{A}_{\mu}\times\boldsymbol{A}_{\nu}\right)$.

[Yang-Mills gauge theory]

Having found that the 2-form F transforms homogeneously under gauge transformation, we are ready to write down *the Yang-Mills Lagrangian*, which is an analogy of the Maxwell Lagrangian:

$$\begin{aligned} L &= -\frac{1}{2g^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = -\frac{1}{4g^2}\left(F_{\mu\nu}^{a}F^{a\mu\nu}\right) \\ &= -\frac{1}{4g^2}\left[\left(\partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a}\right)^2 + f^{abc}A^{b\mu}A^{c\nu}\left(\partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a}\right) + \left(f^{abc}A_{\mu}^{b}A_{\nu}^{c}\right)^2\right], \end{aligned} \tag{IV.95}$$

where the constant g is the Yang-Mills coupling constant. The theory described by the Lagrangian in EQ. (IV.95) is known as pure Yang-Mills theory or non-abelian gauge theory. We note that the quadratic term in EQ. (IV.95) describes the propagation of a massless vector boson with an internal index a, known as *the non-abelian gauge boson* or *the Yang-Mills boson*, which is similar to the case of photons in electromagnetism. The cubic and quartic terms in EQ. (IV.95) are additional terms to electromagnetism, and can be understood as the self-interaction of the non-abelian gauge bosons. The corresponding Feynman rules for these three terms in EQ. (IV.95) are given in Fig. IV.4.1. Unlike the case of electromagnetism where photons do not interact, the Yang-Mills bosons couple to all fields transforming non-trivially under the gauge group, and the Yang-Mills bosons themselves transform non-trivially so that they must couple to themselves. We further note that the self-interaction terms involve the structure coefficients f^{abc} that are determined by group theory. Therefore, the self-interactions of gauge bosons are entirely determined by symmetry.

(*a*) **(*b*)** **(*c*)**

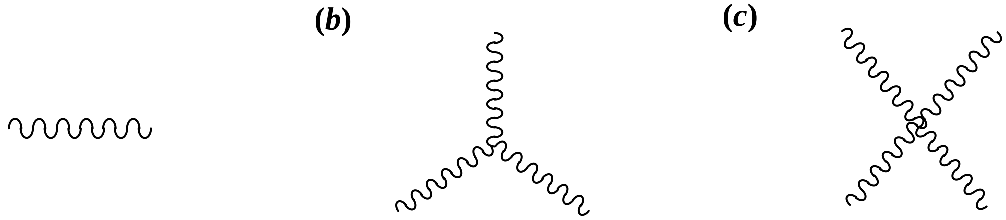

Fig. IV.4.1 The Feynman rules for contributions in pure Yang-Mills theory: **(*a*)** the propagation of massless bosons, **(*b*)** the cubic self-interaction term, and **(*c*)** the quartic self-interaction term.

Finally, we discuss how Yang-Mills bosons couple to matter fields. In general, the prescription to turn a globally symmetric theory into a locally symmetric theory is to replace the ordinary derivative ∂_μ by the covariant derivative

$$D_\mu \varphi = \left(\partial_\mu - iA_\mu^a T_{(\mathcal{R})}^a\right)\varphi\,, \qquad \text{(IV.96)}$$

where $T_{(\mathcal{R})}^a$ denotes the generator associated with a representation $\mathcal{R}$ of the gauge group $\mathcal{G}$. Hence, the coupling of a non-abelian gauge potential to a fermion field of mass *m* is given by

$$L = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi = \bar{\psi}\left(i\gamma^\mu \partial_\mu + \gamma^\mu A_\mu^a T_{(\mathcal{R})}^a - m\right)\psi\,. \qquad \text{(IV.97)}$$

Equation (IV.97) implies that fields couple to the Yang-Mills gauge bosons according to the representation $\mathcal{R}$ of the gauge group $\mathcal{G}$, and those fields belonging to the trivial identity representation cannot couple to the gauge bosons. In the special case of electromagnetism, or equivalently the $U(1)$ gauge theory, the representation $\mathcal{R}$ corresponds to the electric charge of the field. Therefore the fields that transform trivially under the $U(1)$ gauge transformation are electrically neutral.

