

## II.7. Non-relativistic Green's Function Techniques for Many-Body Interactions

As discussed in Part II.6, the Schrödinger's equation can be derived from the relativistic Klein-Gordon equation by taking the non-relativistic limit, and it is an equation linear in time, in contrast to the quadratic time dependence of the Klein-Gordon equation. Hence, in considering Feynman diagrams for most condensed matter physics systems governed by the non-relativistic Schrödinger's equation, the corresponding propagators (and therefore the Green's functions) are expected to take on a slightly different form from what we have seen for the relativistic quantum field theory. In this section, we focus on the non-relativistic Green's function techniques, which play a fundamental role in the treatment of many-body systems. Specifically, various fundamental physical properties of a many-body system, such as the ground state energy, the density of states and the excitation spectrum, the response functions (e.g., conductivity, magnetic susceptibility, dielectric constant, etc.), and the thermodynamic quantities, can be derived from the Green's functions.

### [Basic properties of Green's functions]

Before proceeding with formal treatment of Green's function techniques for many-body systems, it is worthwhile reviewing some basic mathematical properties of Green's functions. Let's first consider the case of time-independent Green's functions. Given a linear, hermitian and time-independent differential operator  $L(\mathbf{r})$  and a complex variable  $z = \lambda + is$ , the Green's function  $G(\mathbf{r}, \mathbf{r}'; z)$  of  $L(\mathbf{r})$  is defined as the solution to the following equation

$$[z - L(\mathbf{r})]G(\mathbf{r}, \mathbf{r}'; z) = \delta(\mathbf{r} - \mathbf{r}'), \quad (\text{II.278})$$

subject to certain homogeneous boundary conditions on the surface  $S$  of the domain  $\Omega$  of  $\mathbf{r}$  and  $\mathbf{r}'$ .

Assuming that  $\{|\phi_n\rangle\}$  is the complete orthonormal set of eigenfunctions of  $L$ , subject to the same conditions on the surface  $S$  as  $G(\mathbf{r}, \mathbf{r}'; z)$ , and that  $\{\lambda_n\}$  is the eigenvalues, we may express the Green's function  $G$  as follows:

$$G(z) = \frac{1}{(z - L)} = \sum'_n \frac{|\phi_n\rangle\langle\phi_n|}{z - \lambda_n} + \int dn \frac{|\phi_n\rangle\langle\phi_n|}{z - \lambda_n}, \quad (\text{II.279})$$

or equivalently

$$G(\mathbf{r}, \mathbf{r}'; z) = \sum'_n \frac{\phi_n(\mathbf{r})\phi_n^*(\mathbf{r}')}{z - \lambda_n} + \int dn \frac{\phi_n(\mathbf{r})\phi_n^*(\mathbf{r}')}{z - \lambda_n}, \quad (\text{II.280})$$

where  $L(\mathbf{r})|\phi_n\rangle = \lambda_n|\phi_n\rangle$ ,  $\sum'_n$  denotes the sum over the eigenfunctions of the discrete spectrum, and  $\int dn$  is an integration over the continuous spectrum. We also note that the following orthogonality relations:

$$\langle\phi_n|\phi_m\rangle = \delta_{nm} = \int d^3r \phi_n^*(\mathbf{r}')\phi_m(\mathbf{r}), \quad \sum_n \phi_n(\mathbf{r})\phi_n^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (\text{II.281})$$

Moreover, since  $L$  is hermitian, all eigenvalues  $\lambda_n$ 's are real, implying that the singularities of  $G(z)$  are all on the real axis. For branch cuts of  $G(z)$ , we may define  $G^+(\lambda)$  and  $G^-(\lambda)$  so that

$$G^\pm(\lambda) \equiv \lim_{s \rightarrow 0^+} G(\lambda \pm is). \quad (\text{II.282})$$

The simple poles and branch cut of  $G(z)$  on the  $\lambda$ - $s$  complex plane are illustrated in Fig. II.7.1.

Once  $G(z)$  is known, we can do the following:

(1) Finding the solutions to the inhomogeneous equation

$$[z - L(\mathbf{r})] u(\mathbf{r}) = f(\mathbf{r}), \quad (\text{II.283})$$

where

$$\begin{aligned} u(\mathbf{r}) &= \int d^3 r' G(\mathbf{r}, \mathbf{r}'; z) f(\mathbf{r}') && \text{if } z \neq \lambda_n, \\ u(\mathbf{r}) &= \int d^3 r' G(\mathbf{r}, \mathbf{r}'; z) f(\mathbf{r}') + \phi(\mathbf{r}) && \text{if } z = \lambda \in \{\text{branch cuts of } G(z)\}, \end{aligned}$$

and  $u(\mathbf{r})$ ,  $\phi(\mathbf{r})$ ,  $G(\mathbf{r}, \mathbf{r}'; z)$ , all satisfy the same boundary conditions on  $S$ , with  $\phi(\mathbf{r})$  being the solution to the homogeneous equation:

$$[z - L(\mathbf{r})] \phi(\mathbf{r}) = 0. \quad (\text{II.284})$$

Here we note that there is no solution to  $u(\mathbf{r})$  if  $z$  coincides with  $\lambda_n$  unless  $f(\mathbf{r})$  is orthogonal to all eigenfunctions of  $\lambda_n$ .

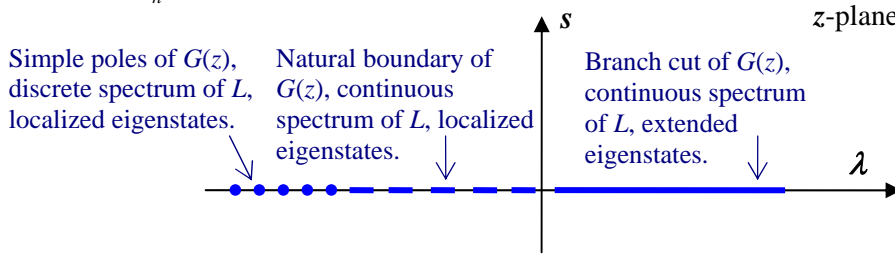


Fig. II.7.1 Green's function solutions on the  $z$ -plane.

(2) Obtaining information for the eigenfunctions and eigenvalues of  $L(\mathbf{r})$ . That is, finding the poles of  $G(z)$ , which correspond to the discrete eigenvalues of  $L(\mathbf{r})$ ; the residues at the poles, which correspond to the products of  $\phi_n(\mathbf{r})\phi_n^*(\mathbf{r}')$ , provided that  $\phi_n(\mathbf{r})$  are not degenerate; and the branch cuts of  $G(\mathbf{r}, \mathbf{r}'; z)$  along the real axis, which correspond to the continuous spectrum of  $L(\mathbf{r})$ .

(3) Deriving the density of states  $\mathcal{N}(\lambda)$  associated with the operator  $L(\mathbf{r})$ :

From  $G^*(\mathbf{r}, \mathbf{r}'; z) = G(\mathbf{r}', \mathbf{r}; z^*)$  and EQ. (II.282), we have  $G^-(\mathbf{r}, \mathbf{r}'; z) = [G^+(\mathbf{r}', \mathbf{r}; z)]^*$ . Consequently,  $\text{Re}\{G^-(\mathbf{r}, \mathbf{r}'; z)\} = \text{Re}\{[G^+(\mathbf{r}, \mathbf{r}'; z)]\}$ ,  $\text{Im}\{G^-(\mathbf{r}, \mathbf{r}'; z)\} = -\text{Im}\{[G^+(\mathbf{r}, \mathbf{r}'; z)]\}$ , and

$$\begin{aligned} \tilde{G}(\mathbf{r}, \mathbf{r}'; \lambda) &\equiv G^+(\mathbf{r}, \mathbf{r}'; \lambda) - G^-(\mathbf{r}, \mathbf{r}'; \lambda) = -2\pi i \sum_n \delta(\lambda - \lambda_n) \phi_n(\mathbf{r}) \phi_n^*(\mathbf{r}') \\ &= -2\pi i \left[ \sum_n' \delta(\lambda - \lambda_n) \phi_n(\mathbf{r}) \phi_n^*(\mathbf{r}') + \int dn \delta(\lambda - \lambda_n) \phi_n(\mathbf{r}) \phi_n^*(\mathbf{r}') \right], \end{aligned} \quad (\text{II.285})$$

where we have used the identity

$$\lim_{s \rightarrow 0^+} \frac{1}{x \pm is} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x). \quad (\text{II.286})$$

In the case of diagonal matrix elements, we have

$$G^\pm(\mathbf{r}, \mathbf{r}; \lambda) = \mathcal{P}\left[\sum_n \frac{\phi_n(\mathbf{r})\phi_n^*(\mathbf{r})}{(\lambda - \lambda_n)}\right] \mp \pi i \sum_n \delta(\lambda - \lambda_n) \phi_n(\mathbf{r}) \phi_n^*(\mathbf{r}), \quad (\text{II.287})$$

which yields the trace:

$$\begin{aligned} \text{Tr}\{G^\pm(\lambda)\} &= \int d^3r G^\pm(\mathbf{r}, \mathbf{r}; \lambda) \\ &= \mathcal{P}\left[\sum_n \frac{1}{(\lambda - \lambda_n)}\right] \mp i\pi \sum_n \delta(\lambda - \lambda_n) \equiv \mathcal{P}\left[\sum_n \frac{1}{(\lambda - \lambda_n)}\right] \mp i\pi \mathcal{N}(\lambda). \end{aligned} \quad (\text{II.288})$$

Therefore,  $[\mathcal{N}(\lambda) d\lambda]$  is the number of states in the interval  $[\lambda, \lambda + d\lambda]$ , and the quantity

$$\rho(\mathbf{r}; \lambda) \equiv \sum_n \delta(\lambda - \lambda_n) \phi_n(\mathbf{r}) \phi_n^*(\mathbf{r}) \quad (\text{II.289})$$

is the density of states per unit volume, so that

$$\mathcal{N}(\lambda) = \int d^3r \rho(\mathbf{r}; \lambda). \quad (\text{II.290})$$

Consequently,

$$\rho(\mathbf{r}; \lambda) = \mp \frac{1}{\pi} \text{Im}\{G^\pm(\mathbf{r}, \mathbf{r}; \lambda)\}, \quad (\text{II.291})$$

and

$$\mathcal{N}(\lambda) = \mp \frac{1}{\pi} \text{Im}\{\text{Tr}[G^\pm(\lambda)]\}. \quad (\text{II.292})$$

Furthermore,

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}'; z) &= \sum_n \frac{\phi_n(\mathbf{r}) \phi_n^*(\mathbf{r}')}{z - \lambda_n} = \int_{-\infty}^{\infty} d\lambda \sum_n \delta(\lambda - \lambda_n) \frac{\phi_n(\mathbf{r}) \phi_n^*(\mathbf{r}')}{z - \lambda}, \\ &= \frac{i}{2\pi} \int_{-\infty}^{\infty} d\lambda \frac{\tilde{G}(\mathbf{r}, \mathbf{r}'; z)}{z - \lambda}. \end{aligned} \quad (\text{II.293})$$

⟨4⟩ Using  $G_0(z) = (z - L_0)^{-1}$  and  $L_1$  to obtain information for the eigenfunctions and eigenvalues of  $L = L_0 + L_1$ . Specifically, if the operator  $L$  is represented by the total Hamiltonian  $\mathcal{H}$  of the system, so that for  $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ ,  $G_0(z) = (z - \mathcal{H}_0)^{-1}$  and  $G(z) = (z - \mathcal{H})^{-1}$ . Hence,

$$\begin{aligned} G &= G_0 \frac{1}{1 - \mathcal{H}_1 G_0} = G_0 + G_0 \mathcal{H}_1 G_0 + G_0 \mathcal{H}_1 G_0 \mathcal{H}_1 G_0 + \dots \\ &= G_0 + G_0 \mathcal{H}_1 G = G_0 + G \mathcal{H}_1 G_0. \end{aligned} \quad (\text{II.294})$$

Next, let's consider time-dependent Green's functions. The Green's function associated with a first-order (in time) partial differential equation of the form

$$\left[ \frac{i}{C} \frac{\partial}{\partial t} - L(\mathbf{r}) \right] \phi(\mathbf{r}, t) = 0, \quad (\text{II.295})$$

is defined as the solution of the following equation:

$$\left[ \frac{i}{C} \frac{\partial}{\partial t} - L(\mathbf{r}) \right] g(\mathbf{r}, t; \mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \quad (\text{II.296})$$

subject to certain boundary conditions on the surface  $S$  of the domain  $\Omega$  of  $\mathbf{r}$  and  $\mathbf{r}'$ ;  $L(\mathbf{r})$  is a linear, hermitian and time-independent operator with a complete set of eigenfunctions  $\{\phi(\mathbf{r})\}$ ; and  $C$  is a constant. If  $C$  is real, it may be taken as positive without losing generality. In this case,  $g(\mathbf{r}, t; \mathbf{r}', t')$  is associated with a

Schrödinger type equation. On the other hand, if  $C$  is imaginary,  $g(\mathbf{r}, t; \mathbf{r}', t')$  is associated with a diffusion type equation.

Generally for time-independent  $L(\mathbf{r})$ ,  $g(\mathbf{r}, t; \mathbf{r}', t')$  may be expressed in terms of the time difference  $\tau \equiv t - t'$ . In this case, the Fourier transform of  $g(\mathbf{r}, \mathbf{r}'; \tau)$  is  $g(\mathbf{r}, \mathbf{r}'; \omega)$ , which is directly related to the time-independent Green's function  $G(\mathbf{r}, \mathbf{r}'; z)$  if we take  $z = \omega/C$  and allow  $\omega$  to be complex. Thus, we have

$$\tilde{g}^{>,<}(\mathbf{r}, \mathbf{r}'; \tau) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} G^{+,-}\left(\mathbf{r}, \mathbf{r}'; \frac{\omega'}{C}\right) e^{-i\omega'\tau}. \quad (\text{II.297})$$

The Green's function associated with a second-order (in time) differential equation is defined as the solution of

$$\left[ -\frac{1}{C^2} \frac{\partial^2}{\partial t^2} - L(\mathbf{r}) \right] g(\mathbf{r}, \mathbf{r}'; \tau) = \delta(\mathbf{r} - \mathbf{r}') \delta(\tau). \quad (\text{II.298})$$

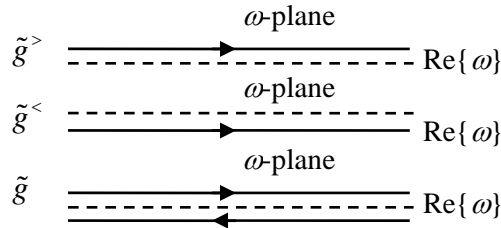
The Fourier transform  $g(\mathbf{r}, \mathbf{r}'; \omega)$  in this case is related to the time-independent Green's function  $G(\mathbf{r}, \mathbf{r}'; z)$  by  $g(\mathbf{r}, \mathbf{r}'; \omega) = G\left(\mathbf{r}, \mathbf{r}'; \frac{\omega^2}{C^2}\right)$ . The special case of  $L = -\nabla^2$  reduces to the wave equation.

Having obtained  $\tilde{g}^>(\tau)$  and  $\tilde{g}^<(\tau)$ , we can:

(1) Solve the homogeneous equation in EQ. (II.295) as

$$\phi(\mathbf{r}, t) = \frac{i}{C} \int d^3r' \tilde{g}(\mathbf{r}, \mathbf{r}', t - t') \phi(\mathbf{r}', t'), \quad (\text{II.299})$$

where  $\tilde{g}(\tau) \equiv \tilde{g}^>(\tau) - \tilde{g}^<(\tau)$ , with the corresponding integration paths in the  $\omega$ -plane illustrated in Fig. II.7.2.



**Fig. II.7.2** Integration paths for Green's functions on the  $\omega$ -plane.

(2) Solve the inhomogeneous equation:

$$\left[ \frac{i}{C} \frac{\partial}{\partial t} - L(\mathbf{r}) \right] \psi(\mathbf{r}, t) = f(\mathbf{r}, t), \quad (\text{II.300})$$

with  $\psi(\mathbf{r}, t)$  given by

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r}, t) + \int d^3r' \int dt' g^+(\mathbf{r}, \mathbf{r}'; t - t') f(\mathbf{r}', t'), \quad (\text{II.301})$$

where  $\phi(\mathbf{r}, t)$  is the solution of the homogenous equation EQ. (III.370).

⟨3⟩ Use  $g_0(\tau)$  and  $L_1(\mathbf{r})$  to obtain information for the solution of

$$\left[ \frac{i}{C} \frac{\partial}{\partial t} - L(\mathbf{r}) \right] \psi(\mathbf{r}, t) = 0,$$

where  $g_0(\tau)$  is the Green's function of solution of  $L_0(\mathbf{r})$ , and  $L = L_0 + L_1$ .

Having reviewed some basic properties of Green's functions, next we consider the effect of temperature, in preparation for applying Green's functions to the derivation of thermodynamic properties.

### [Temperature dependent quantum field theory]

Temperature is one of the most fundamental variables in the description of a condensed matter physics system. The fact that it is conspicuously missing in the quantum field theory that we have discussed so far may have raised concerns for the applicability of quantum field theory to condensed matter physics systems at finite temperatures. In reality, temperature  $T$  can be naturally introduced into the zero-temperature quantum field theory through a Wick rotation  $t = -it_E$  if one identifies the inverse of temperature ( $\beta = 1/T$ ) with the imaginary time. Specifically, the Minkowskian path integral for scalar field theory (with  $\hbar$  restored)

$$Z = \int D\varphi \exp \left[ \frac{i}{\hbar} \int d^d x \left( \frac{1}{2} (\partial\varphi)^2 - \mathcal{V}(\varphi) \right) \right] \quad (\text{II.302})$$

is converted into the Euclidean functional integral

$$Z = \int D\varphi \exp \left[ -\frac{1}{\hbar} \int d_E^d x \left( \frac{1}{2} (\partial\varphi)^2 + \mathcal{V}(\varphi) \right) \right] \equiv \int D\varphi \exp \left[ -\frac{1}{\hbar} \mathcal{E}(\varphi) \right], \quad (\text{II.303})$$

where we have defined  $d^d x \equiv -i d_E^d x \equiv -i dt_E d^{(d-1)} x$ ,  $(\partial\varphi)^2 = (\partial\varphi/\partial t)^2 - (\nabla\varphi)^2$  in EQ. (II.302), and  $(\partial\varphi)^2 = -\left[ (\partial\varphi/\partial t_E)^2 + (\nabla\varphi)^2 \right]$  in EQ. (II.303). The term  $\mathcal{E}(\varphi)$  defined in EQ. (II.303) can be considered as the effective energy functional of the field  $\varphi$ . In this context, if we identify  $\hbar$  as the temperature  $T = 1/\beta$  and recall that the scalar field  $\varphi(x)$  can be mapped onto the coordinates  $q_i$  ( $i = 1, 2, \dots, N$ ) of an  $N$ -particle system, we find that EQ. (II.303) becomes consistent with the partition function of classical statistical mechanics in  $d$ -dimensional space because of the correspondence

$$Z = \prod_i \int dq_i \exp[-\beta \mathcal{H}(q_1, q_2, \dots, q_N)] \leftrightarrow Z = \int D\varphi \exp \left[ -\frac{1}{\hbar} \mathcal{E}(\varphi) \right].$$

Hence, the Euclidean quantum field theory in  $d$ -dimensional spacetime is consistent with classical statistical mechanics in  $d$ -dimensional space.

