II.4. Spinors, Dirac Spinor Field, and Quantization of the Spinor Field

Up to this point we have only considered quantum field theory for bosonic particles (mesons, photons) with integer spins. What would happen if we introduce fermionic particles with half-integer spins? Dirac's ingenious insights provide the answer, which is the focus of our discussion in this section.

[Dirac equation, spinor fields, and γ-matrices]

To describe the half-integer spins, Dirac realized that he needed to introduce a spinor field Ψ that satisfies a relativistic wave equation linear in space-time derivatives $\partial_\mu \equiv \partial / \partial x^\mu$ such as the following:

$$\left(i\gamma^\mu \partial_\mu - m\right)\Psi(x) = 0\,, \qquad \text{(II.121)}$$

where m is just a constant, and the properties of γ^μ are yet to be determined. If we multiple $(i\gamma^\mu\partial_\mu + m)\Psi$ to EQ. (II.121), we obtain $-(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\Psi = 0$. Since derivatives commute, we can rewrite $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu$ into $½\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu$, so that the result becomes $(½\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu + m^2)\Psi = 0$. If we impose the condition

$$\left\{\gamma^\mu, \gamma^\nu\right\} = 2\eta^{\mu\nu} \qquad \text{(II.122)}$$

with $\eta^{\mu\nu}$ being the Minkowski metric, we obtain $(\partial^2 + m^2)\Psi = 0$, which describes a particle of mass m. Consequently, EQ. (II.121) also describes a particle of mass m if γ^μ are defined by the condition given in EQ. (II.122). Noting that $\eta^{00} = 1$ and $\eta^{ij} = -1$, we find that $(\gamma^0)^2 = 1$, $(\gamma^j)^2 = -1$, and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$. From these conditions, it is clear that γ^μ cannot be ordinary numbers.

A set of objects γ^μ satisfying the relation given in EQ. (II.122) is said to form a Clifford algebra, and there are totally d of these objects for a d-dimensional spacetime. Let's now we consider four of the 4×4 γ-matrices in four-dimension that satisfy the Clifford algebra. They are:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = I \otimes \tau_3 \qquad \text{and} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \sigma^i \otimes i\tau_2\,, \qquad \text{(II.123)}$$

where I denotes the 2×2 unit matrix, and σ^i and τ^j denote the standard Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{where } \left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\varepsilon^{ijk}\frac{\sigma^k}{2}\,.$$

We also define $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$ so that $\gamma_\mu\,\partial^\mu = \gamma^\mu\,\partial_\mu$.

Using EQ. (II.123), we obtain $\gamma^i\gamma^j = (\sigma^i \otimes i\tau_2)(\sigma^j \otimes i\tau_2) = (\sigma^i\sigma^j \otimes i^2\tau_2\,\tau_2) = -(\sigma^i\sigma^j \otimes I)$. Consequently, we have $\{\gamma^i, \gamma^j\} = -\{\sigma^i, \sigma^j\} \otimes I = -2\delta^{ij}$, which indeed satisfies the Clifford algebra. In fact, the γ-matrices cannot be smaller than 4×4 matrices, thereby forcing the Dirac spinor to have 4 components.

Next, we transform the Dirac equation in EQ. (II.121) momentum space by using the identity

$$\Psi(x) = \int \frac{d^4 p}{(2\pi)^4}\, e^{-i\,p\,x}\,\Psi(p)\,,$$

so that EQ. (II.121) can be rewritten in momentum space as follows:

$$\left(\gamma^{\mu}p_{\mu}-m\right)\Psi(p)=0\,. \tag{II.124}$$

We'll show later that EQ. (II.124) is in fact Lorentz invariant, so that we can evaluate its physics in any frame, including the rest frame $p^{\mu}=(m,\vec{0})$. In the latter case, EQ. (II.124) becomes:

$$\left(\gamma^{0}-1\right)\Psi(p)=0\,. \tag{II.125}$$

Using EQs. (II.123) and (II.125), we find that $\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}\Psi=0$, which implies that 2 out of the 4 components in the spinor Ψ are zero. This finding is consistent with the fact that the electron only has 2 degrees of freedom. We shall return to the physical significance of EQ. (II.125) later.

For convenience, we'll use in the following the notation introduced by Feynman, $\not{a}\equiv\gamma^{\mu}a_{\mu}$ for any 4-vector a_{μ}. The Dirac equation in EQ. (II.121) is therefore given by $(i\not{\partial}-m)\Psi=0$.

For a Lorentz transformation $x'^{\nu}=\Lambda^{\nu}{}_{\mu}\,x^{\mu}$, the transformation of spinors is determined by the 4 × 4 matrix $S(\Lambda)$ so that $\Psi(x)\rightarrow\Psi'(x')\equiv S(\Lambda)\Psi(x)$. There are 16 components in $S(\Lambda)$ so that we expect 16 linearly independent 4 × 4 matrices. (For more details about the Lorentz group and the related tensor and spinor transformation, you may refer to Supplement_3 notes.) We can immediately identify 5 of the 4 × 4 matrices: the unit matrix and the 4 γ-matrices. In addition, we can define a matrix γ^5:

$$\gamma^{5}\equiv i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}=I\otimes\tau_{1}=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \tag{II.126}$$

such that $\{\gamma^5,\gamma^{\mu}\}=0$. Moreover, we note that $\gamma^{\mu}\gamma^{5}$ for all μ are all different, and there are 4 such independent products. Finally, from $\{\gamma^{\mu},\gamma^{\nu}\}=2\eta^{\mu\nu}$ we find that $\gamma^{\mu}\gamma^{\nu}=\eta^{\mu\nu}+[\gamma^{\mu},\gamma^{\nu}]/2\equiv\eta^{\mu\nu}-i\sigma^{\mu\nu}$, where

$$\sigma^{\mu\nu}\equiv\frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]\,. \tag{II.127}$$

It is useful to express $\sigma^{\mu\nu}$ explicitly as follows:

$$\sigma^{0i}=i\begin{pmatrix}0 & \sigma^{i}\\ \sigma^{i} & 0\end{pmatrix}\qquad\text{and}\qquad\sigma^{ij}=\varepsilon^{ijk}\begin{pmatrix}\sigma^{k} & 0\\ 0 & \sigma^{k}\end{pmatrix}. \tag{II.128}$$

Obviously there are 6 of independent $\sigma^{\mu\nu}$ matrices, and σ^{ij} are simply the Pauli matrices doubly stacked. Therefore, we have obtained a collection of 16 independent matrices given by $\{I,\gamma^{\mu},\gamma^{5},\gamma^{\mu}\gamma^{5},\sigma^{\mu\nu}\}$, and any 4 × 4 matrix can be written as a linear combination of these 16 matrices.