[The Anderson-Higgs mechanism – gauge symmetry breaking]

Having discussed the occurrence of Nambu-Goldstone bosons under spontaneous symmetry breaking of continuous symmetries, it is natural to ask whether there is a similar mechanism in gauge theory. To answer this question, we begin with a simple model of the φ^4-theory in EQ. (IV.28) by gauging it with the replacement $\partial_\mu \to D_\mu = \left(\partial_\mu - ieA_\mu\right)$ so that

$$L = \partial\varphi^\dagger \partial\varphi + \mu^2 \varphi^\dagger \varphi - \lambda\left(\varphi^\dagger\varphi\right)^2 \quad \Rightarrow \quad L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D\varphi^\dagger D\varphi + \mu^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2. \qquad \text{(IV.98)}$$

Taking the polar coordinates $\varphi = \rho e^{i\theta}$, we have $D_\mu\varphi = \left(\partial_\mu - ieA_\mu\right)\varphi = \left[\partial_\mu \rho + i\rho\left(\partial_\mu\theta - eA_\mu\right)\right]e^{i\theta}$ so that

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \rho^2\left(\partial_\mu\theta - eA_\mu\right)^2 + \left(\partial\rho\right)^2 + \mu^2\rho^2 - \lambda\rho^4\,. \qquad \text{(IV.99)}$$

Next, we consider L under a $U(1)$ gauge transformation $\varphi \to \varphi e^{i\alpha}$ and $eA_\mu \to eA_\mu + \partial_\mu \alpha$, so that the combination $A'_\mu \equiv A_\mu - e^{-1}\partial_\mu \theta$ is gauge invariant, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu A'_\nu - \partial_\nu A'_\mu$. Thus, EQ. (IV.99) becomes

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \rho^2 e^2 A'^2_\mu + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4 . \qquad \text{(IV.100)}$$

Upon spontaneous symmetry breaking, we express the fluctuating field amplitude as $\rho = \left(1/\sqrt{2}\right)(v+\chi)$ where $v \equiv \sqrt{\mu^2/\lambda}$, and EQ. (IV.100) becomes

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[(ve)^2 + 2(v\chi)e^2 + (\chi e)^2\right]A'^2_\mu + \frac{1}{2}(\partial\chi)^2 - \mu^2\chi^2 - \mu\sqrt{\lambda}\chi^3 - \frac{\lambda}{4}\chi^4 + \frac{\mu^4}{4\lambda}. \qquad \text{(IV.101)}$$

Comparing EQ. (IV.101) with EQ. (IV.30) for spontaneous symmetry breaking in an ungauged Lagrangian, we find that the theory associated with EQ. (IV.101) consists of a vector field A'_μ of mass $M \equiv ev$ interacting with a scalar field χ with mass $\sqrt{2}\mu$, and the phase field θ that would have been the Nambu-Goldstone boson in the ungauged theory has disappeared. Therefore, the presence of a massless gauge field A_μ has effectively "eaten" the Nambu-Goldstone boson in the ungauged theory and become a massive gauge field A'_μ. This phenomenon of a massless gauge field becoming massive by "eating" a Nambu-Goldstone boson is known in particle physics as *the Higgs mechanism* and in condensed matter physics as *the Anderson mechanism*. The fluctuating field χ is known as *the Higgs field*. We note that this example of Anderson-Higgs mechanism for the appearance of massive gauge bosons due to spontaneous symmetry breaking under a $U(1)$ gauge transformation indicates that the Higgs field can appear for abelian gauge theory.

In the case of non-abelian gauge theory, the Anderson-Higgs mechanism also applies. As an example, let's consider an $O(3)$ gauge theory, which has three generators so that we express the vector field $\boldsymbol{\varphi}$ in terms of φ^a (a = 1, 2, 3). To gauge the Lagrangian of the φ^4-theory, we make the following substitution in the kinetic energy term $(D_\mu\varphi^a)^2/2$:

$$\partial_\mu\varphi^a \to D_\mu\varphi^a = \left(\partial_\mu\varphi^a + g\varepsilon^{abc}A^b_\mu\varphi^c\right). \qquad \text{(IV.102)}$$

Upon spontaneous symmetry breaking so that the vector field is expressed as $\boldsymbol{\varphi} = (0, 0, v+\chi)$, we find

$$\begin{aligned}\frac{1}{2}\left(D_\mu\varphi^a\right)\left(D^\mu\varphi^a\right) &= \frac{1}{2}\left(\partial_\mu\chi\right)\left(\partial^\mu\chi\right) + \frac{1}{2}g^2\left(A^2_\mu v - A^1_\mu v\right)\left(A^{\mu 2}v - A^{\mu 1}v\right)\\ &= \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}(gv)^2\left(A^1_\mu A^{\mu 1} + A^2_\mu A^{\mu 2}\right),\end{aligned} \qquad \text{(IV.103)}$$

which implies that the gauge potential A^1_μ and A^2_μ acquires a finite mass (gv) while A^3_μ remains massless.