Let's now consider the situation in quantum statistical mechanics. The partition function for a system described by the Hamiltonian  $\mathcal{H}$  is

$$Z = \text{Tr} \left\{ e^{-\beta \mathcal{H}} \right\} = \sum_n \langle n | e^{-\beta \mathcal{H}} | n \rangle. \quad (\text{II.304})$$

You may recall that we have derived the integral representation of  $\langle F | e^{-i\mathcal{H}t} | I \rangle$  in Part II.2. Thus, by replacing the time  $t$  in the expression of  $\langle F | e^{-i\mathcal{H}t} | I \rangle$  (see EQ. (II.35)) by  $(-i\beta)$  and by setting  $|I\rangle = |F\rangle = |n\rangle$ , we reach the following:

$$Z = \text{Tr} \left\{ e^{-\beta \mathcal{H}} \right\} = \int_{PBC} Dq \exp \left[ -\int_0^\beta d\tau \mathcal{L}(q) \right], \quad (\text{II.305})$$

where ‘‘PBC’’ refers to periodic boundary condition because of the trace operation, the Lagrangian  $\mathcal{L}(q) = \frac{1}{2}(dq/d\tau)^2 + \mathcal{V}(q)$  under the change of variable from  $dt$  to  $(-id\tau)$  is precisely the Hamiltonian  $\mathcal{H}$  in the Euclidean time  $\tau$ , and the integration over  $\tau$  runs from 0 to  $\beta$ . Since we are taking the trace in EQ. (II.305), the boundary condition  $q(0) = q(\beta)$  must be satisfied. Equation (II.305) can be readily extended to field theory if we take  $\mathcal{H}$  as the Hamiltonian of a quantum field theory in  $D$ -dimensional space (or equivalently,  $d = (D+1)$ -dimensional spacetime) and consider the following correspondence:

$$q \leftrightarrow \varphi, \quad \mathcal{L}(q) \leftrightarrow \int d^D x L(\varphi).$$

Hence, EQ. (II.305) is replaced by the following expression:

$$Z = \text{Tr} \left\{ e^{-\beta \mathcal{H}} \right\} = \int_{PBC} D\varphi \exp \left[ -\int_0^\beta d\tau \int d^D x L(\varphi) \right], \quad (\text{II.306})$$

and for all paths  $\varphi(\mathbf{x}, \tau)$  the condition

$$\varphi(\mathbf{x}, 0) = \varphi(\mathbf{x}, \beta) \quad (\text{II.307})$$

is satisfied.

The expressions given in EQs. (II.305) -- (II.307) suggest that a field theory at finite temperature is equivalent to rotate it to Euclidean space and impose the boundary condition in EQ. (II.307). In other words, a Euclidean quantum field theory in  $(D+1)$ -dimensional spacetime with  $0 \leq \tau < \beta$  is equivalent to quantum statistical mechanics in  $D$ -dimensional space. In the limit of zero temperature where  $\beta \rightarrow \infty$ , EQ. (II.306) becomes the standard Wick rotated quantum field theory over an infinite spacetime, as expected. Thus, the analytic continuation between  $[\exp(-i\mathcal{H}t)]$  in quantum physics and  $[\exp(-\beta\mathcal{H})]$  in thermal physics leads to a remarkable result suggesting that the notion of temperature is in fact equivalent to cyclic imaginary time.

Given EQs. (II.305) -- (II.307), we are able to deal with finite-temperature perturbation theory and Feynman diagrams. In general we still use the Feynman rules as before, except that the boundary condition given in EQ. (II.305) requires that the Euclidean frequency  $\omega$  can only take on discrete values:

$$\omega_n \equiv (2\pi / \beta) n = (2\pi T) n, \quad (n = -\infty, \dots, -1, 0, 1, \dots, \infty). \quad (\text{II.308})$$

Therefore, the propagator of the scalar field becomes  $1/(k_0^2 + \mathbf{k}^2) \rightarrow 1/(\omega_n^2 + \mathbf{k}^2)$  in a relativistic quantum field theory, and the integral associated with a Feynman diagram,  $\int d_E^d k F(k_E^2)$ , is now replaced by:

$$2\pi T \sum_n \int d^D k F \left[ (2\pi T)^2 n^2 + \mathbf{k}^2 \right]. \quad (\text{II.309})$$

In the high temperature limit  $T \rightarrow \infty$ , we find that the dominant contribution to the propagator is from the term  $n = 0$ , which implies that the Feynman diagrams are effectively evaluated in  $D$ -dimensional space. In other words, the Euclidean quantum field theory in  $d$ -dimensional spacetime is equivalent to high-temperature quantum statistical physics in  $D$ -dimensional space. One important application of quantum field theory at finite temperature is to cosmology, because the early universe may be considered as a soup of elementary particles at high temperatures. Finite-temperature quantum field theory is also very important to condensed matter physics. In the remaining part of this chapter, we shall study the application of Green’s

function techniques in quantum field theory to examples of many-body interaction in condensed matter physics, with emphasis on the non-relativistic limit.

### [The physical meanings of Green's functions]

Our earlier discussion of Green's functions in the context of relativistic quantum field theory primarily considers the propagation of particles in vacuum, which is homogeneous spacetime. In condensed matter physics, on the other hand, we are interested in the propagation of particles in a many-body system. Let's begin our consideration with a many-body interacting system in the absence of any external potential, so that the Hamiltonian  $\mathcal{H}$  is also translational invariant in spacetime, and that all Green's functions are only dependent on the difference in the spacetime coordinates ( $x-x'$ ). To incorporate interaction, we define the field operators  $\psi(x) = \psi(\mathbf{r}, t)$  and  $\psi^\dagger(x) = \psi^\dagger(\mathbf{r}, t)$  in the Heisenberg picture as follows (taking  $\hbar = 1$ ):

$$\begin{aligned}\psi(\mathbf{r}, t) &= e^{i\mathcal{H}t} \psi(\mathbf{r}, 0) e^{-i\mathcal{H}t} = \sum_n a_n \psi_n(\mathbf{r}) e^{-i\varepsilon_n t}, \\ \psi^\dagger(\mathbf{r}, t) &= e^{i\mathcal{H}t} \psi^\dagger(\mathbf{r}, 0) e^{-i\mathcal{H}t} = \sum_n a_n^\dagger \psi_n^*(\mathbf{r}) e^{i\varepsilon_n t},\end{aligned}\tag{II.310}$$

and 
$$\mathcal{H} = \sum_n \varepsilon_n a_n^\dagger a_n.\tag{II.311}$$

As before, the creation and annihilation operators  $a_n^\dagger$  and  $a_n$  obey the following commutation relations:

$$a_n a_{n'}^\dagger \mp a_{n'}^\dagger a_n = \delta_{nn'}, \quad a_n a_{n'} \mp a_{n'} a_n = 0, \quad a_n^\dagger a_{n'}^\dagger \mp a_{n'}^\dagger a_n^\dagger = 0,\tag{II.312}$$

where the upper (lower) sign corresponds to bosons (fermions).

The Green's function  $g(\mathbf{r}, t; \mathbf{r}', t') = g(x, x') = g(x-x')$  has the physical significance of a single-particle propagator. For an  $N$ -particle system, each propagator corresponds to the propagation of a particle added to the  $N$ -particle system, which differs from the relativistic single-particle propagator considered in Part II.2 that is referenced to vacuum. As before, the Green's function for the interacting system is defined by the chronological operator  $T$ :

$$g(x-x') \equiv -i \langle T(\psi(x) \psi^\dagger(x')) \rangle,\tag{II.313}$$

where the symbol  $\langle A \rangle$  denotes thermal average of the quantity  $A$  over the grand statistical ensemble:

$$\langle A \rangle \equiv \frac{\sum_i \langle i | A | i \rangle \exp[-\beta(\varepsilon_i - \mu N_i)]}{\sum_i \exp[-\beta(\varepsilon_i - \mu N_i)]} = \frac{\text{Tr} [A e^{-\beta(\varepsilon - \mu N)}]}{\text{Tr} [e^{-\beta(\varepsilon - \mu N)}]},\tag{II.314}$$

$N$  is the total number of particles,  $\mu$  the chemical potential,  $\{|i\rangle\}$  the common eigen-functions, and

$$\begin{aligned}T(\psi(x) \psi^\dagger(x')) &= T(\psi(\mathbf{r}, t) \psi^\dagger(\mathbf{r}', t')) = \psi(\mathbf{r}, t) \psi^\dagger(\mathbf{r}', t'), \quad t > t'; \\ &= \pm \psi^\dagger(\mathbf{r}', t') \psi(\mathbf{r}, t), \quad t < t'.\end{aligned}\tag{II.315}$$

In EQ. (II.315) the upper (lower) sign refers to bosons (fermions). It should be noted that the definition given in EQ. (II.313) has not explicitly included the spin indices of the field operators. For simplicity we temporarily neglect the spin indices, which is valid if we only deal with spin-independent interactions. We shall restore the spin indices later for generality when we discuss the diagrammatic analysis.

It is convenient (as we shall see later) to define different forms of Green's functions besides the one in EQ. (II.313). The customary forms of Green's functions in addition to EQ. (II.313) are listed below:

$$\tilde{g}(x, x') = \tilde{g}(x - x') \equiv -i \left\langle \left[ \psi(x), \psi^\dagger(x') \right]_{\mp} \right\rangle, \quad (\text{II.316})$$

$$\tilde{g}^>(x, x') = \tilde{g}^>(x - x') \equiv -i \left\langle \psi(x) \psi^\dagger(x') \right\rangle, \quad (\text{II.317})$$

$$\tilde{g}^<(x, x') = \tilde{g}^<(x - x') \equiv \mp i \left\langle \psi^\dagger(x') \psi(x) \right\rangle, \quad (\text{II.318})$$

$$g^R(x, x') = g^R(x - x') \equiv -i \theta(t - t') \left\langle \left[ \psi(x), \psi^\dagger(x') \right]_{\mp} \right\rangle, \quad (\text{II.319})$$

$$g^A(x, x') = g^A(x - x') \equiv i \theta(t' - t) \left\langle \left[ \psi(x), \psi^\dagger(x') \right]_{\mp} \right\rangle, \quad (\text{II.320})$$

where the upper (lower) sign refers to bosons (fermions). In the zero temperature limit the thermal average of the physical quantity  $A$  becomes:

$$\langle A \rangle \rightarrow \langle 0 | A | 0 \rangle \quad \text{as } T \rightarrow 0^+,$$

where  $|0\rangle$  is the ground state of the entire many-system instead of vacuum.

In the presence of interactions, the Hamiltonian  $\mathcal{H} (= \mathcal{K} + \mathcal{V})$  of the system consists of both the kinetic energy of particles  $\mathcal{K}$ :

$$\mathcal{K} = \frac{\hbar^2}{2m} \int d^3r (\nabla \psi^\dagger \nabla \psi) = -\frac{\hbar^2}{2m} \int d^3r (\psi^\dagger \nabla^2 \psi) = \sum_k \frac{\hbar^2 k^2}{2m} a_k^\dagger a_k, \quad (\text{II.321})$$

and the interaction potential  $\mathcal{V}$  (restricted to pair interactions for simplicity):

$$\mathcal{V} = \frac{1}{2} \sum_{ij} \mathcal{V}(\mathbf{r}_i, \mathbf{r}_j) = \frac{1}{2} \int d^3r \int d^3r' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') V(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}). \quad (\text{II.322})$$

Consequently, the Schrödinger's equation expressed in terms of the field operators becomes

$$\left( i \frac{\partial}{\partial t} + \frac{\nabla_r^2}{2m} \right) \psi(\mathbf{r}, t) = \left[ \int d^3r' \psi^\dagger(\mathbf{r}', t) V(\mathbf{r}' - \mathbf{r}) \psi(\mathbf{r}', t) \right] \psi(\mathbf{r}, t), \quad (\text{II.323})$$

and the corresponding Green's function  $g(x, x')$  will involve the two-particle Green's function  $g_2$ :

$$\left( i \frac{\partial}{\partial t} + \frac{\nabla_r^2}{2m} \right) g(x, x') = \delta(x - x') \pm i \int d^4x_1 V(\mathbf{r} - \mathbf{r}_1) g_2(x, x_1; x', x_1^+) \Big|_{t_1=t}, \quad (\text{II.324})$$

where  $x_1^+ \equiv \lim_{s \rightarrow 0^+} (\mathbf{r}_1, t_1 + s)$ , and the two-particle Green's function is given by:

$$g_2(x_1, x_2; x'_1, x'_2) = (-i)^2 \left\langle T \left[ \psi(x_1) \psi(x_2) \psi^\dagger(x'_2) \psi^\dagger(x'_1) \right] \right\rangle, \quad (\text{II.325})$$

where the chronological operator  $T$  arranges the operators in chronological order so that the earliest time appears on the far right and the latest time appears on the far left. In addition, for fermions only, a factor  $\pm 1$  is introduced depending on whether the time-ordered product is an even or odd permutation of the original ordering. From EQ. (II.324), we note that the pair interactions in EQ. (II.322) give rise to the added

contribution of the two-particle Green's function to  $g(x, x')$ . Similarly, we expect that the differential equation for  $g_2$  will involve the three-particle Green's function  $g_3$ , and so on. The  $n$ -particle Green's function  $g_n$  is defined as:

$$g_n(x_1, \dots, x_n; x'_1, \dots, x'_n) = (-i)^n \left\langle T \left[ \psi(x_1) \cdots \psi(x_n) \psi^\dagger(x'_n) \cdots \psi^\dagger(x'_1) \right] \right\rangle. \quad (\text{II.326})$$

Thus, the existence of interaction complicates the calculation of the Green's function  $g(x, x')$  in an essential way: While in the non-interacting case  $g(x, x')$  is determined by a single differential equation and proper initial conditions, there is an infinite hierarchy of equations, each connects a Green's function of order  $n$  to one of order  $n + 1$ . Moreover, the chronological order plays an important role in generalizing the Green's functions to finite temperatures.

### [Non-interacting Green's functions]

Before proceeding further with calculations of interacting Green's functions, it is instructive to consider the non-interacting Green's functions associated with the Schrödinger's equation, which differ from those associated with the relativistic quantum field theory. Following EQ. (II.321), in the absence interaction, we have  $\mathcal{H} = \mathcal{K}$  so that

$$\mathcal{H} = \sum_k \varepsilon_k a_k^\dagger a_k, \quad \left( \varepsilon_k = \frac{k^2}{2m} \right).$$

From the general equation of motion  $ida_k(t)/dt = [a_k(t), \mathcal{H}]$  we have  $ida_k(t)/dt = \varepsilon_k a_k(t)$  for the non-interacting system, which leads to

$$a_k(t) = \exp(-i\varepsilon_k t) a_k. \quad (\text{II.327})$$

Let's consider a heuristic example involving one-dimensional non-interacting fermions in a vacuum state  $|\text{vac}\rangle$ . The field operator is given by

$$\psi(x_1, t) = \sum_k a_k(t) e^{ikx_1} = \sum_k a_k e^{-i\varepsilon_k t} e^{ikx_1},$$

so that the one-particle Green's function for the vacuum state is

$$\begin{aligned} G_{\text{vac}}(x_1, t) &= -i \langle \text{vac} | T(\psi(x_1, t) \psi^\dagger(0, 0)) | \text{vac} \rangle, \\ &= -i \langle \text{vac} | T\left(\sum_{k, k'} a_k a_{k'}^\dagger e^{-i\varepsilon_k t} e^{ikx_1}\right) | \text{vac} \rangle, \\ &= -i \sum_k e^{-i\varepsilon_k t} e^{ikx_1} && \text{for } t > 0; \\ &= 0 && \text{for } t < 0. \end{aligned}$$

Hence,

$$G_{\text{vac}}(x_1, 0^+) = -i \sum_k e^{ikx_1} = -i\delta(x_1),$$

and

$$\begin{aligned} G_{\text{vac}}(x_1, t > 0) &= -\frac{i}{2\pi} \int_{-\infty}^{\infty} dk \exp\left[i\left(kx_1 - \frac{\hbar k^2}{2m} t\right)\right], \\ &= e^{-i3\pi/4} \left(\frac{2m\pi}{\hbar t}\right)^{1/2} e^{i(mx_1^2)/(2\hbar t)}, \end{aligned}$$

which is consistent with the differential equation:

$$\left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_1^2} \right) G_{\text{vac}}(x_1, t) = \delta(x_1) \delta(t).$$

Next, we consider the one-particle Green's function for the ground state of non-interacting Fermi sea in one dimension. (N.B.: In reality, one-dimensional fermions are necessarily strongly interacting and form a non-Fermi liquid system known as the Luttinger liquid. However, we shall ignore this complication for now.) In this case, the ground state is a filled Fermi sea (with occupied momentum ranging from  $-k_F$  to  $k_F$ ) rather than vacuum. Therefore, the one-particle Green's function  $G_0(x_1, t > 0)$  deals with a fermion propagating to an unfilled state, whereas  $G_0(x_1, t < 0)$  concerns with a fermion propagating from a filled state. Thus, we have

$$G_0(x_1, t > 0) = -\frac{i}{2\pi} \left[ \int_{-\infty}^{-k_F} dk e^{ikx_1} e^{-i(\hbar k^2 t/2m)} + \int_{k_F}^{\infty} dk e^{ikx_1} e^{-i(\hbar k^2 t/2m)} \right],$$

$$G_0(x_1, t < 0) = -\frac{i}{2\pi} \left[ \int_{-k_F}^{k_F} dk e^{ikx_1} e^{-i(\hbar k^2 t/2m)} \right],$$

so that

$$G_0(x_1, 0^+) = -\frac{i}{\pi} \left[ \pi \delta(x_1) - \frac{\sin(k_F x_1)}{x_1} \right],$$

and

$$G_0(x_1, 0^-) = i \frac{\sin(k_F x_1)}{\pi x_1}.$$

In the case of three-dimensional fermion systems, we may Fourier transform the Green's functions into the following:

$$G_0(\mathbf{k}, t) = \int d^3 r e^{-i\mathbf{k}\cdot\mathbf{r}} G_0(\mathbf{r}, t), \quad \Leftrightarrow \quad G_0(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int d^3 r e^{i\mathbf{k}\cdot\mathbf{r}} G_0(\mathbf{k}, t); \quad (\text{II.328})$$

$$G_0(\mathbf{k}, \omega) = \int dt e^{i\omega t} G_0(\mathbf{k}, t), \quad \Leftrightarrow \quad G_0(\mathbf{k}, t) = \frac{1}{2\pi} \int d\omega e^{-i\omega t} G_0(\mathbf{k}, \omega). \quad (\text{II.329})$$

Consequently, the 3D non-interacting one-particle Green's function  $G_0(\mathbf{k}, t)$  in the ground state  $|0\rangle$  of a filled Fermi sea is given by

$$G_0(\mathbf{k}, t) = -i \langle 0 | T \left( \sum_{\mathbf{k}'} a_{\mathbf{k}'} a_{\mathbf{k}}^\dagger e^{-i\varepsilon_{\mathbf{k}'} t} \int d^3 r e^{-i\mathbf{k}\cdot\mathbf{r}} e^{i\mathbf{k}'\cdot\mathbf{r}} \right) | 0 \rangle$$

$$= -i \langle 0 | a_{\mathbf{k}} a_{\mathbf{k}}^\dagger | 0 \rangle e^{-i\varepsilon_{\mathbf{k}} t} = -i(1 - n_{\mathbf{k}}) e^{-i\varepsilon_{\mathbf{k}} t}, \quad (t > 0); \quad (\text{II.330})$$

$$= i \langle 0 | a_{\mathbf{k}}^\dagger a_{\mathbf{k}} | 0 \rangle e^{-i\varepsilon_{\mathbf{k}} t} = i n_{\mathbf{k}} e^{-i\varepsilon_{\mathbf{k}} t}, \quad (t < 0), \quad (\text{II.331})$$

where  $\langle 0 | a_{\mathbf{k}}^\dagger a_{\mathbf{k}} | 0 \rangle \equiv n_{\mathbf{k}}$  denotes the occupation number of the state  $\mathbf{k}$  and  $n_{\mathbf{k}}$  is either 0 or 1 for fermions. Therefore, we may rewrite EQs. (II.330) and (II.331) into the following:

$$G_0(\mathbf{k}, t) = -i e^{-i\varepsilon_{\mathbf{k}} t} \theta(t) \quad \text{if } |\mathbf{k}| > k_F; \quad (\text{II.332})$$

$$G_0(\mathbf{k}, t) = i e^{-i\varepsilon_{\mathbf{k}} t} \theta(-t) \quad \text{if } |\mathbf{k}| < k_F, \quad (\text{II.333})$$

where  $\theta(t)$  is the step function. Moreover, from EQs. (II.329), (II.332) and (II.333), we have

$$G_0(\mathbf{k}, \omega) = \lim_{\alpha \rightarrow 0^+} \frac{1}{\omega - \varepsilon_{\mathbf{k}} + i\alpha \operatorname{sgn}(\omega - \varepsilon_{\mathbf{k}})}. \quad (\text{II.334})$$

This non-relativistic Green's function for a non-interacting system may be compared with the relativistic Green's function for the fermion propagator given in EQ. (II.218) and that for the boson propagator given in EQ. (II.48).