For spinor representations in an $SO(N)$ group, we have introduced in Supplement_2 note the objects Γ^{μ} that satisfy the Clifford algebra:

$$\left\{\Gamma^{\mu},\Gamma^{\nu}\right\}=2\delta^{\mu\nu}, \tag{II.129}$$

so that the spinor representations of the generators are given by

$$M^{\mu\nu}=\frac{i}{4}\left[\Gamma^{\mu},\Gamma^{\nu}\right], \tag{II.130}$$

and $M^{\mu\nu}$ satisfies the commutation relation

$$\left[M^{\mu\nu}, M^{\lambda\sigma}\right] = i\left(-\delta^{\mu\lambda}M^{\nu\sigma} + \delta^{\nu\lambda}M^{\mu\sigma} + \delta^{\mu\sigma}M^{\nu\lambda} - \delta^{\nu\sigma}M^{\mu\lambda}\right). \tag{II.131}$$

As discussed in Supplement_2, EQ. (II.131) also holds for tensor representations besides the spinor representations. Now using $M^{\mu\nu}$ and further defining

$$U(\theta_{\mu\nu}) = \exp\left(-\frac{i}{2}\theta_{\mu\nu}M^{\mu\nu}\right), \tag{II.132}$$

we find that Γ^{μ} satisfies the following identity:

$$U\left(\theta^{\mu\nu}\right)\Gamma^{\lambda}U^{-1}\left(\theta^{\mu\nu}\right) = \left[O^{-1}\left(\theta^{\mu\nu}\right)\right]^{\lambda\sigma}\Gamma^{\sigma}, \tag{II.133}$$

where O denotes an orthogonal matrix.

Next, we observe that the γ-matrices defined for the Dirac spinor field satisfy the Clifford algebra, and that the matrices $\sigma^{\mu\nu}$ given in EQ. (II.127) take a similar form to the generators given in EQ. (II.130). Therefore we can make the following substitution in EQ. (II.132):

$$M^{\mu\nu} = L^{\mu\nu} + \frac{1}{2}\sigma^{\mu\nu} = L^{\mu\nu} + \frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right], \tag{II.134}$$

where the generators $L^{\mu\nu} \equiv x^{\mu}p^{\nu} - x^{\nu}p^{\mu} = i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right)$ work on the spacetime coordinates, and $\sigma^{\mu\nu}$ work on the spinors. Thus, substituting $\theta_{\mu\nu}$ by $\omega_{\mu\nu}$, we have the matrices Λ and the spinor representations $S(\Lambda)$ in the Lorentz group:

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right), \qquad S(\Lambda) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right). \tag{II.135}$$

The Dirac matrices γ^{μ} transform like vectors under the spinor representations of the Lorentz group, which can be verified by evaluating whether the following relation holds:

$$S(\Lambda)\,\gamma^{\mu}\,S^{-1}(\Lambda) = \left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\,\gamma^{\nu}. \tag{II.136}$$

Using EQ. (II.135) and for infinitesimal ω, we find

$$S(\Lambda)\,\gamma^{\lambda}\,S^{-1}(\Lambda) = \gamma^{\lambda} - \frac{i}{4}\omega_{\mu\nu}\left[\sigma^{\mu\nu}, \gamma^{\lambda}\right], \tag{II.137}$$

and

$$\left[\gamma^{\lambda}, \sigma^{\mu\nu}\right] = \left[\gamma^{\lambda}, \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]\right] = 2i\left(\gamma^{\nu}\eta^{\mu\lambda} - \gamma^{\mu}\eta^{\nu\lambda}\right). \tag{II.138}$$

The last step in EQ. (II.138) can be obtained via the relation $\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma_{\mu} = 4\eta^{\nu\lambda}$. Consequently, we obtain

$$S(\Lambda)\,\gamma^{\lambda}\,S^{-1}(\Lambda) = \gamma^{\lambda} + \gamma^{\mu}\omega^{\lambda}_{\mu} = \left(\delta^{\lambda}_{\mu} + \omega^{\lambda}_{\mu}\right)\gamma^{\mu} = \left(\Lambda^{-1}\right)^{\lambda}{}_{\mu}\,\gamma^{\mu}, \tag{II.139}$$

consistent with EQ. (II.136). Note that in EQ. (II.139) we have used the condition that $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

The result obtained in EQ. (II.139) has an important implication to the Dirac equation given in EQ. (II.121): Noting that the spinor field $\Psi(x)$ in the Lorentz group transforms according to $\Psi'(x') \equiv S(\Lambda)\Psi(x)$, we find that the transformed spinor field $\Psi'(x')$ indeed satisfies the Dirac equation in the primed frame:

$$\left(i\gamma^{\mu}\partial'_{\mu} - m\right)\Psi'(x') = 0, \qquad \text{where } \partial'_{\mu} \equiv \partial/\partial x'^{\mu}.$$

As shown in Supplement_3 that there are six generators J_i and K_i (i = 1, 2, 3) for rotations and boosts in the Lorentz group and that we have introduced the spinor generators $A_i \equiv ½ (J_i + iK_i)$, and $B_i \equiv ½ (J_i - iK_i)$ that satisfy the following commutation relations:

$$\left[A_i, A_j\right] = i\varepsilon_{ijk}A_k, \quad \left[B_i, B_j\right] = i\varepsilon_{ijk}B_k, \quad \left[A_i, B_j\right] = 0. \tag{II.140}$$