As the third example, we consider a more elaborate example of an $SU(5)$ theory relevant to the Grand Unified Theory, in which the Higgs field upon spontaneous breaking leads to 12 massive gauge fields and 12 massless gauge bosons associated with a lowered symmetry $SU(3) \otimes SU(2) \otimes U(1)$. Specifically, in an $SU(5)$

theory the field φ is a (5 × 5) hermitian traceless matrix, as discussed in Supplement 2. To consider a gauged φ^4-theory with *SU*(5) symmetry, we note that the field φ transforms as the adjoint representation of the *SU*(5) group such that $\varphi \to \varphi + i\,\theta^a\left[T^a,\varphi\right]$, where T^a (a = 1, 2, …, 24) represent the 24 generators of *SU*(5). Upon spontaneous symmetry breaking, we can take the vacuum expectation value of φ as a (5 × 5) diagonal and traceless matrix so that $\left\langle \varphi^i_j \right\rangle = v_j \delta^i_j$, where $i, j = 1,\cdots,5$, $\sum_j v_j = 0$, and $\sum_j v_j^2 = \left(\mu^2/\lambda\right)$. Thus, we have

$$D_\mu \varphi^a = \partial_\mu \varphi^a - igA^a_\mu\left[T^a,\varphi\right], \tag{IV.104}$$

and the kinetic energy term in the gauged Lagrangian under spontaneous symmetry breaking becomes

$$\mathrm{Tr}\left\{\left(D_\mu\varphi\right)\left(D^\mu\varphi\right)\right\} \to g^2\mathrm{Tr}\left\{\left[T^a,\langle\varphi\rangle\right]\left[\langle\varphi\rangle,T^b\right]\right\}A^a_\mu A^{\mu b}. \tag{IV.105}$$

Thus, the corresponding gauge boson masses squared can be computed by finding the eigenvalues of the (24 × 24) matrix in EQ. (IV.105).

Although the computation of massive bosons may be laborious for a given $\langle\varphi\rangle$, it is generally easy to find the massless gauge bosons by considering the generators that commute with the diagonal traceless matrix $\langle\varphi\rangle$. For instance, consider in the *SU*(5) theory a vacuum expectation value for the field given by:

$$\langle\varphi\rangle = v\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}. \tag{IV.106}$$

Clearly generators of the forms $\begin{pmatrix} \mathbf{A} & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{B} \end{pmatrix}$ commute with $\langle\varphi\rangle$ given in EQ. (IV.106), where **A** denotes 8 traceless hermitian (3 × 3) matrices known as the Gell-Mann matrices, and **B** represents 3 traceless hermitian (2 × 2) matrices known as the Pauli matrices. In addition, a matrix proportional to that given in EQ. (IV.106) also commutes with $\langle\varphi\rangle$. Consequently, there are overall $\left(3^2-1\right)+\left(2^2-1\right)+1=12$ massless gauge bosons respectively associated with the generators of *SU*(3), *SU*(2) and *U*(1) groups, and the remaining 24 – 12 = 12 gauge bosons become massive due to Anderson-Higgs mechanism.

In general, for a theory with the global symmetry group $\mathcal{G}$, if spontaneous symmetry breaking reduces the symmetry group to S, there will be $\left[n(\mathcal{G})-n(S)\right]$ Nambu-Goldstone bosons, where $n(\mathcal{G})$ and $n(S)$ represent the number of generators of $\mathcal{G}$ and S, respectively. Next, if the group $\mathcal{G}$ is gauged, the initial $n(\mathcal{G})$ massless gauge bosons reduce to $n(S)$ massless gauge bosons and $\left[n(\mathcal{G})-n(S)\right]$ massive gauge bosons upon spontaneous symmetry breaking into a lower symmetry group S.

Finally, we remark that gauge invariance associated with gauge theory is the result of a redundancy in the degrees of freedom involved in the choices of the gauge. The redundancy involves in gauge theory, particularly in non-abelian gauge theory, is relevant to topological field theory.

Further Readings:

1. "*Quantum Field Theory in a Nutshell*", A. Zee, Princeton University Press (2003): Chapters IV.1 – IV.6 and Chapter VII.1.

2. "*An Introduction to Quantum Field Theory*", M. E. Peskin and D. V. Schroeder, HarperCollins Publishers, (1995): Chapters 7, 15, and 20.