In a real Fermi gas, we expect the quasiparticle excitations to only last for a lifetime  $\sim (\Gamma_{\mathbf{k}})^{-1}$  so that EQs. (II.332) and (II.333) are modified into

$$G_0(\mathbf{k}, t) = -i e^{-i\varepsilon_{\mathbf{k}}t} e^{-\Gamma_{\mathbf{k}}t} \theta(t) \quad \text{if } |\mathbf{k}| > k_F; \quad (\text{II.335})$$

$$G_0(\mathbf{k}, t) = i e^{-i\varepsilon_{\mathbf{k}}t} e^{\Gamma_{\mathbf{k}}t} \theta(-t) \quad \text{if } |\mathbf{k}| < k_F, \quad (\text{II.336})$$

and

$$G_0^>(\mathbf{k}, \omega) = \lim_{\Gamma_{\mathbf{k}} \rightarrow 0^+} \frac{1}{\omega - \varepsilon_{\mathbf{k}} + i\Gamma_{\mathbf{k}}} \quad \text{if } |\mathbf{k}| > k_F; \quad (\text{II.337})$$

$$G_0^<(\mathbf{k}, \omega) = \lim_{\Gamma_{\mathbf{k}} \rightarrow 0^+} \frac{1}{\omega - \varepsilon_{\mathbf{k}} - i\Gamma_{\mathbf{k}}} \quad \text{if } |\mathbf{k}| < k_F. \quad (\text{II.338})$$

The results given above for non-interacting fermions in the ground state (at zero temperature) can be further generalized to non-interacting bosons and fermions at finite temperatures if we replace the number operator  $n_{\mathbf{k}}$  by the thermal average number operator  $\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle$ , which, for non-interacting particles with a chemical potential  $\mu$ , is given as follows:

$$\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle = f_{\mp}(\varepsilon_{\mathbf{k}}) = \frac{1}{\exp[\beta(\varepsilon_{\mathbf{k}} - \mu)] \mp 1}, \quad (\text{II.339})$$

where the minus (plus) sign in the denominator refers to boson (fermion) statistics. Thus, we can express various forms of the Green's functions defined according to EQs. (II.316) -- (II.320) into the following (with  $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$ ,  $\tau \equiv t - t'$ ):

$$\begin{aligned} \tilde{g}^>(\mathbf{k}, \tau) &= \int d^3R \left[ e^{-i\mathbf{k}\cdot\mathbf{R}} \tilde{g}^>(\mathbf{r}, t' + \tau; \mathbf{r}', t') \right] = -i \langle a_{\mathbf{k}}(t' + \tau) a_{\mathbf{k}}^\dagger(t') \rangle \\ &= -i \langle e^{-i\varepsilon_{\mathbf{k}}\tau} a_{\mathbf{k}} a_{\mathbf{k}}^\dagger \rangle = -i e^{-i\varepsilon_{\mathbf{k}}\tau} \left( 1 \pm \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle \right). \end{aligned} \quad (\text{II.340})$$

Similarly, it can be easily verified that

$$\tilde{g}^<(\mathbf{k}, \tau) = \mp i e^{-i\varepsilon_{\mathbf{k}}\tau} \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle. \quad (\text{II.341})$$

Following the derivation given above, the other Green's functions can be derived and summarized below:

$$\begin{aligned} g(\mathbf{k}, \tau) &= -i e^{-i\varepsilon_{\mathbf{k}}\tau} \left[ 1 \pm f_{\mp}(\varepsilon_{\mathbf{k}}) \right], \quad (\tau > 0); \\ &= \mp i e^{-i\varepsilon_{\mathbf{k}}\tau} f_{\mp}(\varepsilon_{\mathbf{k}}), \quad (\tau < 0); \end{aligned} \quad (\text{II.342})$$

$$\tilde{g}(\mathbf{k}, \tau) = -i e^{-i\varepsilon_{\mathbf{k}}\tau}; \quad (\text{II.343})$$

$$g^R(\mathbf{k}, \tau) = -i \theta(\tau) e^{-i\varepsilon_{\mathbf{k}}\tau}; \quad (\text{II.344})$$

$$g^A(\mathbf{k}, \tau) = i \theta(-\tau) e^{-i\varepsilon_{\mathbf{k}}\tau}. \quad (\text{II.345})$$

Interestingly, we note that for non-interacting systems, the Green's functions  $\tilde{g}$ ,  $g^R$  and  $g^A$  do not involve either temperature or chemical potential  $\mu$ , and are therefore identical to those associated with a single particle moving in vacuum. In contrast, the Green's functions  $\tilde{g}^>$ ,  $\tilde{g}^<$  and  $g$  involve information pertaining not only to the motion of the added particle (or hole) but also to the state of the system as well. In addition, for fermions in the limit of  $T \rightarrow 0$ , we have  $\mu = \varepsilon_F = k_F^2/(2m)$ . Therefore  $g(\mathbf{k}, \tau)$  at  $T = 0$  becomes

$$\begin{aligned} g(\mathbf{k}, \tau) &= -ie^{-i\varepsilon_{\mathbf{k}}\tau} \theta(|\mathbf{k}| - k_F), & (\tau > 0); \\ &= ie^{-i\varepsilon_{\mathbf{k}}\tau} \theta(k_F - |\mathbf{k}|), & (\tau < 0). \end{aligned} \quad (\text{II.346})$$

On the other hand, for bosons in the  $T \rightarrow 0$  limit, the situation is more complicated because of the phenomenon known as the Bose condensation. We shall consider the case of Bose condensation later.

Next, we take the Fourier transformation with respect to time for the Green's functions given above, and obtain the following expressions:

$$\tilde{g}(\mathbf{k}, \omega) = -2\pi i \delta(\omega - \varepsilon_{\mathbf{k}}); \quad (\text{II.347})$$

$$g^R(\mathbf{k}, \omega) = \lim_{s \rightarrow 0^+} \frac{1}{\omega + is - \varepsilon_{\mathbf{k}}}; \quad (\text{II.348})$$

$$g^A(\mathbf{k}, \omega) = \lim_{s \rightarrow 0^+} \frac{1}{\omega - is - \varepsilon_{\mathbf{k}}}. \quad (\text{II.349})$$

For fermions at  $T = 0$ , we obtain from EQ. (II.346):

$$\begin{aligned} g(\mathbf{k}, \omega) &= \lim_{s \rightarrow 0^+} \left[ \frac{\theta(|\mathbf{k}| - k_F)}{\omega - \varepsilon_{\mathbf{k}} + is} + \frac{\theta(k_F - |\mathbf{k}|)}{\omega - \varepsilon_{\mathbf{k}} - is} \right], \\ &= \lim_{s \rightarrow 0^+} \left[ \frac{1}{\omega - \varepsilon_{\mathbf{k}} + is (\text{sgn}(|\mathbf{k}| - k_F))} \right], \end{aligned} \quad (\text{II.350})$$

where  $\text{sgn}(x) = 1$  for  $x > 0$  and  $\text{sgn}(x) = -1$  for  $x < 0$ . For convenience, we may define

$$G(\mathbf{k}, z) = \frac{1}{z - \varepsilon_{\mathbf{k}}}, \quad (\text{II.351})$$

so that EQs. (II.348) -- (II.350) are rewritten as follows:

$$g^R(\mathbf{k}, \omega) = \lim_{s \rightarrow 0^+} G(\mathbf{k}, \omega + is), \quad (\text{II.352})$$

$$g^A(\mathbf{k}, \omega) = \lim_{s \rightarrow 0^+} G(\mathbf{k}, \omega - is), \quad (\text{II.353})$$

$$g(\mathbf{k}, \omega) = \lim_{s \rightarrow 0^+} G(\mathbf{k}, \omega + is [\text{sgn}(\omega - \varepsilon_F)]). \quad (\text{II.354})$$

### [Interacting Green's functions and Lehmann representation]

To find the form of the Green's function in the exact ground state  $|0\rangle$  of an interacting Fermi gas, we need to express the particle operators in the Heisenberg representation. That is,

$$a_{\mathbf{k}}(t) = e^{i\mathcal{H}t} a_{\mathbf{k}} e^{-i\mathcal{H}t}, \quad (\text{II.355})$$

and

$$\begin{aligned}
 g(\mathbf{k}, t) &= -i \langle 0 | T(a_{\mathbf{k}}(t) a_{\mathbf{k}}^\dagger(0)) | 0 \rangle \\
 &= -i \langle 0 | e^{i\mathcal{H}t} a_{\mathbf{k}} e^{-i\mathcal{H}t} a_{\mathbf{k}}^\dagger | 0 \rangle, & t > 0; \\
 &= i \langle 0 | a_{\mathbf{k}}^\dagger e^{i\mathcal{H}t} a_{\mathbf{k}} e^{-i\mathcal{H}t} | 0 \rangle, & t < 0.
 \end{aligned} \tag{II.356}$$

Assuming the exact ground state energy of the  $N$ -particle system is  $E_0^N$  so that  $\mathcal{H}|0\rangle = E_0^N|0\rangle$ , we obtain

$$\begin{aligned}
 g(\mathbf{k}, t) &= -i \langle 0 | a_{\mathbf{k}} e^{-i\mathcal{H}t} a_{\mathbf{k}}^\dagger | 0 \rangle e^{iE_0^N t}, & t > 0; \\
 &= i \langle 0 | a_{\mathbf{k}}^\dagger e^{i\mathcal{H}t} a_{\mathbf{k}} | 0 \rangle e^{-iE_0^N t}, & t < 0.
 \end{aligned} \tag{II.357}$$

If we further denote the excited states of the system as  $|\ell\rangle$  so that  $|\ell\rangle$  represents the  $(N+1)$ -particle system for  $t > 0$  and the  $(N-1)$ -particle system for  $t < 0$ , we may rewrite EQ. (II.357) into the following form:

$$\begin{aligned}
 g(\mathbf{k}, t) &= -i \sum_{\ell} \langle 0 | a_{\mathbf{k}} e^{-i\mathcal{H}t} |\ell\rangle \langle \ell | a_{\mathbf{k}}^\dagger | 0 \rangle e^{iE_0^N t} = -i \sum_{\ell} |\langle \ell | a_{\mathbf{k}}^\dagger | 0 \rangle|^2 e^{i(E_0^N - E_{\ell}^{N+1})t}, & (t > 0); \\
 &= i \sum_{\ell} \langle 0 | a_{\mathbf{k}}^\dagger e^{i\mathcal{H}t} |\ell\rangle \langle \ell | a_{\mathbf{k}} | 0 \rangle e^{-iE_0^N t} = i \sum_{\ell} |\langle \ell | a_{\mathbf{k}} | 0 \rangle|^2 e^{i(E_{\ell}^{N-1} - E_0^N)t}, & (t < 0).
 \end{aligned} \tag{II.358}$$

Noting that the chemical potential  $\mu$  is given by (for  $N \gg 1$ )

$$\mu \equiv \frac{\partial E}{\partial N} \approx E_0^{N+1} - E_0^N \approx E_0^N - E_0^{N-1},$$

and

$$\begin{aligned}
 -(E_{\ell}^{N+1} - E_0^N) &= -[(E_{\ell}^{N+1} - E_0^{N+1}) + (E_0^{N+1} - E_0^N)] \approx -(\varepsilon_{\ell} + \mu), \\
 (E_{\ell}^{N-1} - E_0^N) &= (E_{\ell}^{N-1} - E_0^{N-1}) + (E_0^{N-1} - E_0^N) \approx (\varepsilon_{\ell} - \mu),
 \end{aligned}$$

we obtain

$$\begin{aligned}
 g(\mathbf{k}, t) &= -i \sum_{\ell} |\langle \ell | a_{\mathbf{k}}^\dagger | 0 \rangle|^2 e^{-i(\varepsilon_{\ell} + \mu)t}, & (t > 0); \\
 &= i \sum_{\ell} |\langle \ell | a_{\mathbf{k}} | 0 \rangle|^2 e^{i(\varepsilon_{\ell} - \mu)t}, & (t < 0).
 \end{aligned} \tag{II.359}$$

Thus, the Fourier transform of EQ. (II.359) becomes:

$$\begin{aligned}
 g(\mathbf{k}, \omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} g(\mathbf{k}, t) = \int_{-\infty}^0 dt e^{i\omega t} g(\mathbf{k}, t) + \int_0^{\infty} dt e^{i\omega t} g(\mathbf{k}, t) \\
 &= \lim_{\alpha \rightarrow 0^+} \left[ -i \int_0^{\infty} dt e^{i\omega t} \sum_{\ell} |\langle \ell | a_{\mathbf{k}}^\dagger | 0 \rangle|^2 e^{-i(\varepsilon_{\ell} + \mu)t - \alpha t} + i \int_{-\infty}^0 dt e^{i\omega t} \sum_{\ell} |\langle \ell | a_{\mathbf{k}} | 0 \rangle|^2 e^{i(\varepsilon_{\ell} - \mu)t + \alpha t} \right] \\
 &= \lim_{\alpha \rightarrow 0^+} \int_0^{\infty} d\omega' \left[ -i \int_0^{\infty} dt e^{i\omega t} \sum_{\ell} |\langle \ell | a_{\mathbf{k}}^\dagger | 0 \rangle|^2 \delta(\omega' - \varepsilon_{\ell}) e^{-i(\omega' + \mu)t} + i \int_{-\infty}^0 dt e^{i\omega t} \sum_{\ell} |\langle \ell | a_{\mathbf{k}} | 0 \rangle|^2 \delta(\omega' - \varepsilon_{\ell}) e^{i(\omega' - \mu)t} \right] \\
 &= \lim_{\alpha \rightarrow 0^+} \int_0^{\infty} d\omega' \left[ \frac{\sum_{\ell} |\langle \ell | a_{\mathbf{k}}^\dagger | 0 \rangle|^2 \delta(\omega' - \varepsilon_{\ell})}{(\omega - \mu) - \omega' + i\alpha} + \frac{\sum_{\ell} |\langle \ell | a_{\mathbf{k}} | 0 \rangle|^2 \delta(\omega' - \varepsilon_{\ell})}{(\omega - \mu) + \omega' - i\alpha} \right] \\
 &\equiv \lim_{\alpha \rightarrow 0^+} \int_0^{\infty} d\omega' \left[ \frac{\rho^+(\mathbf{k}, \omega')}{(\omega - \mu) - \omega' + i\alpha} + \frac{\rho^-(\mathbf{k}, \omega')}{(\omega - \mu) + \omega' - i\alpha} \right],
 \end{aligned} \tag{II.360}$$

where  $\rho^\pm(\mathbf{k}, \omega')$  are known as the spectral density functions. The expression of  $g(\mathbf{k}, \omega)$  in terms of the spectral density functions given in EQ. (II.360) is known as the Lehmann representation.

Using EQ. (II.286) and the Lehmann representation in EQ. (II.360), we find that the imaginary part of  $g(\mathbf{k}, \omega)$  becomes

$$\begin{aligned} \text{Im}[g(\mathbf{k}, \omega)] &= -\pi \rho^+(\mathbf{k}, \omega - \mu) && \text{if } \omega > \mu; \\ &= \pi \rho^-(\mathbf{k}, \mu - \omega) && \text{if } \omega < \mu. \end{aligned} \quad (\text{II.361})$$

Thus,  $\text{Im}[g(\mathbf{k}, \omega)]$  changes sign at  $\omega = \mu$  because the spectral density functions are positive definite. Furthermore, the real part of the Green's function is related to  $\text{Im}[g(\mathbf{k}, \omega)]$  via the following expression:

$$\begin{aligned} \text{Re}[g(\mathbf{k}, \omega)] &= \mathcal{P} \left\{ \int_0^\infty d\omega' \left[ \frac{\rho^+(\mathbf{k}, \omega')}{(\omega - \mu) - \omega'} + \frac{\rho^-(\mathbf{k}, \omega')}{(\omega - \mu) + \omega'} \right] \right\} \\ &= \frac{1}{\pi} \mathcal{P} \left\{ \int_0^\infty d\omega' \left[ \frac{-\text{Im}[g(\mathbf{k}, \omega' + \mu)]}{(\omega - \mu) - \omega'} + \frac{\text{Im}[g(\mathbf{k}, \mu - \omega')]}{(\omega - \mu) + \omega'} \right] \right\} \\ &= -\frac{1}{\pi} \mathcal{P} \left\{ \int_\mu^\infty d\omega' \frac{\text{Im}[g(\mathbf{k}, \omega')]}{(\omega - \omega')} \right\} + \frac{1}{\pi} \mathcal{P} \left\{ \int_{-\infty}^\mu d\omega' \frac{\text{Im}[g(\mathbf{k}, \omega')]}{(\omega - \omega')} \right\} \\ &= \frac{1}{\pi} \mathcal{P} \left\{ \left[ \int_\mu^\infty d\omega' - \int_{-\infty}^\mu d\omega' \right] \frac{\text{Im}[g(\mathbf{k}, \omega')]}{(\omega' - \omega)} \right\}. \end{aligned} \quad (\text{II.362})$$

The above consideration of ground state Green's functions at  $T = 0$  can be generalized to finite temperatures by introducing a statistical operator

$$\rho_m = \exp[-\beta(\mathcal{E}_m - \mu N_m)] / \text{Tr} \left\{ \exp[-\beta(\mathcal{H} - \mu N)] \right\}, \quad (\text{II.363})$$

where  $\mathcal{H}|m\rangle = \mathcal{E}_m|m\rangle$ ,  $N|m\rangle = N_m|m\rangle$ ,  $|m\rangle$  denotes the eigen-functions of  $\mathcal{H}$  and  $N$ , and  $\beta = T^{-1}$ .