We have noted that EQ. (II.140) implies that the matrices A_i and B_t form two separate $SU(2)$ algebras, so that the $SO(3,1)$ Lorentz group is isomorphic to $SU(2) \otimes SU(2)$. The representations of $SU(2)$ are labeled by j = 0, ½, 1, 3/2, …, and each representation can be thought of as consisting of $(2j+1)$ objects ψ_m with $m = -j, -j+1, \ldots, j-1, j$ that transform into each other under the symmetry operations of $SU(2)$. Following the result in EQ. (II.140), we label the representations of $SO(3,1)$ by (j^+, j^-), and each j^+ and j^- takes on the values 0, ½, 1, 3/2, …, so that there are $(2j^+{+}1)(2j^-{+}1)$ objects of $\psi_{m+,m-}$ with $m^+ = -j^+, -j^++1, \ldots, j^+-1, j^+$ and $m^- = -j^-, -j^-+1, \ldots, j^--1, j^-$. Therefore the representations of $SO(3,1)$ are (0,0), (1/2,0), (0,1/2), (1,0), (0,1), (1/2,1/2), (2,0), (0,2), …, in order of increasing dimension. The one-dimensional representation (0,0) is a Lorentz scalar, and the four-dimensional representation (1/2,1/2) is a Lorentz vector, the defining representation of the Lorentz group. It is also worth noting that the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ transform in an Lorentz group as (1,0) ⊕ (0,1).

We have also considered the two-dimensional representations (1/2,0) and (0,1/2) by decomposing the 4-dimensional Dirac spinor field Ψ into the Weyl spinors ψ_α and $\bar{\chi}^{\dot{\alpha}}$ (α = 1, 2) through two reducible representations Γ_+ and Γ_-, such that

$$\Gamma_+\Psi = c_1\begin{pmatrix}\psi_\alpha \\ 0\end{pmatrix}, \qquad \Gamma_-\Psi = c_1\begin{pmatrix}0 \\ \bar{\chi}^{\dot{\alpha}}\end{pmatrix}, \qquad \text{and } \Psi = \begin{pmatrix}\psi_\alpha \\ \bar{\chi}^{\dot{\alpha}}\end{pmatrix}, \tag{II.141}$$

where c_1 is to be determined. The representation (1/2,0) and (0,1/2) can be thought of as the irreducible representations corresponds to operating Γ_+ on ψ_α to yield ½ σ_i while operating Γ_- on ψ_α to yield 0. Here σ_i denotes the Pauli matrices. Similarly, operating Γ_+ on $\bar{\chi}^{\dot{\alpha}}$ yields 0 while operating Γ_- on $\bar{\chi}^{\dot{\alpha}}$ yields ½ σ_i. Therefore from EQ. (II.141) we expect that Γ_+ and Γ_- have the following structures:

$$\Gamma_+ = \begin{pmatrix}\frac{1}{2}\boldsymbol{\sigma} & 0 \\ 0 & 0\end{pmatrix}, \qquad \Gamma_- = \begin{pmatrix}0 & 0 \\ 0 & \frac{1}{2}\boldsymbol{\sigma}\end{pmatrix}, \tag{II.142}$$

Moreover, we expect Γ_+ and Γ_- to transform into each other under parity operation $\boldsymbol{x} \to -\boldsymbol{x}$ and $\boldsymbol{p} \to -\boldsymbol{p}$. Thus, we may write $\Gamma_+ = ½ (\Gamma_1 + i\Gamma_2)$ and $\Gamma_- = ½ (\Gamma_1 - i\Gamma_2)$, so that:

$$\Gamma_1 = \begin{pmatrix}\frac{1}{2}\boldsymbol{\sigma} & 0 \\ 0 & \frac{1}{2}\boldsymbol{\sigma}\end{pmatrix}, \qquad i\Gamma_2 = \begin{pmatrix}\frac{1}{2}\boldsymbol{\sigma} & 0 \\ 0 & -\frac{1}{2}\boldsymbol{\sigma}\end{pmatrix}, \tag{II.143}$$

and $\Gamma_1 \to \Gamma_1$ and $\Gamma_2 \to -\Gamma_2$ under parity. We can identify Γ_1 as the generators of rotation for the spinor Ψ, and $i\Gamma_2$ as the generators of the Lorentz boosts for Ψ. Traditionally, we use $\boldsymbol{J} \leftrightarrow \Gamma_1$ and $i\boldsymbol{K} \leftrightarrow i\Gamma_2$, to be consistent with the notations of the Lorentz operators for tensors. However, the notations used for spinors in EQ. (II.143) are not to be confused with the $\boldsymbol{J}$ and $\boldsymbol{K}$ matrices defined in Supplement_3 for tensors.

We have seen that under parity the representations (1/2,0) and (0,1/2) transform into each other. Therefore parity dictates that we have a 4-component spinor. On the other hand, we know that the electron only has two physical degrees of freedom. So if we choose the rest frame, where the 4-vector momentum is given by $p_r = (m, \vec{0}\,)$, we have $(p^0\gamma^0 - m)\Psi(p_r) = m(\gamma^0 - 1)\Psi(p_r) = 0$. Noting that $\psi_\alpha = \bar{\chi}^{\dot{\alpha}}$ in the rest frame, we find that the γ^0 matrix is given by:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{II.144}$$

if we associate the two components in the Dirac spinor with parity. Note that this definition of γ^0 differs from previous definition in EQ. (II.123), and is identical to that of γ^5 defined earlier. This change in the basis is for convenience and really does not change any of the physics. We can now generalize the result to any moving frame with a Lorentz boost to the spinor $\Psi(\,p_r)$, such that the new spinor $\Psi(\,p_r)$ is given by

$$\Psi(\boldsymbol{p}) = e^{-i\boldsymbol{\theta}\boldsymbol{K}}\,\Psi(\boldsymbol{p}_r)\,. \tag{II.145}$$