Given the statistical operator we can express the Green's functions in their finite-temperature forms. For instance, the quantity  $\tilde{g}^>(\mathbf{k}, t)$ :

$$\begin{aligned} \tilde{g}^>(\mathbf{k}, t) &= -i \langle a_{\mathbf{k}}(t) a_{\mathbf{k}}^\dagger(0) \rangle = -i \sum_m \rho_m \langle m | a_{\mathbf{k}}(t) a_{\mathbf{k}}^\dagger(0) | m \rangle, \\ &= -i \sum_{m\ell} \rho_m \langle m | e^{i\mathcal{H}t} a_{\mathbf{k}} e^{-i\mathcal{H}t} | \ell \rangle \langle \ell | a_{\mathbf{k}}^\dagger | m \rangle, \\ &= -i \sum_{m\ell} \rho_m e^{-i(\mathcal{E}_\ell - \mathcal{E}_m)t} \left| \langle \ell | a_{\mathbf{k}}^\dagger | m \rangle \right|^2. \end{aligned} \quad (\text{II.364})$$

From EQ. (II.364) we obtain  $\tilde{g}^>(\mathbf{k}, \omega)$  as

$$\begin{aligned} \int_0^\infty dt e^{i\omega t} \tilde{g}^>(\mathbf{k}, t) &= \int_{-\infty}^\infty dt e^{i\omega t} \tilde{g}^>(\mathbf{k}, t) \theta(t) \\ &= \lim_{\alpha \rightarrow 0^+} \left[ -i \int_0^\infty dt e^{i\omega t} \sum_{m\ell} \rho_m \left| \langle \ell | a_{\mathbf{k}}^\dagger | m \rangle \right|^2 e^{-i(\mathcal{E}_\ell - \mathcal{E}_m)t - \alpha t} \right] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{m\ell} \rho_m |\langle \ell | a_{\mathbf{k}}^\dagger | m \rangle|^2 \frac{1}{\omega - (\varepsilon_\ell - \varepsilon_m) + i\alpha} \\
 &= \sum_{m\ell} \rho_m |\langle m | a_{\mathbf{k}} | \ell \rangle|^2 \frac{1}{\omega - (\varepsilon_\ell - \varepsilon_m) + i\alpha} \\
 (\ell \Leftrightarrow m) \quad &= \sum_{\ell m} \rho_\ell |\langle \ell | a_{\mathbf{k}} | m \rangle|^2 \frac{1}{\omega + (\varepsilon_\ell - \varepsilon_m) + i\alpha}. \tag{II.365}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \int_{-\infty}^0 dt e^{i\omega t} g(\mathbf{k}, t) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \theta(-t) \tilde{g}^<(\mathbf{k}, t) = \sum_{m\ell} \rho_m |\langle \ell | a_{\mathbf{k}} | m \rangle|^2 \frac{1}{\omega + (\varepsilon_\ell - \varepsilon_m) - i\alpha} \\
 &= \sum_{\ell m} \rho_\ell e^{\beta[\mu(N_m - N_\ell) + (\varepsilon_\ell - \varepsilon_m)]} |\langle \ell | a_{\mathbf{k}} | m \rangle|^2 \frac{1}{\omega + (\varepsilon_\ell - \varepsilon_m) - i\alpha}. \tag{II.366}
 \end{aligned}$$

If the temperature is not very high, it is reasonable to assume that the matrix elements  $|\langle \ell | a_{\mathbf{k}} | m \rangle|$  are dominated by those with  $N_m - N_\ell = 1$ . Hence, we may consolidate the expressions in EQs. (II.365) and (II.366), yielding

$$\begin{aligned}
 g(\mathbf{k}, \omega) &= \sum_{\ell m} \rho_\ell |\langle \ell | a_{\mathbf{k}} | m \rangle|^2 \left[ \frac{1}{\omega + (\varepsilon_\ell - \varepsilon_m) + i\alpha} + \frac{e^{\beta[\mu + (\varepsilon_\ell - \varepsilon_m)]}}{\omega + (\varepsilon_\ell - \varepsilon_m) - i\alpha} \right] \\
 &= \sum_{\ell m} \rho_\ell |\langle \ell | a_{\mathbf{k}} | m \rangle|^2 \left\{ \mathcal{P} \left[ \frac{e^{\beta[\mu + (\varepsilon_\ell - \varepsilon_m)]} + 1}{\omega + (\varepsilon_\ell - \varepsilon_m)} \right] + i\pi \left( e^{\beta[\mu + (\varepsilon_\ell - \varepsilon_m)]} - 1 \right) \delta[\omega + (\varepsilon_\ell - \varepsilon_m)] \right\}. \tag{II.367}
 \end{aligned}$$

Thus, we find that

$$\tilde{g}^<(\mathbf{k}, \omega) = e^{\beta(\mu - \omega)} \tilde{g}^>(\mathbf{k}, \omega), \tag{II.368}$$

so that at  $T = 0$ ,

$$\tilde{g}^<(\mathbf{k}, \omega) = 0 \text{ for } \omega > \mu, \quad \tilde{g}^>(\mathbf{k}, \omega) = 0 \text{ for } \omega < \mu. \tag{II.369}$$

Moreover, EQ. (II.367) implies that

$$\text{Re}[g(\mathbf{k}, \omega)] = \frac{1}{\pi} \mathcal{P} \left\{ \int_{-\infty}^{\infty} d\omega' \coth \left[ \frac{\beta(\omega' - \mu)}{2} \right] \frac{\text{Im}[g(\mathbf{k}, \omega')]}{(\omega' - \omega)} \right\}. \tag{II.370}$$

Given the properties of  $\tilde{g}^<(\mathbf{k}, \omega)$  and  $\tilde{g}^>(\mathbf{k}, \omega)$ , we may define a quantity  $A(\mathbf{k}, \omega)$  that can be shown to represent the generalized density of states in the  $(\mathbf{k}, \omega)$ -space:

$$A(\mathbf{k}, \omega) \equiv i\tilde{g}(\mathbf{k}, \omega) = i[\tilde{g}^>(\mathbf{k}, \omega) - \tilde{g}^<(\mathbf{k}, \omega)]. \tag{II.371}$$

Using EQ. (II.371), we find that  $\tilde{g}^<(\mathbf{k}, \omega)$  and  $\tilde{g}^>(\mathbf{k}, \omega)$  can be rewritten as:

$$\tilde{g}^>(\mathbf{k}, \omega) = -iA(\mathbf{k}, \omega)[1 \pm f_{\mp}(\omega)], \tag{II.372}$$

$$\rightarrow -iA(\mathbf{k}, \omega)\theta(\omega - \mu) \quad \text{for fermions at } T \rightarrow 0; \tag{II.373}$$

$$\tilde{g}^<(\mathbf{k}, \omega) = \mp iA(\mathbf{k}, \omega)f_{\mp}(\omega), \tag{II.374}$$

$$\rightarrow iA(\mathbf{k}, \omega)\theta(\mu - \omega) \quad \text{for fermions at } T \rightarrow 0; \quad (\text{II.375})$$

where

$$f_{\mp}(\omega) \equiv \frac{1}{\exp[\beta(\omega - \mu)] \mp 1}. \quad (\text{II.376})$$

As stated previously, the situation for bosons at  $T = 0$  is more complicated due to the phenomenon of Bose condensation, and therefore is not explicitly expressed in EQs. (II.373) and (II.375). We further note that  $A(\mathbf{k}, \omega)$  is real, which is consistent with the fact that is the generalized density of states of the many-body system under consideration. Furthermore, we show in the following that  $A(\mathbf{k}, \omega)$  satisfies the sum rule:

$$\int \frac{d\omega}{2\pi} A(\mathbf{k}, \omega) = 1. \quad (\text{II.377})$$

Equation (II.377) can be easily proven as follows:

$$\begin{aligned} \int \frac{d\omega}{2\pi} A(\mathbf{k}, \omega) &= i \int \frac{d\omega}{2\pi} \int dt e^{i\omega t} [\tilde{g}^>(\mathbf{k}, t) - \tilde{g}^<(\mathbf{k}, t)], \\ &= i [\tilde{g}^>(\mathbf{k}, 0) - \tilde{g}^<(\mathbf{k}, 0)] = \langle a_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} \mp a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \rangle = 1. \end{aligned}$$

In addition, we find that the function  $G(\mathbf{k}, \omega)$  is related to  $A(\mathbf{k}, \omega)$ :

$$G(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{A(\mathbf{k}, \omega')}{\omega - \omega'}. \quad (\text{II.378})$$

The function  $G(\mathbf{k}, \omega)$  is analytic in the complex  $\omega$ -plane and has singularities (branch cuts, in general) along the portions of the real axis where  $A(\mathbf{k}, \omega) \neq 0$ .

From EQs. (II.352) -- (II.354) and (II.378), also using  $\lim_{s \rightarrow 0^+} (x + is)^{-1} = \mathcal{P}(x^{-1}) - i\pi\delta(x)$ , we obtain:

$$\text{Re}[g^R(\mathbf{k}, \omega)] = \text{Re}[g^A(\mathbf{k}, \omega)] = \mathcal{P} \left\{ \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{A(\mathbf{k}, \omega')}{\omega - \omega'} \right\}, \quad (\text{II.379})$$

$$\text{Im}[g^R(\mathbf{k}, \omega)] = -\text{Im}[g^A(\mathbf{k}, \omega)] = -\frac{1}{2} A(\mathbf{k}, \omega). \quad (\text{II.380})$$

Noting that  $g = g^R + \tilde{g}^<$ , we find:

$$\text{Re}[g(\mathbf{k}, \omega)] = \text{Re}[g^R(\mathbf{k}, \omega)] = \text{Re}[g^A(\mathbf{k}, \omega)] = \mathcal{P} \left\{ \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{A(\mathbf{k}, \omega')}{\omega - \omega'} \right\}, \quad (\text{II.381})$$

$$\text{Im}[g(\mathbf{k}, \omega)] = -\frac{1}{2} A(\mathbf{k}, \omega) [1 \pm 2f_{\mp}(\omega)] = \text{Im}[g^R(\mathbf{k}, \omega)] \coth \left[ \frac{\beta(\omega - \mu)}{2} \right] \quad \text{for bosons;} \quad (\text{II.382})$$

$$= \text{Im}[g^R(\mathbf{k}, \omega)] \tanh \left[ \frac{\beta(\omega - \mu)}{2} \right] \quad \text{for fermions.} \quad (\text{II.383})$$

From EQ. (II.383), it follows that a fermion system in the limit of  $T \rightarrow 0$  satisfies the relation

$$\begin{aligned}\text{Im}[g(\mathbf{k}, \omega)] &= \text{sgn}(\omega - \mu) \text{Im}[g^R(\mathbf{k}, \omega)], \\ &= -\frac{1}{2} \text{sgn}(\omega - \mu) A(\mathbf{k}, \omega) = \lim_{s \rightarrow 0^+} \text{Im}[G(k, \omega + is \text{sgn}(\omega - \mu))].\end{aligned}\quad (\text{II.384})$$

In other words, at  $T = 0$ ,  $\text{Im}[g(\mathbf{k}, \omega)]$  can be determined directly from experimental measurements of the general density of states  $A(\mathbf{k}, \omega < \mu)$ .

### [Relating Green's functions to physical observables]

Next, we investigate the relation of Green's functions to various important physical properties in condensed matter systems besides the density of states. For instance, consider the second quantization formalism for a first-quantization one-particle operator  $F(\mathbf{r})$ :

$$F = \int d^3 r \psi^\dagger(\mathbf{r}) F(\mathbf{r}) \psi(\mathbf{r}). \quad (\text{II.385})$$

Examples of operators  $F$  include the kinetic energy  $\mathcal{K}$  with  $\mathcal{K}(\mathbf{r}) = -\nabla^2/(2m)$ , the total number of particles  $N$  with  $N(\mathbf{r}) = 1$ , and the density operator  $n(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_0)$ . It follows from the definition of  $\tilde{g}^<(x, x')$  that the thermal average of  $F$  is given by:

$$\langle F \rangle = \pm i \int d^3 r F(\mathbf{r}) \tilde{g}^<(\mathbf{r}, t; \mathbf{r}, t) = \pm i \lim_{\mathbf{r}' \rightarrow \mathbf{r}} \int d^3 r F(\mathbf{r}) \lim_{t' \rightarrow t^+} g(\mathbf{r}, t; \mathbf{r}', t'). \quad (\text{II.386})$$

More explicitly, using EQ. (II.386) we can obtain the thermal average of the kinetic energy:

$$\begin{aligned}\langle \mathcal{K} \rangle &= \pm i \lim_{\mathbf{r}' \rightarrow \mathbf{r}} \int d^3 r \lim_{t' \rightarrow t^+} \left[ -\frac{\nabla_{\mathbf{r}}^2}{2m} g(\mathbf{r}, t; \mathbf{r}', t') \right] \\ &= \pm i \sum_{\mathbf{k}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{g}^<(\mathbf{k}, \omega) \frac{k^2}{2m} = \sum_{\mathbf{k}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{k}, \omega) f_{\mp}(\omega) \frac{k^2}{2m}.\end{aligned}\quad (\text{II.387})$$

Similarly, the thermal average of particle density becomes:

$$\langle n(\mathbf{r}) \rangle = \pm i \lim_{\mathbf{r}' \rightarrow \mathbf{r}} \lim_{t' \rightarrow t^+} g(\mathbf{r}, t; \mathbf{r}', t') = \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{k}, \omega) f_{\mp}(\omega), \quad (\text{II.388})$$

and the density in  $\mathbf{k}$ -space becomes

$$\langle n(\mathbf{k}) \rangle = \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{k}, \omega) f_{\mp}(\omega). \quad (\text{II.389})$$

We can also obtain the Hamiltonian for an interacting many-body system. Recall the pair interaction potential given in EQ. (II.323), which is reproduced below:

$$\left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) \psi(\mathbf{r}, t) = \left[ \int d^3 r_1 \psi^\dagger(\mathbf{r}_1, t) V(\mathbf{r} - \mathbf{r}_1) \psi(\mathbf{r}_1, t) \right] \psi(\mathbf{r}, t), \quad (\text{II.323})$$

and its adjoint:

$$\left(-i\frac{\partial}{\partial t'} + \frac{\nabla'^2}{2m}\right)\psi^\dagger(\mathbf{r}', t') = \psi^\dagger(\mathbf{r}', t') \left[ \int d^3 r_2 \psi^\dagger(\mathbf{r}_2, t') V(\mathbf{r}' - \mathbf{r}_2) \psi(\mathbf{r}_2, t') \right]. \quad (\text{II.390})$$

Multiplying EQ. (II.323) from the left by  $\psi^\dagger(\mathbf{r}', t')/4$  and EQ. (II.390) from the right by  $\psi(\mathbf{r}, t)/4$ , subtracting the resulting equations, putting  $\mathbf{r}' = \mathbf{r}$ ,  $t' = t$ , and integration over  $\mathbf{r}$ , we obtain

$$\frac{1}{4} \int d^3 r \left[ \left( i\frac{\partial}{\partial t} - i\frac{\partial}{\partial t'} \right) \psi^\dagger(\mathbf{r}, t') \psi(\mathbf{r}, t) \right]_{t'=t} = \frac{1}{2} \langle \mathcal{K} \rangle + \langle V \rangle. \quad (\text{II.391})$$

From EQs. (II.387) and (II.391), the thermal average of the interaction potential becomes

$$\langle V \rangle = \pm \frac{i}{2} \int d^3 r \left[ \left( i\frac{\partial}{\partial t} + \frac{\nabla_{\mathbf{r}}^2}{2m} \right) g(x, x') \right]_{\mathbf{r}' \rightarrow \mathbf{r}, t' \rightarrow t+} = \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \frac{1}{2} \left( \omega - \frac{k^2}{2m} \right) A(\mathbf{k}, \omega) f_{\mp}(\omega). \quad (\text{II.392})$$

Combining EQs. (II.387) and (II.392), we obtain the Hamiltonian

$$\langle \mathcal{H} \rangle = \pm \frac{i}{2} \int d^3 r \left[ \left( i\frac{\partial}{\partial t} - \frac{\nabla_{\mathbf{r}}^2}{2m} \right) g(x, x') \right]_{\mathbf{r}' \rightarrow \mathbf{r}, t' \rightarrow t+} = \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \frac{1}{2} \left( \omega + \frac{k^2}{2m} \right) A(\mathbf{k}, \omega) f_{\mp}(\omega). \quad (\text{II.393})$$

To obtain all other thermodynamic quantities, it is sufficient to calculate the grand canonical partition function  $Z_G = \text{Tr}\{\exp[-\beta(\mathcal{H} - \mu N)]\}$  as function of the volume  $\Omega$ , the chemical potential  $\mu$ , and the temperature  $T = \beta^{-1}$ . The grand canonical partition function  $Z_G$  is directly related to the pressure  $P$  by the thermodynamic equation

$$Z_G = \exp(\beta\Omega P), \quad (\text{II.394})$$

and the pressure can be expressed in terms of the density through the relation

$$P(\beta, \mu) = \int_{-\infty}^{\mu} d\mu' [n(\beta, \mu')] = \int_{-\infty}^{\mu} d\mu' \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{k}, \omega) f_{\mp}(\omega), \quad (\text{II.395})$$

where we have inserted the expression given in EQ. (II.388) for the density, and we note that both  $A(\mathbf{k}, \omega)$  and  $f(\omega)$  depend on the chemical potential  $\mu$ , and  $f(\omega)$  depends on the temperature.

However, in general the dependence on  $\mu$  may not be easily attainable. Hence, it is more convenient to take an alternative approach to calculate  $Z_G$ : Using the general thermodynamic relation

$$\left\langle \frac{\partial \mathcal{H}}{\partial \alpha} \right\rangle = -\Omega \left. \frac{\partial P}{\partial \alpha} \right|_{T, \mu}, \quad (\text{II.396})$$

where  $\alpha$  is a parameter in the Hamiltonian:

$$\mathcal{H} = \mathcal{K} + \alpha V, \quad (\text{II.397})$$

with  $\alpha = 0$  representing the non-interacting system and  $\alpha = 1$  representing the actual interacting system, we obtain

$$-\beta \langle V \rangle = \frac{\partial}{\partial \alpha} [\ln(Z_G)]. \quad (\text{II.398})$$

Integrating EQ. (II.398) over  $\alpha$  and using EQ. (II.392), we have

$$\ln(Z_G) = \beta P \Omega = \beta P_0 \Omega - \beta \int_0^1 d\alpha \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \frac{1}{2} \left( \omega - \frac{k^2}{2m} \right) A_\alpha(\mathbf{k}, \omega) f_{\mp}(\omega), \quad (\text{II.399})$$

where the subscript  $\alpha$  in  $A(\mathbf{k}, \omega)$  denotes that  $A_\alpha(\mathbf{k}, \omega)$  corresponds to the Hamiltonian in EQ. (II.397), and  $P_0$  is the pressure for the non-interacting system.