The Dirac equation in a general Lorentz frame is therefore given by

$$\begin{aligned} e^{-i\boldsymbol{\theta}\boldsymbol{K}}\left(\gamma^0 - 1\right)\Psi(\boldsymbol{p}_r) &= 0 = e^{-i\boldsymbol{\theta}\boldsymbol{K}}\left(\gamma^0 - 1\right)e^{i\boldsymbol{\theta}\boldsymbol{K}}\Psi(\boldsymbol{p}) \\ &= \left(e^{-i\boldsymbol{\theta}\boldsymbol{K}}\gamma^0 e^{i\boldsymbol{\theta}\boldsymbol{K}} - 1\right)\Psi(\boldsymbol{p}) = \left(\frac{\gamma^\mu p_\mu}{m} - 1\right)\Psi(\boldsymbol{p})\,. \end{aligned} \tag{II.146}$$

Therefore we find that $\left(\gamma^\mu p_\mu / m\right) = e^{-i\boldsymbol{\theta}\boldsymbol{K}}\gamma^0 e^{i\boldsymbol{\theta}\boldsymbol{K}}$ according to EQ. (II.146).

There is in fact a deeper reason for the difference set of basis used in our discussion about parities of electrons as compared with our previous choice of basis in EQ. (II.123). Let's investigate this issue a bit further. If we return to the definition of the γ-matrices as given in EQ. (II.123), we would in fact get two components in the Dirac spinor different from those given by the Weyl spinors ψ_α and $\bar{\chi}^{\dot{\alpha}}$, which we shall refer to as ϕ and χ:

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \tag{II.147}$$

Now we can solve the Dirac equation explicitly. Noting that

$$\left(\not{p} - m\right)\Psi(p) = \left(\gamma^\mu p_\mu - m\right)\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0\,, \tag{II.148}$$

and taking the direction of the momentum to be along the z-axis, we find that EQ. (II.148) becomes:

$$\begin{aligned} \left(\gamma^0 p_0 - \gamma^3 p - m\right)\begin{pmatrix} \phi \\ \chi \end{pmatrix} &= \left[E\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - p\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}\right]\begin{pmatrix} \phi \\ \chi \end{pmatrix} - m\begin{pmatrix} \phi \\ \chi \end{pmatrix} \\ &= \begin{pmatrix} E - m & -p\sigma^3 \\ p\sigma^3 & -E - m \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0\,. \end{aligned} \tag{II.149}$$

Therefore we obtain the following relationship between ϕ and χ:

$$\chi = \left[\frac{p}{(E+m)} \right] \sigma^3 \phi \,, \tag{II.150}$$

$$(E-m)\phi - p\sigma^3 \chi = 0 \,. \tag{II.151}$$

For a slow moving electron, $p << (E + m) \approx 2m$, so that $\chi \approx (p/2m)\sigma^3\phi$, so that χ is smaller than ϕ by a factor of $(p/2m)$ from EQ. (II.150). We also have $(E - m - p^2/2m)\phi = 0$ from EQ. (II.151), which is simply the relation between energy and momentum in the non-relativistic limit. In the electron rest frame, we obtain $\chi = 0$, which is consistent with the finding described earlier in EQ. (II.125). Consequently, we find that in the slow electron limit, the basis defined in EQ. (II.123) is indeed more convenient.

In contrast, if we consider the relativistic limit with $p >> m$, EQ. (II.148) becomes

$$\not{p}\Psi(p) = 0 \,. \tag{II.152}$$

Noting that γ^5 anti-commutes with γ^μ, we find that $\gamma^5\Psi$ is also a solution to the Dirac equation. Since $(\gamma^5)^2 = 1$, it is convenient to define two projection operators using γ^5: $P_L \equiv ½ (1 - \gamma^5)$ and $P_R \equiv ½ (1 + \gamma^5)$, so that the operators satisfy $P_L^2 = P_L$, $P_R^2 = P_R$, and $P_L P_R = 0$. If we further define $\Psi_L = ½ (1 - \gamma^5)\Psi$ and $\Psi_R = ½ (1 + \gamma^5)\Psi$, we have $P_L\Psi = \Psi_L$, $P_R\Psi = \Psi_R$, $\gamma^5\Psi_L = -\Psi_L$, and $\gamma^5\Psi_R = +\Psi_R$. The physical significance for the presence of Ψ_L and Ψ_R is that a relativistic electron has two degrees of freedom, either spinning clockwise or counterclockwise. In hindsight of our earlier discussion of the Weyl spinors, it is convenient to take a basis different from EQ. (II.123), with γ^0 given by EQ. (II.144) instead. If we keep all γ^i ($i = 1, 2, 3$) the same, we find that the new γ^5 for this new "Weyl basis" is

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{II.153}$$

Comparing the relations $P_L\Psi = \Psi_L$ and $P_R\Psi = \Psi_R$ with EQ. (II.141), we identify Ψ_L and Ψ_R with the Weyl spinors ψ_α and $\bar{\chi}^{\dot{\alpha}}$. We also note that both $(\gamma^0)^2 = 1$ and $(\gamma^5)^2 = 1$ while $(\gamma^i)^2 = -1$. Hence, it is not surprising that the basis can be chosen with γ^0 and γ^5 interchanged while keeping all γ^i intact. The choice of either the Dirac or the Weyl basis is made based on convenience: For non-relativistic spin ½ particles, we prefer the Dirac basis with γ^0 diagonal, whereas for relativistic spin ½ particles, we prefer the Weyl basis with γ^5 diagonal.