As seen in EQs. (II.387), (II.388), (II.392), (II.393), (II.395) and (II.399), various thermodynamic quantities that we have discussed so far all involve an integral of the following type:

$$I = \lim_{\sigma \rightarrow 0^-} I(\sigma) = \lim_{\sigma \rightarrow 0^-} \int \frac{d\omega}{2\pi} e^{-\omega\sigma} F(\omega) A(\mathbf{k}, \omega) f_{\mp}(\omega), \quad (\text{II.400})$$

where  $F(\omega)$  is a polynomial, and the term  $\exp(-\omega\sigma)$  is introduced to assist the computation of  $I$  in the complex  $\omega$ -plane. Noting that  $A(\mathbf{k}, \omega) = i\tilde{g}(\mathbf{k}, \omega) = i[G(\mathbf{k}, \omega + is) - G(\mathbf{k}, \omega - is)]$ , we can express the integration over  $\omega$  in EQ. (II.400) by the sum of two integrands, one along the upper  $\omega$ -plane and the other along the lower  $\omega$ -plane:

$$I(\sigma) = i \int_{C_R} \frac{d\omega}{2\pi} e^{-\omega\sigma} F(\omega) G(\mathbf{k}, \omega) f_{\mp}(\omega) - i \int_{C_A} \frac{d\omega}{2\pi} e^{-\omega\sigma} F(\omega) G(\mathbf{k}, \omega) f_{\mp}(\omega), \quad (\text{II.401})$$

where the paths  $C_R$  and  $C_A$  refer to the integration over the upper and lower half-planes, respectively, and the subscripts  $R$  and  $A$  are consistent with the those used in the definitions of Green's functions, denoting respectively the "retarded" and "advanced" chronological orders. The paths  $C_R$  and  $C_A$  are chosen to avoid the poles of  $f_{\mp}(\omega)$ , which are imposed by EQ. (II.308) as discussed earlier. More specifically, the poles  $z_\nu$  are slightly different between the cases for bosons and fermions, and are given explicitly below:

$$z_\nu = \mu + \frac{i\pi\nu}{\beta}; \quad \begin{array}{l} \nu = 2n \quad \text{for bosons,} \\ \nu = 2n - 1 \quad \text{for fermions,} \end{array} \quad (\text{II.402})$$

and  $n$  is positive for the path  $C_R$  and negative for the path  $C_A$ . Thus, for  $-\beta < \sigma < 0$ , EQ. (II.401) becomes

$$\begin{aligned} I(\sigma) &= I_R(\sigma) - I_A(\sigma) = \left[ \mp \frac{1}{\beta} \sum_{\nu > 0} e^{-z_\nu \sigma} F(z_\nu) G(\mathbf{k}, z_\nu) \right] - \left[ \pm \frac{1}{\beta} \sum_{\nu < 0} e^{-z_\nu \sigma} F(z_\nu) G(\mathbf{k}, z_\nu) \right] \\ &= \mp \frac{1}{\beta} \sum_{\nu} e^{-z_\nu \sigma} F(z_\nu) G(\mathbf{k}, z_\nu), \end{aligned}$$

and therefore

$$I = \mp \frac{1}{\beta} \sum_{\nu} F(z_\nu) G(\mathbf{k}, z_\nu), \quad (\text{II.403})$$

which suggests that the thermodynamic quantities of a many-body system can be obtained from the values of  $G(\mathbf{k}, \omega)$  at the special points  $z_\nu$  specified in EQ. (II.402). This result is convenient because generally  $G(\mathbf{k}, \omega)$  is easier to calculate than  $A(\mathbf{k}, \omega)$ .

In addition to the density of states and thermodynamic quantities, another class of important physical quantities related to the Green's functions is the linear response functions. The linear response functions

describe the response of a many-body system to a weak external perturbation, which differ from the thermodynamic quantities derived from thermal equilibrium conditions. In general the linear response functions are also manifestations of the fluctuation-dissipation theorem. We shall consider these issues later in Part II.9. We also remark that in the absence of interactions and for a translational invariant system, the quantity  $A(\mathbf{k}, \omega)$  is related to the density of states of the non-interacting system via the following:

$$A(\mathbf{k}, \omega) / 2\pi = \rho(\mathbf{k}, \omega) = \delta(\omega - \varepsilon_{\mathbf{k}}^0), \quad \varepsilon_{\mathbf{k}}^0 \equiv \frac{k^2}{2m}. \quad (\text{II.404})$$

Hence, we may interpret  $A(\mathbf{k}, \omega) / 2\pi$  as the generalized density of states per unit frequency and per unit volume in  $k$ -space. In addition, from EQ. (II.374)  $\tilde{g}^<(\mathbf{k}, \omega)$  can be interpreted (apart from a factor of  $\pm i$ ) as the thermally averaged number of particles per unit volume in the  $(\mathbf{k}, \omega)$ -space. Similarly, from EQ. (II.372)  $\tilde{g}^>(\mathbf{k}, \omega)$  can be interpreted (apart from a factor of  $\pm i$ ) as the thermally averaged number of states (per unit volume in the  $(\mathbf{k}, \omega)$ -space) available for the addition of an extra particle to the system.

Having seen the importance of Green's functions to the descriptions of various physical properties of many-body systems at all temperatures, we briefly discuss the equation of state method used to calculate the Green's functions of an interacting many-body system, followed by more elaborate studies of the perturbative diagrammatic method.

As mentioned earlier, the Green's functions for an interacting many-body system obey a hierarchy of equations, and the first of which in EQ. (II.323) is reproduced below:

$$\left( i \frac{\partial}{\partial t} + \frac{\nabla_{\mathbf{r}}^2}{2m} \right) g(x, x') = \delta(x - x') \pm i \int d^4 x_1 V(\mathbf{r} - \mathbf{r}_1) g_2(x, x_1; x', x_1^+) \Big|_{t_1=t}, \quad (\text{II.323})$$

In order to obtain an explicit solution, the hierarchy has to be terminated at some point by employing approximate relation that connects  $g_n$  with  $g_{n-1}$ ,  $g_{n-2}$ , etc. The simplest approximation is to express  $g_2$  in terms of  $g$ , which is equivalent to the Hartree approximation:

$$g_2(x_1, x_2; x_1', x_2') \approx g(x_1, x_1') g(x_2, x_2'). \quad (\text{II.405})$$

Equation (III.405) implies that the added two particles propagate independently, one from  $x_1$  to  $x_1'$  and the other from  $x_2$  to  $x_2'$ . The Hartree approximation does not satisfy the basic symmetry property that requires  $(g_2)^2$  to be invariant under the exchange of  $x_1 \leftrightarrow x_2$  or  $x_1' \leftrightarrow x_2'$ , and is therefore oversimplified.

Substituting EQ. (II.405) into EQ. (II.323) and using EQ. (II.378), we obtain

$$\begin{aligned} \left( i \frac{\partial}{\partial t} + \frac{\nabla_{\mathbf{r}}^2}{2m} \right) g(x, x') &= \delta(x - x') \pm i g(x, x') \int d^3 r_1 V(\mathbf{r} - \mathbf{r}_1) g(x_1, x_1^+), \\ &= \delta(x - x') \pm i g(x, x') \int d^3 r_1 V(\mathbf{r} - \mathbf{r}_1) \langle n(\mathbf{r}_1) \rangle \equiv \delta(x - x') + i g(x, x') V_{\text{eff}}(\mathbf{r}), \end{aligned} \quad (\text{II.406})$$

where we have introduced an effective one-body potential  $V_{\text{eff}}(\mathbf{r})$ . Thus, we can rewrite EQ. (II.406) into the following simple form:

$$\left( i \frac{\partial}{\partial t} + \frac{\nabla_{\mathbf{r}}^2}{2m} - V_{\text{eff}}(\mathbf{r}) \right) g(x, x') = \delta(x - x'), \quad (\text{II.407})$$

which implies that the added particle (or hole) moves independently in the averaged potential  $V_{\text{eff}}(\mathbf{r})$  created by all particles in the many-body system. For a translational invariant system, the density  $\langle n(\mathbf{r}_1) \rangle$  is a constant  $n_0$ . As a result,  $V_{\text{eff}}(\mathbf{r})$  is a constant given by  $n_0 V_0$ , where  $V_0 = i \int d^3 r V(\mathbf{r})$ . Consequently, EQ. (II.407) becomes a simple differential equation, and its Fourier transformation yields:

$$\left( \omega - \frac{k^2}{2m} - n_0 V_0 \right) g(\mathbf{k}, \omega) = 1. \quad (\text{II.408})$$

In reference to EQ. (II.384), we obtain the Green's function solution for fermions at  $T = 0$ :

$$g(\mathbf{k}, \omega) = \lim_{s \rightarrow 0^+} \left[ \frac{1}{\omega - (k^2/2m) - n_0 V_0 + is \operatorname{sgn}(\omega - \mu)} \right]. \quad (\text{III.409})$$

To obtain the Green's function for  $T \neq 0$ , we note that  $g(\mathbf{k}, \omega)$  is not the limit of an analytic function, so that we cannot arbitrarily continue EQ. (II.409) in the complex  $\omega$ -plane without further analysis. However, we can still use the "imaginary time" trick to obtain the Green's function for  $T \neq 0$ . The procedure is as follows. Consider the casual Green's function  $g(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)$ . If we substitute  $t_1$  by  $-i\sigma_1$  and  $t_2$  by  $-i\sigma_2$ , the time ordering by the  $\sigma$ -ordering, and confine  $\sigma_1$  and  $\sigma_2$  within the interval  $[0, \beta]$ , we have:

$$\begin{aligned} g(\mathbf{r}_1, -i\sigma_1; \mathbf{r}_2, -i\sigma_2) &= \tilde{g}^>(\mathbf{r}_1, -i\sigma_1; \mathbf{r}_2, -i\sigma_2), & \text{if } \sigma_1 > \sigma_2; \\ &= \tilde{g}^<(\mathbf{r}_1, -i\sigma_1; \mathbf{r}_2, -i\sigma_2), & \text{if } \sigma_2 > \sigma_1. \end{aligned} \quad (\text{II.410})$$

For the systems of our consideration,  $g(\mathbf{r}_1, -i\sigma_1; \mathbf{r}_2, -i\sigma_2)$  is a function of  $\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$  and  $\sigma = (\sigma_1 - \sigma_2)$ . Using the relation

$$\tilde{g}^<(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \mp i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3 k}{(2\pi)^3} e^{-i\omega(t_1 - t_2)} e^{i\mathbf{k}\cdot\mathbf{r}} A(\mathbf{k}, \omega) f_{\mp}(\omega),$$

we obtain the following for  $\sigma < 0$ :

$$\tilde{g}^<(\mathbf{r}_1, -i\sigma_1; \mathbf{r}_2, -i\sigma_2) = \mp i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3 k}{(2\pi)^3} e^{-\omega\sigma} e^{i\mathbf{k}\cdot\mathbf{r}} A(\mathbf{k}, \omega) f_{\mp}(\omega). \quad (\text{II.411})$$

Equation (II.411) together with EQ. (II.403) gives

$$g(\mathbf{r}, -i\sigma) = \mp i \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \left[ \frac{\mp 1}{\beta} \sum_{\nu} e^{-z_{\nu}\sigma} g(\mathbf{k}, z_{\nu}) \right], \quad (\text{II.412})$$

and

$$g(\mathbf{k}, -i\sigma) = \left[ \frac{i}{\beta} \sum_{\nu} e^{-z_{\nu}\sigma} g(\mathbf{k}, z_{\nu}) \right], \quad (\text{II.413})$$

where the poles  $z_{\nu}$  have been given in EQ. (II.402). We may also invert EQ. (II.413) to obtain an explicit expression for  $g(\mathbf{k}, z_{\nu})$ , which yields:

$$g(\mathbf{k}, z_{\nu}) = -\frac{i}{2} \int_{-\beta}^{\beta} d\sigma g(\mathbf{k}, -i\sigma) e^{z_{\nu}\sigma}, \quad (\text{II.414})$$

Substituting EQ. (II.413) into EQ. (II.407) using the imaginary times, we obtain

$$\left( z_\nu - \frac{k^2}{2m} - n_0 V_0 \right) g(\mathbf{k}, z_\nu) = 1 \rightarrow g(\mathbf{k}, z_\nu) = \frac{1}{z_\nu - (k^2/2m) - n_0 V_0}. \quad (\text{II.415})$$

The solution in EQ. (II.415) gives rise to EQ. (II.409) through analytic continuation.

In the Hartree approximation, we find that the average energy of the system, using EQs. (II.393) and (II.415), is given by

$$\langle \mathcal{H} \rangle = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} f_{\mp}(\varepsilon_{\mathbf{k}}) - \frac{1}{2} \Omega n_0^2 V_0, \quad (\text{II.416})$$

which is equivalent to the free particle energy with an added energy  $n_0 V_0$  per particle, as expected.

An improvement over the Hartree approximation can be made by taking into account the symmetry or anti-symmetry of  $g_2$  under the exchange of  $x_1 \leftrightarrow x_2$  or  $x_1' \leftrightarrow x_2'$  while still keeping the added particles as moving independently of each other. This improved approximation is known as the Hartree-Fock approximation:

$$g_2(x_1, x_2; x_1', x_2') \approx g(x_1, x_1') g(x_2, x_2') \pm g(x_1, x_2') g(x_2, x_1'). \quad (\text{II.417})$$

By following the same procedure outlined above for the Hartree approximation, we obtain within the framework of the Hartree-Fock approximation the Green's function:

$$g(\mathbf{k}, \omega) = \frac{1}{\omega - \varepsilon_{\mathbf{k}}}, \quad (\text{II.418})$$

where

$$\varepsilon_{\mathbf{k}} = \frac{k^2}{2m} + n_0 V_0 \pm \int \frac{d^3 k'}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}') \langle n(\mathbf{k}') \rangle, \quad (\text{II.419})$$

and  $V(\mathbf{k}) = \int d^3 r \exp(-i\mathbf{k} \cdot \mathbf{r}) V(\mathbf{r})$  is the Fourier transform of the potential  $V(\mathbf{r})$ . We note that the quasi-particle energy  $\varepsilon_{\mathbf{k}}$  in the Hartree-Fock approximation depends implicitly on the temperature through its dependence on the thermal average of the density  $\langle n(\mathbf{k}') \rangle$ .

Next, we want to evaluate the Green's functions of interacting many-body systems using diagrammatic analysis of perturbation theory. Similar to our earlier consideration of relativistic quantum field theory, it is generally convenient to apply Wick's theorem to evaluate the Green's functions in the perturbation expansion. In the following we revisit Wick's theorem by introducing the definitions of chronological ordering, normal ordering, and contractions.

### [Wick's theorem -- revisited]

The reason why Wick's theorem is important in the perturbative diagrammatic analysis of Green's functions is due to the need to perform time ordering of field operators in the interaction picture. Specifically, while the Green's functions are given in terms of field operators in the Heisenberg picture, it is not practical to calculate the Green's functions for realistic physical systems in the Heisenberg picture. The strategy to evaluate the Green's functions of an interacting system is to employ perturbation theory, which is best performed in the interaction picture. Therefore, it is necessary to reformulate operators in the Heisenberg picture into equivalent expressions in the interaction picture.

Let's assume that in the interaction picture the total Hamiltonian  $\mathcal{H}$  can be expressed by  $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ , where  $\mathcal{H}_0$  denotes the unperturbed Hamiltonian and  $\mathcal{H}_I$  the interaction perturbation. Moreover, we denote the exact interacting ground state  $|0\rangle$  by  $|\Psi_0\rangle$  and the eigenstate of  $\mathcal{H}_0$  by  $|\Phi_0\rangle$ . We shall first prove that the matrix element of a Heisenberg operator  $O_H$  evaluated in the Heisenberg picture can be converted into the interaction picture according to the following relation:

$$\frac{\langle \Psi_0 | O_H(t) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \frac{1}{\langle \Phi_0 | U_\varepsilon(\infty, -\infty) | \Phi_0 \rangle} \langle \Phi_0 | \sum_{\nu=0}^{\infty} \left( \frac{-i}{\hbar} \right)^\nu \frac{1}{\nu!} \int_{-\infty}^{\infty} dt_1 \dots \int_{-\infty}^{\infty} dt_\nu \times e^{-\varepsilon(|t_1| + \dots + |t_\nu|)} T[H_I(t_1) \dots H_I(t_\nu) O_I(t)] | \Phi_0 \rangle. \quad (\text{II.420})$$

Equation (II.420) can be further extended to converting the matrix element of two Heisenberg operators into the interaction picture as follows:

$$\frac{\langle \Psi_0 | O_H(t) O_H(t') | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \frac{1}{\langle \Phi_0 | U_\varepsilon(\infty, -\infty) | \Phi_0 \rangle} \langle \Phi_0 | \sum_{\nu=0}^{\infty} \left( \frac{-i}{\hbar} \right)^\nu \frac{1}{\nu!} \int_{-\infty}^{\infty} dt_1 \dots \int_{-\infty}^{\infty} dt_\nu \times e^{-\varepsilon(|t_1| + \dots + |t_\nu|)} T[H_I(t_1) \dots H_I(t_\nu) O_I(t) O_I(t')] | \Phi_0 \rangle. \quad (\text{II.421})$$

It is clear from EQs. (II.420) and (II.421) that the evaluation of Green's functions in the interaction picture requires performing the time ordering of products of field operators where Wick's theorem becomes helpful.

To prove EQ. (II.420), recall the Gell-Mann and Low theorem in Part I that expresses the exact ground state of an interacting system in the interaction picture:

$$\frac{|\Psi_0\rangle}{\langle \Phi_0 | \Psi_0 \rangle} = \frac{U_\varepsilon(0, \pm\infty) |\Phi_0\rangle}{\langle \Phi_0 | U_\varepsilon(0, \pm\infty) | \Phi_0 \rangle}. \quad (\text{II.422})$$

From EQ. (II.422), we obtain

$$\begin{aligned} \frac{\langle \Psi_0 | \Psi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} &= \frac{\langle \Phi_0 | U_\varepsilon(0, \infty)^\dagger U_\varepsilon(0, -\infty) | \Phi_0 \rangle}{\langle \Phi_0 | U_\varepsilon(0, \infty) | \Phi_0 \rangle^* \langle \Phi_0 | U_\varepsilon(0, -\infty) | \Phi_0 \rangle} = \frac{\langle \Phi_0 | U_\varepsilon(\infty, -\infty) | \Phi_0 \rangle}{\langle \Phi_0 | U_\varepsilon(0, \infty) | \Phi_0 \rangle^* \langle \Phi_0 | U_\varepsilon(0, -\infty) | \Phi_0 \rangle}, \\ \Rightarrow \frac{1}{\langle \Psi_0 | \Psi_0 \rangle} &= \frac{\langle \Phi_0 | U_\varepsilon(\infty, 0) | \Phi_0 \rangle \langle \Phi_0 | U_\varepsilon(0, -\infty) | \Phi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} \frac{1}{\langle \Phi_0 | U_\varepsilon(\infty, -\infty) | \Phi_0 \rangle}. \end{aligned} \quad (\text{II.423})$$

Similarly, the numerator on the left side of EQ. (II.420) can be rewritten as:

$$\begin{aligned} \langle \Psi_0 | O_H(t) | \Psi_0 \rangle &= \langle \Phi_0 | U_\varepsilon(0, \infty)^\dagger O_H(t) U_\varepsilon(0, -\infty) | \Phi_0 \rangle \frac{|\langle \Phi_0 | \Psi_0 \rangle|^2}{\langle \Phi_0 | U_\varepsilon(\infty, 0) | \Phi_0 \rangle \langle \Phi_0 | U_\varepsilon(0, -\infty) | \Phi_0 \rangle} \\ &= \langle \Phi_0 | U_\varepsilon(\infty, 0) U_\varepsilon(0, t) O_I(t) U_\varepsilon(t, 0) U_\varepsilon(0, -\infty) | \Phi_0 \rangle \frac{|\langle \Phi_0 | \Psi_0 \rangle|^2}{\langle \Phi_0 | U_\varepsilon(\infty, 0) | \Phi_0 \rangle \langle \Phi_0 | U_\varepsilon(0, -\infty) | \Phi_0 \rangle} \\ &= \langle \Phi_0 | U_\varepsilon(\infty, t) O_I(t) U_\varepsilon(t, -\infty) | \Phi_0 \rangle \frac{|\langle \Phi_0 | \Psi_0 \rangle|^2}{\langle \Phi_0 | U_\varepsilon(\infty, 0) | \Phi_0 \rangle \langle \Phi_0 | U_\varepsilon(0, -\infty) | \Phi_0 \rangle}. \end{aligned} \quad (\text{II.424})$$

Combining EQs. (II.423) and (II.424), we have

$$\frac{\langle \Psi_0 | O_H(t) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \frac{\langle \Phi_0 | U_\varepsilon(\infty, t) O_I(t) U_\varepsilon(t, -\infty) | \Phi_0 \rangle}{\langle \Phi_0 | U_\varepsilon(\infty, -\infty) | \Phi_0 \rangle}. \quad (\text{II.425})$$

Next, we want to express the product of operators in the numerator on the right side of EQ. (II.425) explicitly. Following the discussion in Part I for the time-evolution  $U$ -operators, we obtain

$$\begin{aligned} U_\varepsilon(\infty, t) O_I(t) U_\varepsilon(t, -\infty) &= \sum_{n=0}^{\infty} \left( \frac{-i}{\hbar} \right)^n \frac{1}{n!} \int_t^\infty dt_1 \dots \int_t^\infty dt_n e^{-\varepsilon(|t_1| + \dots + |t_n|)} T[H_I(t_1) \dots H_I(t_n)] \\ &\times O_I(t) \sum_{m=0}^{\infty} \left( \frac{-i}{\hbar} \right)^m \frac{1}{m!} \int_{-\infty}^t dt_1 \dots \int_{-\infty}^t dt_m e^{-\varepsilon(|t_1| + \dots + |t_m|)} T[H_I(t_1) \dots H_I(t_m)]. \end{aligned} \quad (\text{II.426})$$

On the other hand, we note that the numerator of the right side of EQ. (II.420) can be rewritten if we have  $\nu = n + m$  and if for the  $\nu$ -th term in the sum we divide the integration variables into  $n$  factors with  $t_i > t$  and  $m$  factors with  $t_i < t$ . Clearly there are a total of  $\nu!/(n!m!)$  different ways of such divisions for each  $\nu$ . Thus, the product of operators on the right side of EQ. (II.420) becomes:

$$\begin{aligned} \sum_{\nu=0}^{\infty} \left( \frac{-i}{\hbar} \right)^\nu \frac{1}{\nu!} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{\nu, m+n} \frac{\nu!}{n!m!} \int_t^\infty dt_1 \dots \int_t^\infty dt_n e^{-\varepsilon(|t_1| + \dots + |t_n|)} T[H_I(t_1) \dots H_I(t_n)] \\ \times O_I(t) \int_{-\infty}^t dt_1 \dots \int_{-\infty}^t dt_m e^{-\varepsilon(|t_1| + \dots + |t_m|)} T[H_I(t_1) \dots H_I(t_m)], \end{aligned} \quad (\text{II.427})$$

which is essentially identical to EQ. (II.426) after we apply the Kronecker delta function. Thus, we have proven EQ. (II.420). Similar procedure can be applied to prove EQ. (II.421), except that we need to partition the integration variables into three distinct groups.