Regardless of either the Dirac or Weyl basis, you can easily verify that our following discussion of the properties of the Dirac spinor Ψ is in fact independent of choice of the basis. For convenience, we shall assume that the Dirac basis is used unless specified. Given that $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$, a consequence is that γ^0 is hermitean while γ^i is anti-hermitean. [You may recall that A is hermitean if $A^\dagger \equiv (A^*)^T \equiv A$ and is anti-hermitean if $A^\dagger \equiv (A^*)^T \equiv -A$.] A convenient form to express this fact is given below:

$$\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0 \,. \tag{II.154}$$

[You may want to check that it is indeed equivalent to replace γ^0 in EQ. (II.154) by γ^5.] Next, we define

$$\overline{\Psi} \equiv \Psi^\dagger\gamma^0 \,, \tag{II.155}$$

and therefore we find the bilinear $\overline{\Psi}\gamma^\mu\Psi$ (rather than $\Psi^\dagger\gamma^\mu\Psi$ as you might have guessed) is hermitean. It follows that the hermitean of $\sigma^{\mu\nu}$ given in EQ. (II.127) satisfies

$$\left(\sigma^{\mu\nu}\right)^{\dagger}=\gamma^0\sigma^{\mu\nu}\gamma^0 . \tag{II.156}$$

Therefore, the (4 × 4) matrix $S(\Lambda)$ that transforms $\Psi(x)$ to $\Psi'(x')$ according to $\Psi'(x')\equiv S(\Lambda)\Psi(x)$ has its hermitian given by

$$S(\Lambda)^{\dagger}=\gamma^0\exp\left(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\gamma^0 , \tag{II.157}$$

and we obtain

$$\begin{aligned}\overline{\Psi'}(x')&=\overline{S(\Lambda)\Psi(x)}=\left[S(\Lambda)\Psi(x)\right]^{\dagger}\gamma^0=\Psi(x)^{\dagger}S(\Lambda)^{\dagger}\gamma^0\\&=\Psi(x)^{\dagger}\gamma^0\exp\left(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\gamma^0\gamma^0=\overline{\Psi(x)}\exp\left(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right),\end{aligned} \tag{II.158}$$

so that

$$\overline{\Psi'(x')}\Psi'(x')=\overline{\Psi(x)}\exp\left(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\Psi(x)=\overline{\Psi(x)}\Psi(x). \tag{II.159}$$

The result in EQ. (II.159) indicates that $\overline{\Psi(x)}\Psi(x)$ (rather than $\Psi(x)^{\dagger}\Psi(x)$) transforms like a Lorentz scalar.

Recall that there are 16 linearly independent matrices for the Lorentz group, it is obvious that we also have 16 bilinears $\overline{\Psi(x)}\,\Gamma\,\Psi(x)$. You can examine how various bilinears transform as an exercise. For instance, you can verify that $\overline{\Psi(x)}\,\gamma^{\mu}\,\Psi(x)$ transforms like a Lorentz vector by examining its transformation laws using EQ. (S2.8).

[The Dirac Lagrangian]

Having seen various important properties of the Dirac equation, we want to move further to discuss quantum field theory associated with the Dirac equation. We want to know what Lagrangian can give rise to the Dirac equation. The answer is

$$\mathcal{L}=\overline{\Psi}\left(i\partial\!\!\!/-m\right)\Psi . \tag{II.160}$$

Noting that $\overline{\Psi(x)}$ and $\Psi(x)$ are complex and independent, we can vary either $\overline{\Psi(x)}$ or $\Psi(x)$ in EQ. (II.160) independently to obtain the Euler-Lagrangian equation of motion. For instance, by varying $\overline{\Psi(x)}$ we obtain:

$$\left(\delta\mathcal{L}/\delta\overline{\Psi}\right)-\partial_{\mu}\left(\delta\mathcal{L}/\delta\partial_{\mu}\overline{\Psi}\right)=\left(i\partial\!\!\!/-m\right)\Psi-0=0 ,$$

which is exactly the Dirac equation. On the other hand, by varying EQ. (II.160) relative to $\Psi(x)$, we have:

$$\partial_{\mu}\left(\delta\mathcal{L}/\delta\partial_{\mu}\Psi\right)-\left(\delta\mathcal{L}/\delta\Psi\right)=\partial_{\mu}\left(i\overline{\Psi}\gamma^{\mu}\right)+m\overline{\Psi}=0 . \tag{II.161}$$

We can rewrite EQ. (II.161) into

$$\gamma^0\left[\partial_{\mu}\left(i\overline{\Psi}\gamma^{\mu}\right)+m\overline{\Psi}\right]^{\dagger}=\gamma^0\left[-i\partial_{\mu}\left(\gamma^{\mu\dagger}\overline{\Psi}^{\dagger}\right)+m\overline{\Psi}^{\dagger}\right]=\gamma^0\left[-i\partial_{\mu}\left(\gamma^0\gamma^{\mu}\gamma^0\gamma^0\Psi\right)+m\gamma^0\Psi\right]=-\left(i\partial_{\mu}\gamma^{\mu}-m\right)\Psi=0$$

which is indeed the Dirac equation.

In general, we can decompose Ψ into the left and right handed fields Ψ_L and Ψ_R as follows:

$$\Psi(x)=\Psi_L(x)+\Psi_R(x)\equiv\frac{1}{2}\left(1-\gamma^5\right)\Psi(x)+\frac{1}{2}\left(1+\gamma^5\right)\Psi(x). \tag{II.162}$$

Inserting EQ. (II.161) into the Dirac Lagrangian in EQ. (II.160), one finds that

$$\mathcal{L} = \overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi = \overline{\Psi}_L\, i\partial\!\!\!/\,\Psi_L + \overline{\Psi}_R\, i\partial\!\!\!/\,\Psi_R - m\left(\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L\right). \tag{II.163}$$

In addition, we find that the transformation $\Psi \rightarrow \exp(i\theta)\ \Psi$ leaves the Lagrangian invariant. Similarly, the left and right handed fields under the transformation $\Psi_L \rightarrow \exp(i\theta)\ \Psi_L$ and $\Psi_R \rightarrow \exp(i\theta)\ \Psi_R$ also leaves the Lagrangian invariant. We can also apply the Noether's theorem to obtain a conserved current $J^\mu = \overline{\Psi}\gamma^\mu\Psi$. Finally, if $m = 0$, the Dirac Lagrangian enjoys an additional symmetry, known as the chiral symmetry, so that the massless Dirac Lagrangian remains invariant under the transformation $\Psi \rightarrow \exp(i\theta\gamma^5)\ \Psi$. [You can easily verify this statement.]