Now we are ready to evaluate Green's functions in the interaction picture with the aid of EQ. (II.421). That is, by identifying the operators with the fermion field operators in the interaction picture so that  $O_I(t) = \psi_\alpha(x)$  and  $O_I(t') = \psi_\beta^\dagger(x')$  where  $\alpha$  and  $\beta$  refer to the spin indices, we obtain the Green's function in the interaction picture as follows:

$$i\bar{g}_{\alpha\beta}(x, x') = \sum_{\nu=0}^{\infty} \left( \frac{-i}{\hbar} \right)^\nu \frac{1}{\nu!} \int_{-\infty}^{\infty} dt_1 \dots \int_{-\infty}^{\infty} dt_\nu \frac{\langle \Phi_0 | T[H_I(t_1) \dots H_I(t_\nu) \psi_\alpha(x) \psi_\beta^\dagger(x')] | \Phi_0 \rangle}{\langle \Phi_0 | U_\varepsilon(\infty, -\infty) | \Phi_0 \rangle}. \quad (\text{II.428})$$

If we further assume a simple interaction potential given by

$$U(x_1, x_2) = V(\mathbf{r}_1, \mathbf{r}_2) \delta(t_1 - t_2), \quad (\text{II.429})$$

the numerator of the Green's function in EQ. (II.428) is simplified into:

$$\begin{aligned} i\bar{g}_{\alpha\beta}(x, x') &= iG_{\alpha\beta}^0(x, x') + \left( \frac{-i}{\hbar} \right) \sum_{\lambda\lambda', \mu\mu'} \frac{1}{2} \int_{-\infty}^{\infty} d^4x_1 d^4x'_1 U(x_1, x'_1)_{\lambda\lambda', \mu\mu'} \\ &\times \langle \Phi_0 | T[\psi_\lambda^\dagger(x_1) \psi_\mu^\dagger(x'_1) \psi_{\mu'}(x'_1) \psi_{\lambda'}(x_1) \psi_\alpha(x) \psi_\beta^\dagger(x')] | \Phi_0 \rangle. \end{aligned} \quad (\text{II.430})$$

It is clear from EQ. (II.430) that the evaluation of requires finding the expectation values in the non-interacting ground state of time-ordered products of creation and annihilation operators of the following form:

$$\langle \Phi_0 | T[\psi^\dagger \dots \psi \psi_\alpha(x) \psi_\beta^\dagger(x')] | \Phi_0 \rangle. \quad (\text{II.431})$$

We'll rely on Wick's theorem to provide a general procedure for evaluating matrix elements of the form given in EQ. (II.431).

Before stating Wick's theorem in the context of time-ordered operators, we summarize below three different definitions for products of field operators  $(ABCD\dots)$ .

⟨1⟩ Time ordering:

The time ordering of products of field operators is defined as ordering the field operators with the latest time on the far left and including an additional factor of  $(-1)$  for each interchange of fermion operators. That is,

$$T(ABCD\dots) = (-1)^p T(CADB\dots), \quad (\text{II.432})$$

where  $p$  is the number of permutations of fermion operators needed to rearrange the product given on the left side of EQ. (II.432) to agree with the order on the right side.

⟨2⟩ Normal ordering:

The normal ordering of products of field operators is defined as placing all the annihilation operators to the right of all creation operators and including a factor  $(-1)$  for each interchange of fermion operators. Hence, we have

$$N(ABCD\dots) = (-1)^p N(CABD\dots), \quad (\text{II.433})$$

so that the field operators within a normal-ordered product either commute (for bosons) or anti-commute (for fermions). We remark that the normal-ordering notation  $N(ABCD\dots)$  in EQ. (II.433) is equivalent to the notation  $:ABCD\dots:$  given in some textbooks of relativistic quantum field theory.

The introduction of normal-ordered products of field operators is motivated by the fact that their expectation values in the unperturbed ground state vanish identically. This result remains true even if the product consists of all creation operators, because the expectation value in the unperturbed ground state of its adjoint product vanishes identically. It is therefore convenient to reduce the time-ordering products into combinations of normal-ordering products plus extra terms due to commutation or anti-commutation of bosonic or fermionic operators. For this purpose, it is also convenient to decompose the field operator into a destruction part  $\psi^{(a)}$  that annihilates the non-interacting ground state and a creation part  $\psi^{(c)}$ . Hence,

$$\psi(x) = \psi^{(a)}(x) + \psi^{(c)}(x) \quad \text{and} \quad \psi^\dagger(x) = \psi^{(a)\dagger}(x) + \psi^{(c)\dagger}(x), \quad (\text{II.434})$$

where  $\psi^{(a)}(x)|\Phi_0\rangle = 0$  and  $\psi^{(c)\dagger}(x)|\Phi_0\rangle = 0$ . (II.435)

As an explicit example, we may consider the decomposition of the free fermion field of a many-body system:

$$\psi(x) = \Omega^{-1/2} \left[ \sum_{|\mathbf{k}| > k_F, \lambda} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t)} \eta_\lambda a_{\mathbf{k}\lambda} + \sum_{|\mathbf{k}| < k_F, \lambda} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t)} \eta_\lambda b_{-\mathbf{k}\lambda}^\dagger \right] \equiv \psi^{(a)}(x) + \psi^{(c)}(x), \quad (\text{II.436})$$

where  $a_{\mathbf{k}\lambda}$  and  $b_{-\mathbf{k}\lambda}^\dagger$  are respectively the particle and hole operators for  $|\mathbf{k}| > k_F$  and  $|\mathbf{k}| < k_F$ ,  $\eta_\lambda$  is the spinor operator, and  $\Omega$  is the volume of the system.

⟨3⟩ Contraction:

The contraction of two operators  $A$  and  $B$  is denoted by  $A \cdot B$  and is defined as follows:

$$A \cdot B = T(AB) - N(AB). \quad (\text{II.437})$$

Given the definition in EQ. (II.437), it can be easily verified that all the following contractions of operators vanish:

$$\psi^{(a)} \cdot \psi^{(c)} = \psi^{(a)\dagger} \cdot \psi^{(c)\dagger} = \psi^{(a)\dagger} \cdot \psi^{(c)} = \psi^{(a)} \cdot \psi^{(c)\dagger} = 0. \quad (\text{II.438})$$

Therefore, most contractions are zero. In particular, the contraction of two creation parts or two annihilation parts always vanish. The only non-zero contractions are given by the following:

$$\begin{aligned} \psi^{(a)}(x) \cdot \psi^{(a)\dagger}(y) &= iG^{(0)}(x, y) \quad \text{for } t_x > t_y, \\ &= 0 \quad \text{for } t_x < t_y; \\ \psi^{(c)}(x) \cdot \psi^{(c)\dagger}(y) &= 0 \quad \text{for } t_x > t_y, \\ &= iG^{(0)}(x, y) \quad \text{for } t_x < t_y. \end{aligned} \quad (\text{II.439})$$

We note that the Green's function in EQ. (II.439) for free fermions is given by (with spin indices restored)

$$iG_{\alpha\beta}^{(0)}(x, y) = \delta_{\alpha\beta} \Omega^{-1} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \left[ \theta(t_x - t_y) \theta(|\mathbf{k}| - k_F) - \theta(t_y - t_x) \theta(k_F - |\mathbf{k}|) \right].$$

Moreover, the contractions are  $c$  numbers in the occupation-number Hilbert space rather than operators. In the case of a product of operators consisting of more than one pair of contraction, we denote the first pair by a pair of single dots, the second pair by a pair of double dots, etc.

Having introduced the definitions of time-ordering, normal-ordering and contraction, we are ready to prove the following basic lemma for Wick's theorem.

⟨4⟩ Lemma:

For a normal-ordered product  $N(AB \cdots XY)$ , if  $Z$  is an operator labeled with time earlier than the times of all operators in the normal-ordered product, then the following relation is satisfied:

$$\begin{aligned} N(AB \cdots XY)Z &= N(AB \cdots XY \cdot Z) + N(AB \cdots X \cdot YZ) + \cdots \\ &\quad + N(A \cdot B \cdots XYZ) + N(AB \cdots XYZ). \end{aligned} \quad (\text{II.440})$$

[Proof] We first note that for the special case of  $Z$  being an annihilation operator, we have  $T(UZ) = N(UZ)$  for any operator  $U$  in the normal-ordered product so that all contractions vanish and EQ. (II.440) is proven. We may also assume that the operator product  $AB \cdots XY$  is already normal-ordered for simplicity. Otherwise we can always reorder the operators on both sides of EQ. (II.440) so that the same signature factor occurs in each term of EQ. (II.440) and therefore cancels identically. To further simplify the problem in the proof, we may assume next that  $Z$  is a creation operator and  $AB \cdots XY$  are all annihilation operators, because if we can prove this special case for EQ. (II.440), for situations with creation operators within the normal-ordered product, we may simply multiply the creation operators to the left side of  $AB \cdots XY$ , and the resulting additional contractions with  $Z$  are identically zero because the contractions of two creation operators are always zero and EQ. (II.440) remains valid.

Now we may prove EQ. (II.440) by induction for the case of  $Z$  being a creation operator and  $AB \cdots XY$  being all annihilation operators. First, we note that EQ. (II.440) apparently holds for two operators because by definition,

$$AZ = T(AZ) = A \cdot Z \cdot + N(AZ). \quad (\text{II.441})$$

Next, if we assume that EQ. (II.440) holds for  $n$  operators, we want to prove that it is also valid for  $(n+1)$  operators. If we multiply to the left of the product of operators in EQ. (II.440) another annihilation operator  $D$  that has a time later than that of  $Z$ , we obtain

$$\begin{aligned} DN(AB \cdots XY)Z &= N(DAB \cdots XY)Z = N(DAB \cdots XY \cdot Z \cdot) + N(DAB \cdots X \cdot YZ \cdot) + \cdots \\ &\quad + N(DA \cdot B \cdots XYZ \cdot) + DN(AB \cdots XYZ). \end{aligned} \quad (\text{II.442})$$

The last term in EQ. (II.442) can be further analyzed as follows:

$$\begin{aligned} DN(AB \cdots XYZ) &= (-1)^p DZAB \cdots XY = (-1)^p T(DZ)AB \cdots XY \\ &= (-1)^p D \cdot Z \cdot AB \cdots XY + (-1)^p N(DZ)AB \cdots XY \\ &= (-1)^p D \cdot Z \cdot AB \cdots XY + (-1)^{p+q} N(ZD)AB \cdots XY \\ &= \left[ (-1)^p \right]^2 D \cdot AB \cdots XYZ \cdot + \left[ (-1)^{p+q} \right]^2 N(DAB \cdots XYZ) \\ &= N(D \cdot AB \cdots XYZ \cdot) + N(DAB \cdots XYZ). \end{aligned} \quad (\text{II.443})$$

By inserting EQ. (II.443) into EQ. (II.442) we therefore prove EQ. (II.440). Now we are ready to state and prove Wick's theorem for product of field operators.

#### ⟨5⟩ Wick's theorem:

The time-ordered product of operators can be expressed in terms of the normal-ordered product of these operators plus the normal-ordering of all possible pair contractions of these operators. That is,

$$\begin{aligned} T(ABC \cdots XYZ) &= N(ABC \cdots XYZ) + N(A \cdot B \cdot C \cdots XYZ) + N(A \cdot BC \cdot \cdots XYZ) + \cdots + N(A \cdot B \cdot C \cdot \cdots X \cdot Y \cdot Z \cdot) \\ &= N(ABC \cdots XYZ) + N(\text{sum over all possible pairs of contractions}). \end{aligned} \quad (\text{II.444})$$

[Proof] We shall prove this theorem by induction. First, EQ. (II.444) apparently holds for two operators by definition. That is,

$$T(AB) = N(AB) + N(A \cdot B \cdot) = N(AB) + A \cdot B \cdot. \quad (\text{II.445})$$

Next, we assume that EQ. (II.444) holds for  $n$  operators and we multiply on the right of EQ. (II.444) by an operator  $\Omega$  with a time earlier than all other operators  $ABC \cdots XYZ$ . Thus, we find that

$$\begin{aligned} T(ABC \cdots XYZ)\Omega &= T(ABC \cdots XYZ\Omega) = N(ABC \cdots XYZ)\Omega + N(A \cdot B \cdot C \cdots XYZ)\Omega + \cdots \\ &= N(ABC \cdots XYZ\Omega) + N(\text{sum over all possible pairs of contractions}), \end{aligned} \quad (\text{II.446})$$

where in the second line of EQ. (II.446) we have used the lemma given by EQ. (II.440). Thus, we have proven Wick's theorem in EQ. (II.444) by induction.

Wick's theorem is useful for evaluating ground-state averaged quantities of the form  $\langle \Phi_0 | \dots | \Phi_0 \rangle$ , where all uncontracted normal-ordered products of operators vanish. Therefore, the Green's function given in EQ. (II.428) consists of all possible fully contracted terms of the operators in the interaction picture.

### [Application of Green's functions to diagrammatic analysis of perturbation theory]

Now let's consider the diagrammatic method for calculating the Green's functions. This method is applicable to the important case where the total Hamiltonian  $\mathcal{H}$  can be decomposed as the sum of an unperturbed part ( $\mathcal{H}_0$ ) and the perturbation part ( $\mathcal{H}_I$ ); the unperturbed part is such that the Green's functions corresponding to  $\mathcal{H}_0$  can be easily calculated, and the perturbation part can be expressed in the second quantization form as:

$$\mathcal{H}_I = \frac{1}{2} \int d^3 r \int d^3 r' \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}') V(\mathbf{r} - \mathbf{r}')_{\alpha\alpha', \beta\beta'} \psi_\beta(\mathbf{r}') \psi_\alpha(\mathbf{r}), \quad (\text{II.447})$$

where we have generalized our consideration for fermion systems by restoring the spin indices  $\alpha$  and  $\beta$  to the field operators, interaction potential, and the Green's functions. For simplicity, we have assumed in EQ. (II.447) that the interaction potential  $U(x, x')_{\alpha\alpha', \beta\beta'} = V(\mathbf{r} - \mathbf{r}')_{\alpha\alpha', \beta\beta'} \delta(t - t')$  only involves instant-time interaction. The Green's function is now defined as

$$i g_{\alpha\beta}(x, y) = \frac{\langle \Psi_0 | T [\psi_{H\alpha}(x) \psi_{H\beta}^\dagger(y)] | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}, \quad (\text{II.448})$$

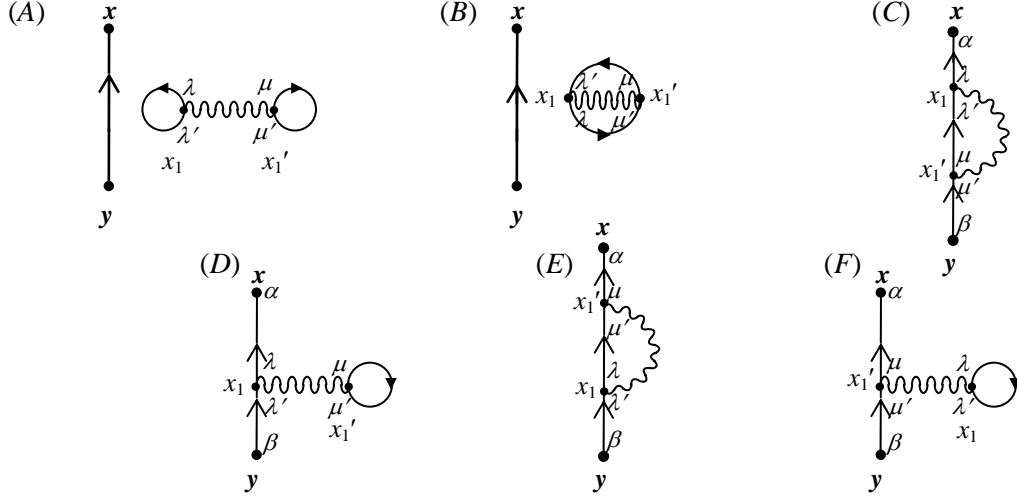
where  $\alpha$  and  $\beta$  denote spin indices and all other notations remain the same as before.

While in general it is more convenient to compute Green's functions in momentum and frequency space, as we shall focus on later, it is instructive to consider Green's functions in spacetime first to illustrate the utility of Wick's theorem. As an example, let us consider the first-order contributions to in EQ. (II.430) for a pair interaction potential given by  $U(x, x')_{\lambda\lambda', \mu\mu'}$ . In this case, we have three pairs of creation and annihilation operators so that there are 6 fully contracted and non-vanishing products of field operators between  $\psi$  and  $\psi^\dagger$ . Hence, to the first order EQ. (II.430) becomes:

$$\begin{aligned} i \bar{g}_{\alpha\beta}(x, y) = & \left( \frac{-i}{\hbar} \right) \frac{1}{2} \sum_{\lambda\lambda', \mu\mu'} \int d^4 x_1 d^4 x'_1 U(x_1, x'_1)_{\lambda\lambda', \mu\mu'} \\ & \times \left\{ iG_{\alpha\beta}^0(x, y) \left[ \underset{(A)}{iG_{\mu'\mu}^0(x'_1, x'_1)} \underset{(B)}{iG_{\lambda'\lambda}^0(x_1, x_1)} - iG_{\mu'\lambda}^0(x'_1, x_1) iG_{\lambda'\mu}^0(x_1, x'_1) \right] \right. \\ & + iG_{\alpha\lambda}^0(x, x_1) \left[ \underset{(C)}{iG_{\lambda'\mu}^0(x_1, x'_1)} iG_{\mu'\beta}^0(x'_1, y) - iG_{\lambda'\beta}^0(x_1, y) iG_{\mu'\mu}^0(x'_1, x'_1) \right] \\ & \left. + iG_{\alpha\mu}^0(x, x'_1) \left[ \underset{(E)}{iG_{\mu'\lambda}^0(x'_1, x_1)} iG_{\lambda'\beta}^0(x_1, y) - iG_{\mu'\beta}^0(x'_1, y) iG_{\lambda'\lambda}^0(x_1, x_1) \right] \right\}. \quad (\text{II.449}) \end{aligned}$$

The corresponding Feynman diagrams are illustrated in Fig. II.7.3, which is essentially the same result as that discussed in Part II.3, except that each interaction potential is now explicitly given by a wavy line connecting with two solid lines (i.e. two propagators  $G^0$ 's) rather than simply one vertex point as shown in Part II.3. It is clear from Fig. II.7.3 that graphs *C* and *E* are equivalent and similarly *D* and *F* are equivalent. Moreover, graphs *A* and *B* are disconnected graphs, and are therefore not contributing to the total Green's function in

EQ. (II.428) because they are identically cancelled by contributions from the denominator  $\langle \Phi_0 | U_\varepsilon(\infty, -\infty) | \Phi_0 \rangle$ . This fact is consistent with our earlier discussion of the generating functional in relativistic quantum field theory, where we have shown using the path integral formalism that the only relevant Feynman diagrams contributing to the propagators are connected diagrams. We demonstrate in the following that the same concept applies to the canonical formalism in the interaction picture.



**Figure II.7.3** First-order diagrammatic contributions to  $\bar{g}_{\alpha\beta}(x, y)$

The contribution of a disconnected graph to the Green's function factors because it closes on itself. Therefore, we may express the  $\nu$ th-order term of the numerator in EQ. (II.430) as a product of connected and disconnected parts:

$$i\bar{g}_{\alpha\beta}^{(\nu)}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{-i}{\hbar} \right)^{n+m} \delta_{\nu, m+n} \frac{1}{\nu!} \frac{\nu!}{n!m!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_m \langle \Phi_0 | T [H_I(t_1) \cdots H_I(t_m) \psi_\alpha(x) \psi_\beta^\dagger(y)] | \Phi_0 \rangle_{\text{connected}} \\ \times \int_{-\infty}^{\infty} dt_{m+1} \cdots \int_{-\infty}^{\infty} dt_\nu \langle \Phi_0 | T [H_I(t_{m+1}) \cdots H_I(t_\nu)] | \Phi_0 \rangle. \quad (\text{II.450})$$

Summing over all  $\nu$ th-order terms, we find that EQ. (II.430) becomes

$$i\bar{g}_{\alpha\beta}(x, y) = \sum_{m=0}^{\infty} \left( \frac{-i}{\hbar} \right)^m \frac{1}{m!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_m \langle \Phi_0 | T [H_I(t_1) \cdots H_I(t_m) \psi_\alpha(x) \psi_\beta^\dagger(y)] | \Phi_0 \rangle_{\text{connected}} \\ \times \sum_{n=0}^{\infty} \left( \frac{-i}{\hbar} \right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \langle \Phi_0 | T [H_I(t_1) \cdots H_I(t_n)] | \Phi_0 \rangle. \\ = \sum_{m=0}^{\infty} \left( \frac{-i}{\hbar} \right)^m \frac{1}{m!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_m \langle \Phi_0 | T [H_I(t_1) \cdots H_I(t_m) \psi_\alpha(x) \psi_\beta^\dagger(y)] | \Phi_0 \rangle_{\text{connected}} \\ \times \langle \Phi_0 | U_\varepsilon(\infty, -\infty) | \Phi_0 \rangle, \quad (\text{II.451})$$

so that EQ. (II.428) reduces to

$$ig_{\alpha\beta}(x, y) = \sum_{m=0}^{\infty} \left( \frac{-i}{\hbar} \right)^m \frac{1}{m!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_m \langle \Phi_0 | T [H_I(t_1) \cdots H_I(t_m) \psi_\alpha(x) \psi_\beta^\dagger(y)] | \Phi_0 \rangle_{\text{connected}}. \quad (\text{II.452})$$

In this context, graphs *A* and *B* in Fig. II.7.3 do not contribute to first-order Green's function.

It is also instructive to comment on a special case of a solid line closed on itself, which corresponds to a Green's function evaluated at the same time. While the time ordering of field operators is not well defined in the equal Green's function, the single-particle Green's function  $g_{\alpha\beta}(x, x)$  arises from a contraction of two field operators within the interaction Hamiltonian  $\mathcal{H}_I$ , and the field operators appear in the form of  $\psi_{\beta}^{\dagger}(x)\psi_{\alpha}(x)$ . Therefore, the Green's function at equal times must be interpreted as:

$$\begin{aligned} iG_{\alpha\beta}^0(x, x) &= \lim_{t' \rightarrow t^+} \langle \Phi_0 | T [\psi_{\alpha}(\mathbf{r}, t) \psi_{\beta}^{\dagger}(\mathbf{r}, t')] | \Phi_0 \rangle \\ &= -\langle \Phi_0 | \psi_{\beta}^{\dagger}(\mathbf{r}) \psi_{\alpha}(\mathbf{r}) | \Phi_0 \rangle \\ &= -(2s+1)^{-1} \delta_{\alpha\beta} n^0(\mathbf{r}) \end{aligned} \quad (\text{II.453})$$

for a system of spin- $s$  fermions, where  $n^0(\mathbf{r})$  denotes the particle density in the unperturbed ground state, which need not be the same as the particle density  $n(\mathbf{r})$  in the interacting system. Thus, the terms denoted by graphs *D* and *F* represent the lowest-order direct interaction with all the particles that constitute the non-interacting ground state. On the other hand, the terms *C* and *E* represent the lowest-order exchange interaction.

Based on the above discussions, we summarize the following Feynman rules in spacetime for the  $n$ th-order contribution to the single-particle Green's function  $g_{\alpha\beta}(x, y)$ :

1) Draw all topologically distinct connected diagrams with  $n$  interaction lines  $U$  and  $(2n+1)$  direct Green's functions  $G^0$ .

-- There are  $2n$ -pairs of internal Green's functions for  $n$  interaction lines and one external pair of Green's function. For the internal Green's functions denoted by the set of variables  $(x_1, x'_1) \cdots (x_n, x'_n)$ , there are  $n!$  different possibilities of choosing the variables for each of these diagrams, and Wick's theorem is used to verify the enumeration.

2) Label each vertex with a four-dimensional spacetime point  $x_i$ .

3) Each solid line represents a Green's function  $G_{\alpha\beta}^0(x, y)$  running from  $y$  to  $x$ .

4) Each wavy line represents an interaction given by

$$U(x, y)_{\lambda\lambda', \mu\mu'} = V(\mathbf{r} - \mathbf{r}')_{\lambda\lambda', \mu\mu'} \delta(t_x - t_y),$$

where the association of matrix indices is illustrated in Fig. II.7.4.

5) Integrate all internal variables  $(x_1, x'_1) \cdots (x_n, x'_n)$  over space and time.

6) Associate a spin matrix product with each continuous fermion line as well as with the potentials at each vertex.

7) Assign a factor  $(-1)^F$  to a diagram that contains  $F$  closed fermion loops.

-- Every time a fermion line closes on itself, there is an extra sign change because the fields contracted into a closed loop may be first arranged into  $[\psi^\dagger(1)\psi(1)] [\psi^\dagger(2)\psi(2)] \dots [\psi^\dagger(N)\psi(N)]$  without changing sign and then followed by moving the last field operator over to the far left via an odd number of interchanges, so that there is a net sign change.

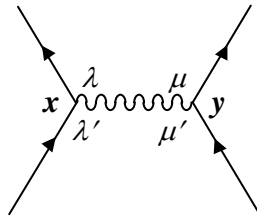
8) Assign a factor  $(i/\hbar)^n$  to each  $n$ th-order term  $ig_{\alpha\beta}^{(n)}(x, y)$  of the Green's function.

-- The  $n$ th-order term contains an explicit numerical factor  $(-i/\hbar)^n$ . In addition, the  $(2n+1)$  contractions of field operators contribute an additional factor  $(i)^{2n+1}$ , so that the computation of  $g(x, y)$  involves a net numerical factor  $(-i)(-i/\hbar)^n(i)^{2n+1} = (i/\hbar)^n$  to the  $n$ th-order term.

9) A Green's function with equal time variables must be interpreted as  $G_{\alpha\beta}^0(\mathbf{r}, t; \mathbf{r}, t^+)$ .

As an example, we apply the above Feynman rules to express the first-order contribution to  $G(x, y)$ . There are two internal variables and two internal Green's functions, so that we have

$$g_{\alpha\beta}^{(1)}(x, y) = \left(\frac{i}{\hbar}\right) \int d^4x_1 d^4x_1' \left\{ (-1) G_{\alpha\lambda}^{(0)}(x, x_1) U(x_1, x_1')_{\lambda\lambda', \mu\mu'} G_{\mu\mu'}^{(0)}(x_1', x_1') G_{\lambda'\beta}^{(0)}(x_1, y) + G_{\alpha\lambda}^{(0)}(x, x_1) U(x_1, x_1')_{\lambda\lambda', \mu\mu'} G_{\lambda'\mu}^{(0)}(x_1, x_1') G_{\mu\beta}^{(0)}(x_1', y) \right\}. \quad (\text{II.454})$$



**Figure II.7.4** Illustration of the matrix indices for the interaction potential  $U(x, y)_{\lambda\lambda', \mu\mu'}$ , where  $\lambda, \lambda', \mu$  and  $\mu'$  denote the spin indices.

Next, we consider the Feynman diagrams in momentum space. If we assume a uniform and isotropic system, and also assume that the Hamiltonian is time independent so that

$$U(x, x')_{\lambda\lambda', \mu\mu'} = V(\mathbf{r} - \mathbf{r}')_{\lambda\lambda', \mu\mu'} \delta(t - t'),$$

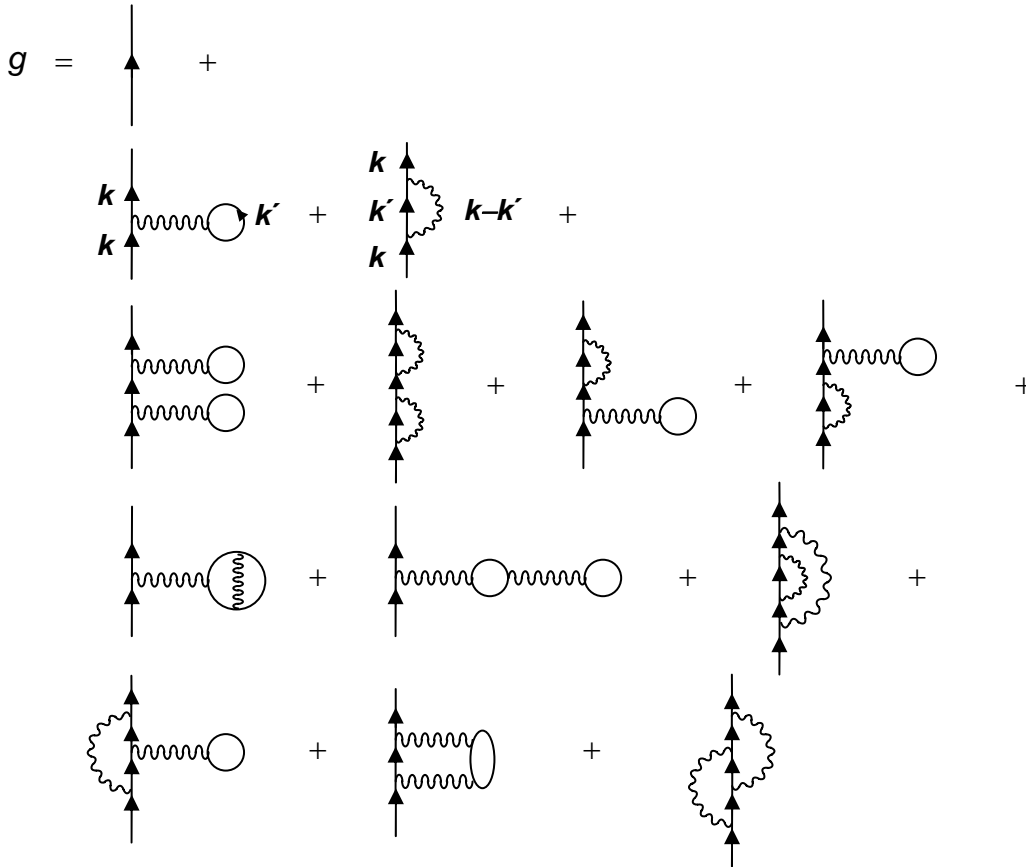
the exact Green's function takes the form

$$G_{\alpha\beta}(x, y) = \delta_{\alpha\beta} G_{\alpha\beta}(x - y)$$

and the Feynman rules in momentum space are analogous to those described in Part II.3, except that now we have to replace the relativistic propagators for the directed lines by the non-relativistic many-body Green's function

$$G_{\alpha\beta}^0(\mathbf{k}, \omega) = \delta_{\alpha\beta} \lim_{\alpha \rightarrow 0^+} \left[ \frac{1}{\omega - \varepsilon_{\mathbf{k}} + i\alpha \operatorname{sgn}(\omega - \mu)} \right] = \delta_{\alpha\beta} \lim_{\alpha \rightarrow 0^+} \left[ \frac{\theta(|\mathbf{k}| - k_F)}{\omega - \varepsilon_{\mathbf{k}} + i\alpha} + \frac{\theta(k_F - |\mathbf{k}|)}{\omega - \varepsilon_{\mathbf{k}} - i\alpha} \right] \quad (\text{for } T = 0).$$

and each  $G^0(\mathbf{k}, \omega)$  line that either starts from and ending at the same point or is linked by the same interaction line should be interpreted as being  $\tilde{G}_0^<(\mathbf{k}, \omega)$  where  $\text{Im}[\tilde{G}_0^<(\mathbf{k}, \omega)] = 2\pi i \delta(\omega - \varepsilon_{\mathbf{k}}^0) \theta(k_F - |\mathbf{k}|)$ . Also, as for Green's function in the spacetime coordinates, we need to multiply a factor of  $(-1)$  for each closed loop in the diagram. In addition, each interaction (wavy) line should be labeled by  $iV(\mathbf{q}) = i \int d^3r V(\mathbf{r}) \exp(-i\mathbf{q} \cdot \mathbf{r})$ , where  $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$  is the momentum change involved. The diagrams contributing to the Green's function up to the second order are illustrated in Fig. II.7.5.



**Figure II.7.5** Feynman diagrams for  $g$  of zero-order (first line), first-order (second line), and second-order (third, fourth and fifth lines).

For the consideration of spin-1/2 particles, we distinguish in the following two possibilities for the interaction potential. The first case is for spin-independent interaction, where the interaction potential has the form of  $[\mathbf{1}(1) \mathbf{1}(2)]$  in spin space, which represents the unit spin matrix with respect to both particles 1 and 2. Therefore, we have

$$U(\mathbf{q})_{\lambda\lambda', \mu\mu'} = U(\mathbf{q}) \delta_{\lambda\lambda'} \delta_{\mu\mu'}, \quad (\text{II.455})$$

and the matrix elements become

$$U(\mathbf{q})_{\lambda\lambda', \mu\mu} = 2U(\mathbf{q}) \delta_{\lambda\lambda'} \quad \text{and} \quad U(\mathbf{q})_{\lambda\kappa, \kappa\mu'} = U(\mathbf{q}) \delta_{\lambda\mu'}. \quad (\text{II.456})$$

On the other hand, if the interaction is spin dependent of the form  $\boldsymbol{\sigma}(1) \cdot \boldsymbol{\sigma}(2)$ , we find

$$U(\mathbf{q})_{\lambda\lambda',\mu\mu'} = U(\mathbf{q}) \boldsymbol{\sigma}(1)_{\lambda\lambda'} \cdot \boldsymbol{\sigma}(2)_{\mu\mu'}. \quad (\text{II.457})$$

Furthermore,

$$\boldsymbol{\sigma}_{\lambda\lambda'} \cdot \boldsymbol{\sigma}_{\mu\mu'} = 0 \quad \text{and} \quad \boldsymbol{\sigma}_{\lambda\kappa} \cdot \boldsymbol{\sigma}_{\kappa\mu'} = \left[ (\boldsymbol{\sigma})^2 \right]_{\lambda\mu'} = 3\delta_{\lambda\mu'}, \quad (\text{II.458})$$

because  $\text{Tr}[\boldsymbol{\sigma}] = 0$  and  $\text{Tr}[\mathbf{1}] = 2$ . Finally, if the interaction potential  $V$  consists of a spin-independent unperturbed potential  $V_0$  and a spin-dependent interaction potential  $V_1$ , we may express  $V$  as follows:

$$V(\mathbf{r}_1 - \mathbf{r}_2) = V_0(|\mathbf{r}_1 - \mathbf{r}_2|) \mathbf{1}(1) \mathbf{1}(2) + V_1(|\mathbf{r}_1 - \mathbf{r}_2|) \boldsymbol{\sigma}(1) \cdot \boldsymbol{\sigma}(2). \quad (\text{II.459})$$

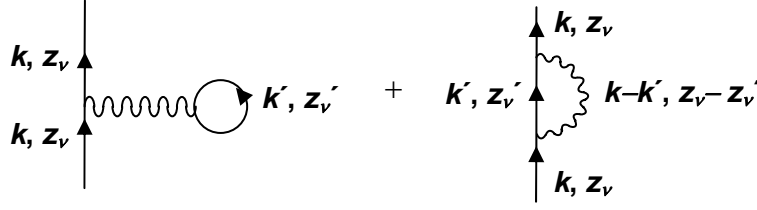
According to the aforementioned rules, the contributions from the first-order diagrams to the Green's function  $g(\mathbf{k})$  appropriate for a spin-independent interaction  $\mathcal{H}_I$  given in EQ. (II.447) become:

$$\begin{aligned} & -i \int \frac{d^4 k'}{(2\pi)^4} \tilde{G}_0^<(\mathbf{k}') V(0) G_0(\mathbf{k}) G_0(\mathbf{k}) + i \int \frac{d^4 k'}{(2\pi)^4} G_0(\mathbf{k}) G_0(\mathbf{k}) \tilde{G}_0^<(\mathbf{k}') V(\mathbf{k} - \mathbf{k}') \\ & = G_0^2(\mathbf{k}) \left[ V(0) \int \frac{d^3 k'}{(2\pi)^3} \theta(k_F - |\mathbf{k}'|) - \int \frac{d^3 k'}{(2\pi)^3} V(|\mathbf{k}| - |\mathbf{k}'|) \theta(k_F - |\mathbf{k}'|) \right]. \end{aligned} \quad (\text{II.460})$$

The aforementioned diagrammatic method provides descriptions for the fermionic causal Green's function  $g$  at  $T = 0$  in terms of expansions of  $G_0$  and  $V$ . For finite temperatures, the corresponding Green's function has to be obtained through the "imaginary time" trick. Upon Fourier transformation to the  $(k, z_\nu)$ -space with  $z_\nu$  denoting the poles in the  $\omega$ -space, we have the following rules for obtaining the  $n$ th order contribution to the function  $g(\mathbf{k}, z_\nu)$  given in EQ. (II.414):

- 1) Draw all topologically distinct connected diagrams with  $n$  interaction lines, two external points, and  $(2n+1)$  directed lines.
- 2) Label each line with a four-momentum  $(\mathbf{k}, z_\nu)$ ; conserve momentum and  $\text{Im}(z_\nu)$  at each vertex.
- 3) For each directed line labeled with a four-momentum  $(\mathbf{k}, z_\nu)$  write a factor  $G^0(\mathbf{k}, z_\nu) = \frac{1}{z_\nu - \epsilon_k^0}$ .
- 4) For each interaction (wavy) line labeled by a four-momentum  $(\mathbf{k}, z_\nu)$  write a factor  $-V(\mathbf{k})$ .
- 5) Integrate over all internal independent momenta  $\mathbf{k}$  through  $\left[ (2\pi)^{-3} \int d^3 k \right]$  and sum over all internal independent discrete frequencies through  $\left[ \beta^{-1} \sum_\nu \dots \right]$ .
- 6) Multiply by  $(-1)^F$  where  $F$  denotes the number of closed fermion loops.
- 7) Whenever a directed line either closes on itself or is joined by the same interaction line, insert a convergence factor  $\exp(-z_\nu \sigma)$  with  $\sigma \rightarrow 0^-$ .