Given the Dirac Lagrangian, we can now consider the interaction of Fermions with scalar or vector fields. For instance, if the Dirac spinor couples to a scalar field φ via a term $g\varphi\overline{\Psi}\Psi$ with g being a coupling constant, the Lagrangian becomes:

$$\mathcal{L} = \overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi + \frac{1}{2}\left[\left(\partial\varphi\right)^2 - m^2\varphi^2\right] + g\varphi\overline{\Psi}\Psi\ . \tag{II.164}$$

Similarly, we can couple a vector field by adding the term $eA_\mu\overline{\Psi}\gamma^\mu\Psi$ and by introducing $D_\mu = \partial_\mu - ieA_\mu$ to the Lagrangian:

$$\mathcal{L} = \overline{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu\,, \tag{II.165}$$

where μ is the mass of the vector field. If the mass μ vanishes, EQ. (II.165) becomes the Lagrangian for quantum electrodynamics (QED). Varying the QED Lagrangian with respect to $\overline{\Psi}$, we obtain the Dirac equation in the presence of electromagnetic fields as follows:

$$\left(i\gamma^\mu D_\mu - m\right)\Psi = \left[i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right]\Psi = 0\,. \tag{II.166}$$

[γ-matrices and discrete symmetries of parity, charge conjugation and time reversal]

The Dirac γ-matrices can in fact be associated with three important discrete symmetries: parity, charge conjugate, and time reversal. Let's consider parity first. A point in space-time transforms under the parity operation according to the following:

$$x^\mu = (x^0, \boldsymbol{x}) \rightarrow x'^\mu = (x^0, -\boldsymbol{x})\ .$$

Multiplying the Dirac equation by γ^0, and using the commutation relations of γ-matrices, we obtain

$$\gamma^0\left(i\gamma^\mu\partial_\mu - m\right)\Psi(x) = \left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\gamma^0\Psi(x) = \left(i\gamma^\mu\partial'_\mu - m\right)\gamma^0\Psi(x) \equiv \left(i\gamma^\mu\partial'_\mu - m\right)\Psi'(x') = 0\,,$$

where $\partial'_\mu \equiv \partial/\partial x'^\mu$. Consequently, we can define the parity transformation of the Dirac spinor as:

$$\Psi'(x') = \gamma^0\Psi(x)\,. \tag{II.167}$$

Let us examine a couple of bilinears under the parity transformation. Consider, for instance, $\overline{\Psi'(x')}\Psi'(x')$. Using EQ. (II.167), we find that

$$\overline{\Psi'(x')}\,\Psi'(x') = \overline{\gamma^0\Psi(x)}\,\gamma^0\Psi(x) = \left[\gamma^0\Psi(x)\right]^\dagger \gamma^0\gamma^0\Psi(x) = \left[\Psi(x)^\dagger\gamma^0\right]\Psi(x) = \overline{\Psi(x)}\,\Psi(x),$$

so that $\overline{\Psi(x)}\Psi(x)$ transforms like a scalar under parity. On the other hand, the bilinear $\overline{\Psi'(x')}\,\gamma^5\,\Psi'(x')$ follows

$$\overline{\Psi'(x')}\,\gamma^5\,\Psi'(x') = \overline{\gamma^0\Psi(x)}\;\gamma^5\gamma^0\Psi(x) = \left[\gamma^0\Psi(x)\right]^\dagger\gamma^0\gamma^5\gamma^0\Psi(x) = -\left[\Psi(x)^\dagger\gamma^0\right]\gamma^5\,\Psi(x) = -\overline{\Psi(x)}\,\gamma^5\,\Psi(x).$$

Therefore $\overline{\Psi(x)}\,\gamma^5\,\Psi(x)$ transforms like a pseudo-scalar.

The concept of charge conjugation is important for considering interaction of Fermions with electromagnetic fields and for antimatter. Taking the complex conjugate of EQ. (II.166), the Dirac equation for QED, we have

$$\left[-i\,\gamma^{\mu *}\left(\partial_\mu + ieA_\mu\right) - m\right]\Psi^* = 0. \tag{II.168}$$

From EQ. (II.123) we note that $-\gamma^{\mu *}$ also satisfy the Clifford algebra. Therefore, there exists a matrix $C\gamma^0$ such that

$$-\gamma^{\mu *} = \left(C\gamma^0\right)^{-1}\gamma^\mu\left(C\gamma^0\right). \tag{II.169}$$

Inserting EQ. (II.169) to EQ. (II.168), we obtain

$$\left[-i\,\gamma^{\mu *}\left(\partial_\mu + ieA_\mu\right) - m\right]\Psi^* = \left(C\gamma^0\right)^{-1}\left[i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right]\left(C\gamma^0\right)\Psi^* \equiv \left(C\gamma^0\right)^{-1}\left[i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right]\Psi_c = 0$$

The charge conjugation is therefore defined through the expression

$$\Psi_c \equiv C\gamma^0\Psi^*, \tag{II.170}$$

where C denotes the charge conjugation matrix, Ψ_c is the field of a particle with a charge opposite to the field Ψ, and Ψ_c satisfies the Dirac equation:

$$\left[i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right]\Psi_c = 0. \tag{II.171}$$

To gain more insights into the charge conjugation matrix C, we rewrite EQ. (II.169) such that

$$\left(C\gamma^0\right)\gamma^{\mu *}\left(C\gamma^0\right)^{-1} = C\gamma^0\gamma^{\mu *}\gamma^0C^{-1} = \left(-\gamma^\mu\right)\left(C\gamma^0\right)\left(C\gamma^0\right)^{-1} = \left(-\gamma^\mu\right). \tag{II.172}$$

Recalling the expression for the hermitean of γ^μ and using EQ. (II.172), we have

$$\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0 = \left(\gamma^\mu\right)^{T*} = \left(\gamma^0\gamma^{\mu *}\gamma^0\right)^* = \left(-C^{-1}\gamma^\mu C\right)^*,$$

so that

$$\left(\gamma^\mu\right)^T = -C^{-1}\gamma^\mu C. \tag{II.173}$$

Also, from EQ. (II.169), we find that

$$\begin{aligned}&\left(C\gamma^0\right)\left(-\gamma^{2*}\right) = \left(C\gamma^0\right)\left(\gamma^2\right) = \gamma^2\left(C\gamma^0\right),\\ &\left(C\gamma^0\right)\left(-\gamma^{\mu *}\right) = \left(C\gamma^0\right)\left(-\gamma^\mu\right) = \gamma^\mu\left(C\gamma^0\right) \qquad \text{for } \mu \neq 2.\end{aligned} \tag{II.174}$$

Hence, $C\gamma^0$ commutes with γ^2 and anti-commutes with other three γ-matrices. Therefore we have

$$C = \eta\,\gamma^2\gamma^0\,, \tag{II.175}$$

where η is a constant. Moreover, from EQs. (II.170) and (II.175), we find a simple relation

$$\Psi_c \equiv \gamma^2\Psi^*\,. \tag{II.176}$$

We further note that Ψ_c also transforms like a spinor, which can be proven as follows: The spinor Ψ under Lorentz transformation is given by

$$\Psi \to \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\Psi\,, \qquad \text{so that} \qquad \Psi^* \to \exp\left(\frac{i}{4}\omega_{\mu\nu}\left(\sigma^{\mu\nu}\right)^*\right)\Psi^*\,.$$

Therefore we find:

$$\Psi_c = \gamma^2\Psi^* \to \gamma^2\exp\left(\frac{i}{4}\omega_{\mu\nu}\left(\sigma^{\mu\nu}\right)^*\right)\Psi^* = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\gamma^2\Psi^* = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\Psi_c\,,$$

where we have used the fact that

$$\gamma^2\left(\sigma^{\mu\nu}\right)^* = -\frac{i}{2}\gamma^2\left[\gamma^\mu,\gamma^\nu\right] = -\frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]\gamma^2 = -\sigma^{\mu\nu}\gamma^2\,.$$

The concept of time reversal readily appears in non-relativistic quantum physics problems. As we shall discuss later in Part II.6, in the non-relativistic limit, the Klein-Gordon relation reduces to the Schrödinger's equation. For simplicity, let's consider time reversal transformation for a simple one-particle non-relativistic quantum physics problem. That is, we take the Schrödinger's equation

$$i\frac{\partial}{\partial t}\Psi(t) = \mathcal{H}\Psi(t) \qquad \text{where} \quad \mathcal{H} = -\frac{1}{2m}\nabla^2 + \mathcal{V}(\boldsymbol{x})\,,$$

and we have suppressed the dependence of Ψ on $\boldsymbol{x}$ for simplicity. Now consider the transformation $t \to t' = -t$. Our objective is to find $\Psi'(t') \equiv \mathcal{T}\Psi(t)$ such that $i\left(\partial/\partial t'\right)\Psi'(t') = \mathcal{H}\Psi'(t')$, and $\mathcal{T}$ is an operator to be determined. Thus,

$$i\frac{\partial}{\partial(-t)}\mathcal{T}\Psi(t) = \mathcal{H}\mathcal{T}\Psi(t)\,,$$

and

$$\mathcal{T}^{-1}(-i)\frac{\partial}{\partial t}\mathcal{T}\Psi(t) = \mathcal{T}^{-1}(-i)\mathcal{T}\frac{\partial}{\partial t}\Psi(t) = \mathcal{T}^{-1}\mathcal{H}\mathcal{T}\Psi(t) = \mathcal{H}\mathcal{T}^{-1}\mathcal{T}\Psi(t) = \mathcal{H}\Psi(t)\,,$$

because $\mathcal{H}$ does not contain any time dependence so commute with $\mathcal{T}$ and $\mathcal{T}^{-1}$. We therefore conclude that

$$\mathcal{T}^{-1}(-i)\mathcal{T} = i\,. \tag{II.177}$$

The sign change in EQ. (II.177) is suggestive of the nature of complex conjugate on in the time reversal operation. We can therefore define $\mathcal{T} = UK$, where K is an operator that complex conjugates everything to its right, and U is an operator yet to be determined, we find that $\mathcal{T}^{-1} = KU^{-1}$, and that $U^{-1}iU = i$ from EQ. (II.177), provided that the operator U is an ordinary unitary operator that does nothing to i. The presence of K in the operator $\mathcal{T}$ is said to make the time reversal operator "anti-unitary".

Let's first consider applying $\mathcal{T}$ to a spinless particle in a plane wave state $\Psi(t) = \exp\left[i\left(\boldsymbol{k}\bullet\boldsymbol{x} - Et\right)\right]$. We find that $\Psi'(t') = \mathcal{T}\Psi(t) = UK\Psi(t) = U\exp\left[-i\left(\boldsymbol{k}\bullet\boldsymbol{x} - Et\right)\right]$, or equivalently $\Psi'(t) = U\exp\left[-i\left(\boldsymbol{k}\bullet\boldsymbol{x} + Et\right)\right]$. Therefore $\Psi'(t)$ describes a plane wave moving in the opposite direction. Also,

$$\mathcal{T}^2 = UKUK = UU^*K^2 = +1 . \tag{II.178}$$

holds for spinless particles. What about applying $\mathcal{T}$ to a spin-1/2 particle?

For a non-relativistic spin-1/2 particle, we want to obtain a spin-down state by acting $\mathcal{T}$ on the spin-up state, and vise versa. That is,

$$\mathcal{T}\begin{pmatrix}1\\0\end{pmatrix} = UK\begin{pmatrix}1\\0\end{pmatrix} = U\begin{pmatrix}1\\0\end{pmatrix} = i\eta\begin{pmatrix}0\\1\end{pmatrix},$$

where η is a constant. The above can be accomplished by taking $U = \eta\sigma_2$. You can easily verify that the operation of $\mathcal{T}$ on the spin down state also yields the spin up state. Furthermore, we find that

$$\mathcal{T}^2 = \eta\sigma_2 K\eta\sigma_2 K = \eta\eta^*\sigma_2\sigma_2{}^*K^2 = -1 , \tag{II.179}$$

where we have chosen $\eta\eta^* = 1$.