As an example, we calculate the contributions of the first-order diagrams shown in Fig. II.7.6.



**Figure II.7.6** First-order Feynman diagrams for  $g(\mathbf{k}, z_v)$ .

Following the rules outlined above, we obtain:

$$G_0^2(\mathbf{k}, z_v) \left[ \mp V(0) \int \frac{d^3 k'}{(2\pi)^3} \frac{1}{\beta} \sum_{v'} e^{-z_v \sigma} G_0(\mathbf{k}', z_v') - \int \frac{d^3 k'}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}') \frac{1}{\beta} \sum_{v'} e^{-z_v \sigma} G_0(\mathbf{k}', z_v') \right]. \quad (\text{II.461})$$

Using EQ. (II.403),  $A(\mathbf{k}, \omega) = 2\pi\delta(\omega - \varepsilon_{\mathbf{k}}^0)$ , and the identities

$$\begin{aligned} \lim_{\eta \rightarrow 0^+} \sum_n \frac{e^{i\omega_n \eta}}{i\omega_n - x} &= -\frac{\beta\hbar}{e^{\beta\hbar x} - 1}, & \omega_n &= \frac{2\pi n}{\beta\hbar} \quad \text{for bosons;} \\ \lim_{\eta \rightarrow 0^+} \sum_n \frac{e^{i\omega_n \eta}}{i\omega_n - x} &= \frac{\beta\hbar}{e^{\beta\hbar x} + 1}, & \omega_n &= \frac{\pi(2n+1)}{\beta\hbar} \quad \text{for fermions;} \end{aligned}$$

EQ. (II.461) becomes:

$$G_0^2(\mathbf{k}, z_v) \left[ V(0) \int \frac{d^3 k'}{(2\pi)^3} f_{\mp}(\varepsilon_{\mathbf{k}'}^0) \pm \int \frac{d^3 k'}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}') f_{\mp}(\varepsilon_{\mathbf{k}'}^0) \right]. \quad (\text{II.462})$$

We emphasize that the results outlined above are only applicable to normal fermion systems, which must be modified for bosons at low temperatures and for fermion systems that are not normal, such as in the case of superfluids and superconductors. We shall not cover these more complicated cases here.

In general, it is rare that a small number of lowest-order diagrams would be a good approximation for the Green's function  $g$ . Therefore in practical calculations one either tries to obtain certain general results without employing approximations, or, whenever specific results are sought, to find a class of diagrams such that the contribution from this whole class is both calculable and dominant. However, it is not very common that such ideal situation occurs.

The general strategy in practical calculations is to reorganize the expansion through certain partial summations in a graphical way. One may begin with a few simple diagrams (called "skeletons"), and then perform all possible summations of lines and vertices (corresponding to "putting flesh to the skeletons") to obtain the Green's function  $g$ . To facilitate such calculations, we introduce in the following a few definitions and important relations. First, we define the self-energy  $\Sigma$  as the sum of contributions from all parts of the diagrams that are connected to the rest by two directed lines, one in and the other out. This definition can be explicitly expressed by the following relation:

$$g(\mathbf{k}, z_v) = G_0(\mathbf{k}, z_v) + G_0(\mathbf{k}, z_v) \Sigma(\mathbf{k}, z_v) G_0(\mathbf{k}, z_v). \quad (\text{II.463})$$

The definition in EQ. (II.463) can also be expressed graphically, as shown in Fig. II.7.7 (a). We also define the proper self-energy  $\Sigma^*$ , which involves only the self-energy parts that cannot be separated into two pieces by cutting a single particle line. The proper self-energy is related to the self energy via the relation:

$$\Sigma = \Sigma^* + \Sigma^* G_0 \Sigma^* + \Sigma^* G_0 \Sigma^* G_0 \Sigma^* + \dots = \Sigma^* + \Sigma^* G_0 \Sigma = \Sigma^* + \Sigma G_0 \Sigma^* . \quad (\text{II.464})$$

Combining EQs. (II.463) and (II.464), we obtain:

$$g = G_0 + G_0 \Sigma^* g = G_0 + g \Sigma^* G_0 . \quad (\text{II.465})$$

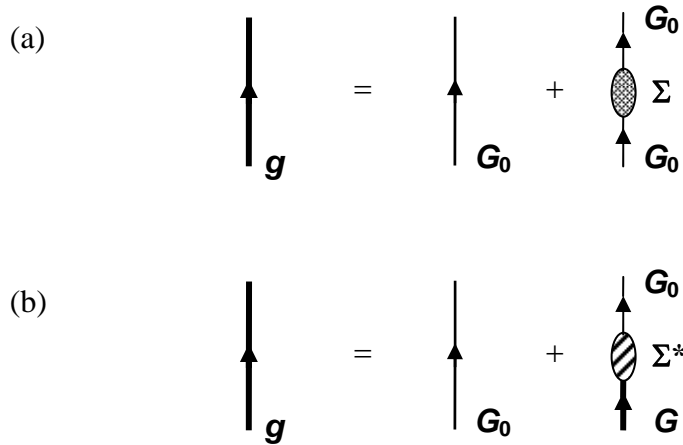
In the  $(\mathbf{k}, \omega)$ –space, EQ. (II.465) can be written as:

$$g(\mathbf{k}, \omega) = \frac{G_0(\mathbf{k}, \omega)}{1 - G_0(\mathbf{k}, \omega) \Sigma^*(\mathbf{k}, \omega)} = \frac{1}{\omega - \epsilon_{\mathbf{k}}^0 - \Sigma^*(\mathbf{k}, \omega)} \leftrightarrow g^{-1}(\mathbf{k}, \omega) = G_0^{-1}(\mathbf{k}, \omega) - \Sigma^*(\mathbf{k}, \omega) . \quad (\text{II.466})$$

The diagrammatic expression for EQ. (II.466) is illustrated in Fig. II.7.7 (b).

The proper self energy  $\Sigma^*$  contains an infinite number of diagrams, which can be expressed in terms of  $g$  and a quantity  $\Gamma$ , known as the vertex part. The vertex part is defined as the sum of the contributions from all parts that are connected with four (two in and two out) directed  $G_0$  lines to the rest, which cannot be decomposed into disconnected parts. The vertex part  $\Gamma$  is related to the two-particle Green’s function  $g_2$ , as shown in Fig. II.7.8 and also expressed explicitly below:

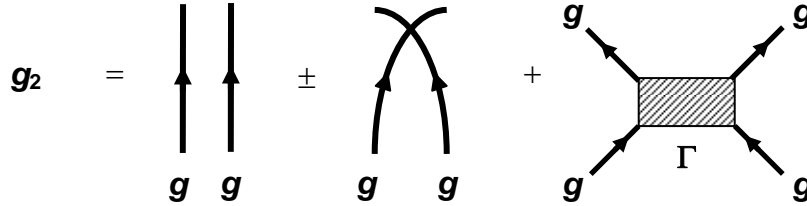
$$g_2(x_1, x_2; x'_1, x'_2) = g(x_1, x'_1) g(x_2, x'_2) \pm g(x_1, x'_2) g(x_2, x'_1) + \int d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}'_1 d\tilde{x}'_2 \Gamma(\tilde{x}_1, \tilde{x}_2; \tilde{x}'_1, \tilde{x}'_2) g(x_1, \tilde{x}_1) g(x_2, \tilde{x}_2) g(\tilde{x}'_1, x'_1) g(\tilde{x}'_2, x'_2) . \quad (\text{II.467})$$



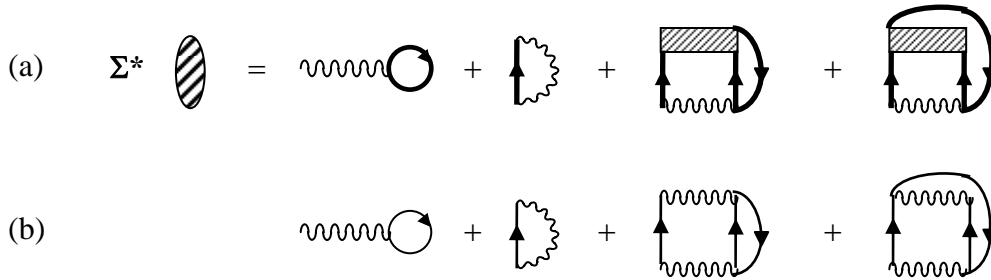
**Figure II.7.7** (a) Relation between the Green’s function  $g$  (thick line), the unperturbed Green’s function  $G_0$  (thin line), and the self energy  $\Sigma$ . (b) Relation between the Green’s function  $g$  (thick line), the unperturbed Green’s function  $G_0$  (thin line), and the proper self energy  $\Sigma^*$ .

From EQs. (II.465) and (II.467), it can be shown that the proper self energy  $\Sigma^*$  can be calculated from the four diagrams shown in Fig. II.7.9 (a), which result from “fleshing” the four skeleton diagrams in Fig. II.7.9 (b) – The “fleshing” is done by replacing the  $G_0$ -lines by  $g$ -lines and one of the two interaction

lines by the vertex part  $\Gamma$ . Moreover, we note that the diagrammatic expression given in Fig. II.7.9 (a) is in fact consistent with EQ. (II.324) given by the equation of motion method. The basic expression for the Green's function in EQ. (II.465) together with the diagrams in Fig. II.7.9 (a) for the proper self energy is known as the Dyson's equation.



**Figure II.7.8.** Diagrammatic expression for the relation between the two-particle Green's function  $g_2$ , the Green's function  $g$ , and the vertex part  $\Gamma$ .

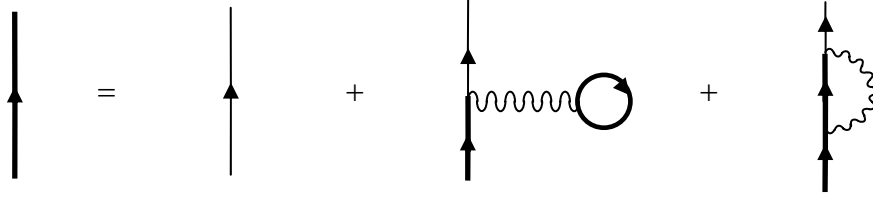


**Figure II.7.9** The proper self energy  $\Sigma^*$  is consistent with the sum of the four diagrams in (a), which are obtained by “fleshing” the four skeleton diagrams in (b).

However, the vertex part  $\Gamma$  generally cannot be expressed in terms of a closed form involving  $g$  and  $\Gamma$  because the skeleton diagrams for  $\Gamma$  are infinite in number. A typical approach is to end the infinite hierarchy of relations by approximately expressing  $\Gamma$  in a closed form involving  $g$  and  $\Gamma$ . The simplest of such approximations is to set  $\Gamma = 0$ , which is equivalent to the Hartree-Fock approximation that we have discussed earlier. The proper self energy in the Hartree-Fock approximation is given by:

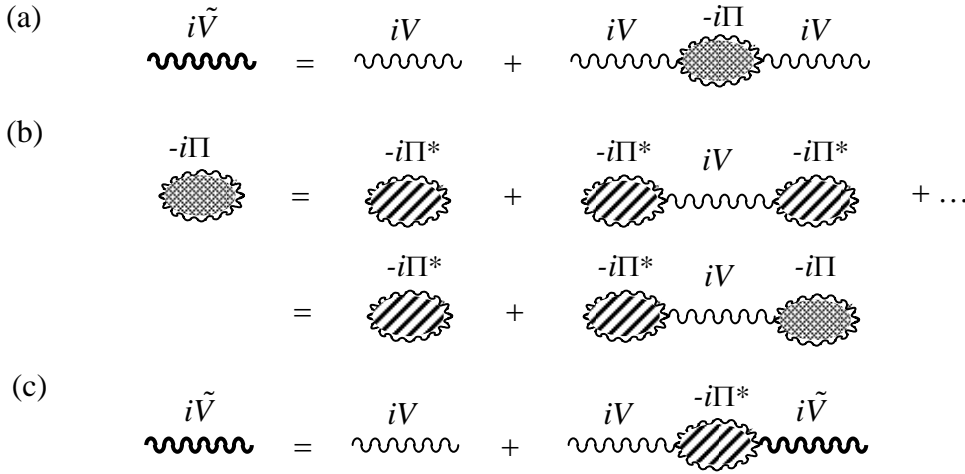
$$\begin{aligned} \Sigma^*(\mathbf{k}, z_\nu) &= \mp V(0) \int \frac{d^3 k'}{(2\pi)^3} \frac{1}{\beta} \sum_{\nu'} e^{-z'_\nu \sigma} G_0(\mathbf{k}', z'_\nu) - \int \frac{d^3 k'}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}') \frac{1}{\beta} \sum_{\nu'} e^{-z'_\nu \sigma} G_0(\mathbf{k}', z'_\nu) \\ &= V(0) \int \frac{d^4 k'}{(2\pi)^4} A(\mathbf{k}', \omega') f_{\mp}(\omega') \pm \int \frac{d^4 k'}{(2\pi)^4} V(\mathbf{k} - \mathbf{k}') A(\mathbf{k}', \omega') f_{\mp}(\omega') = \Sigma^*(\mathbf{k}). \end{aligned} \quad (\text{II.468})$$

Inserting EQ. (II.468) into EQ. (II.466), we recover EQs. (II.418) and (II.419) for the Green's function and eigen-energy in the Hartree-Fock approximation, as expected. The diagrammatic representation of the Hartree-Fock approximation is shown in Fig. II.7.10.



**Figure II.7.10** Feynman diagrams for the Hartree-Fock approximation.

Next, we introduce the concept of an effective inter-particle interaction  $\tilde{V}$  as the sum of the bare interaction potential  $V$  plus contributions (apart from a factor  $i$  or  $-1$  for the case of  $T = 0$  or  $T \neq 0$ , respectively) from all parts of the diagrams that have two external interaction lines. The effective interaction  $\tilde{V}$  can be expressed diagrammatically in terms of the polarization  $\Pi$ , the latter is defined as the sum of the contributions (apart from a factor  $i$  or  $-1$  for the case of  $T = 0$  or  $T \neq 0$ , respectively) of all parts that are connected to the rest by two interaction lines, as illustrated in Fig. II.7.11 (a). Following an analysis similar to that given for the self energy, we can easily obtain the polarization  $\Pi$  in terms of the proper polarization  $\Pi^*$ , as shown in Fig. II.7.11 (b), where the proper polarization is defined as the sum of the contributions (apart from a factor  $i$  or  $-1$  for the case of  $T = 0$  or  $T \neq 0$ , respectively) of all parts that are connected to the rest by two external interaction lines, which cannot be separated into two pieces by cutting a single interaction line.



**Figure II.7.11** Diagrammatic expressions for the relation between (a)  $\tilde{V}$ ,  $V$ , and  $\Pi$ ; (b)  $V$ ,  $\Pi$  and  $\Pi^*$ ; (c)  $\tilde{V}$ ,  $V$ , and  $\Pi^*$  for fermions at  $T = 0$ . For  $T \neq 0$ , the imaginary time case,  $i\tilde{V}$ ,  $iV$ ,  $i\Pi$  and  $i\Pi^*$  must be replaced by  $-\tilde{V}$ ,  $-V$ ,  $-\Pi$  and  $-\Pi^*$ , respectively.

For a rotational invariant system, if we define the momentum transfer due to the interaction as  $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ , the diagrammatic expressions in Fig. II.7.11 can be written as follows:

$$\tilde{V}(\mathbf{q}) = V(\mathbf{q}) + V^2(\mathbf{q})\Pi(\mathbf{q}), \quad (\text{II.469})$$

$$\Pi(\mathbf{q}) = \Pi^*(\mathbf{q}) + \Pi^*(\mathbf{q})V(\mathbf{q})\Pi^*(\mathbf{q}) + \dots = \Pi^*(\mathbf{q}) + \Pi^*(\mathbf{q})V(\mathbf{q})\Pi(\mathbf{q}). \quad (\text{II.470})$$

Hence, we obtain

$$\Pi(\mathbf{q}) = \frac{\Pi^*(\mathbf{q})}{1 - V(\mathbf{q})\Pi^*(\mathbf{q})}, \quad (\text{II.471})$$

and

$$\tilde{V}(\mathbf{q}) = V(\mathbf{q}) + V(\mathbf{q})\Pi^*(\mathbf{q})\tilde{V}(\mathbf{q}), \quad (\text{II.472})$$

which leads to

$$\tilde{V}(\mathbf{q}) = \frac{V(\mathbf{q})}{1 - V(\mathbf{q})\Pi^*(\mathbf{q})}. \quad (\text{II.473})$$

If we define a dielectric function  $\varepsilon^c(\mathbf{k}, \omega)$  by the following relation:

$$\tilde{V}(\mathbf{q}, \omega) = \frac{V(\mathbf{q}, \omega)}{\varepsilon^c(\mathbf{q}, \omega)}, \quad (\text{II.474})$$

we can express the dielectric function in terms of the polarization  $\Pi$  and proper polarization  $\Pi^*$ :

$$\frac{1}{\varepsilon^c(\mathbf{q}, \omega)} = 1 + V(\mathbf{q}, \omega)\Pi(\mathbf{q}, \omega) = \frac{1}{1 - V(\mathbf{q}, \omega)\Pi^*(\mathbf{q}, \omega)}. \quad (\text{II.475})$$

The polarization  $\Pi$  defined here is the result of the dielectric response of a many-body system to the inter-particle interactions, which we shall elaborate in Part II.8. For comparison, we note that similar polarization tensor can be introduced in the case of relativistic quantum field theory for quantum fluctuations associated with photon propagation in vacuum.

For classic references on the application of Green's functions to condensed matter physics, you may consider the following books:

- “Green’s Functions for Solid State Physicists”, S. Doniach and E. H. Sondheimer, Imperial College Press (1998).
- “Methods of Quantum Field Theory in Statistical Physics”, A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, revised English edition.
- “Quantum Theory of Many-Particle Systems”, A. L. Fetter and J. D. Walecka.