The result in EQ. (II.179) for spin-1/2 particles contrasts that in EQ. (II.178) for spinless particles, which also has an important implication, known as the Kramer's degeneracy. That is, for a system with an odd number of electrons in an electric field, there is a fundamental degree of two-fold degeneracy associated with each energy level. The proof for the Kramer's degeneracy is as follows. If there were no Kramer's degeneracy, the states Ψ and $\mathcal{T}\Psi$ would have the same eigen-energy and would have only differ by at most a phase. That is, $\mathcal{T}\Psi = \exp(i\alpha)\Psi$. However, this assumption leads to $\mathcal{T}^2\Psi = \mathcal{T}\exp(i\alpha)\Psi = \exp(-i\alpha)\,\mathcal{T}\Psi = \Psi$, which contradicts to the assertion in EQ. (II.179) that $\mathcal{T}^2\Psi = -\Psi$. Therefore Ψ and $\mathcal{T}\Psi$ must represent two distinct states.

Next, we want to understand how the time-reversal operator affects the Dirac equation. Combining the Dirac equation with the Schrödinger's equation, we obtain

$$i\left(\gamma^0\partial_0 - \gamma^i\partial_i - m\right)\Psi = \left[\gamma^0\mathcal{H} - i\gamma^i\partial_i - m\right]\Psi = \left[\mathcal{H} - i\gamma^0\left(\gamma^i\partial_i\right) - \gamma^0 m\right]\Psi = 0 ,$$

so that the Hamiltonian for a spin ½ particle becomes:

$$\mathcal{H} = i\gamma^0\gamma^i\partial_i + \gamma^0 m . \tag{II.180}$$

Our objective is to find the corresponding time-reversal operator $\mathcal{T}$ (= UK) such that $\Psi'(t') \equiv \mathcal{T}\Psi(t)$ and $i\left(\partial/\partial t'\right)\Psi'(t') = \mathcal{H}\Psi'(t')$. Following similar steps as before, we find $\mathcal{T}^{-1}\mathcal{H}\,\mathcal{T} = KU^{-1}\mathcal{H}\,UK = \mathcal{H}$, which, when compared with EQ. (II.180), yields two relations:

$$KU^{-1}\left(i\gamma^0\gamma^i\right)UK = i\gamma^0\gamma^i \ , \tag{II.181}$$

$$KU^{-1}\gamma^0\,UK = \gamma^0 \ . \tag{II.182}$$

Multiplying by K on the left and on the right of EQs. (II.181) and (II.182), we have

$$U^{-1}\gamma^0\,U = \gamma^{0*} \ , \tag{II.183}$$

and $U^{-1}\left(\gamma^0\gamma^i\right)U = -\gamma^{0*}\gamma^{i*} = -U^{-1}\gamma^0 U\gamma^{i*}$, so that

$$U^{-1}\gamma^i U = -\gamma^{i*} \; . \tag{II.184}$$

In both the Dirac and Weyl's bases, only γ^2 is complex. Using EQs. (II.183) and (II.184), we find that γ^0 and γ^2 commute with U and that γ^1 and γ^3 anti-commute with U. We can therefore construct the matrix U such that

$$U = \eta\gamma^1\gamma^3 = \eta\left(\sigma^1 \otimes i\tau_2\right)\left(\sigma^3 \otimes i\tau_2\right) = \eta\, i\sigma^2 \otimes 1, \tag{II.185}$$

where η is an arbitrary phase factor. The time-reversal operator becomes:

$$\mathcal{T} = UK = \eta\gamma^1\gamma^3 K \; . \tag{II.186}$$

It is then easily verified that $\mathcal{T}^2\,\Psi = -\Psi$.

We summarize how the γ-matrices are related to the discrete symmetry operations of parity, charge conjugation and time-reversal in the following:

$$\begin{array}{lll} \text{Parity:} & \Psi'(x') = \eta\gamma^0\Psi(x)\,; & \mathcal{P} = \eta\gamma^0 . \\ \text{Charge conjugation:} & \Psi_c(x) \equiv \gamma^2\Psi^*(x)\,; & C = \eta\gamma^2\gamma^0 . \\ \text{Time reversal:} & \Psi'(t') = \eta\gamma^1\gamma^3 K\,\Psi(t)\,; & \mathcal{T} = \eta\gamma^1\gamma^3 K . \end{array} \tag{II.187}$$

A couple of remarks on the discrete symmetries of parity ($\mathcal{P}$), charge conjugation (C) and time reversal ($\mathcal{T}$): In nature, these discrete symmetries are often violated. For instance, parity is maximally violated by the weak interactions, and the combination of $C\mathcal{P}$ is violated in *K*-meson decays. However, there is a remarkable theorem that states that any quantum field theory can be invariant under the combined operation of $C\mathcal{P}\mathcal{T}$:

$$\left(C\mathcal{P}\mathcal{T}\right)\mathcal{H}(x)\left(C\mathcal{P}\mathcal{T}\right)^{-1} = \mathcal{H}(x'), \tag{II.188}$$

if the following two conditions are satisfied:

1. The theory must be local, possess a hermitian Lagrangian, and be invariant under Lorentz transformation.
2. The theory must be quantized with commutators for integer spin fields and quantized with anti-commutators for half-integer spin fields.

We shall not go into various theoretical proofs for the *CPT* theorem other than emphasizing that the spin 0, ½, and 1 fields that we have investigated so far in fact obey the *CPT* theorem.

[Quantization of the Dirac field]

Now that we become familiar with properties of the Dirac spinor field and the γ-matrices, we can proceed to quantize the Dirac field by applying the canonical formalism to the Dirac Lagrangian. We'll first introduce the creation and annihilation operators of Fermions, similar to what we have done in Part II.1 for the bosons. An important difference here is that the creation and annihilation operators of Fermions satisfy anti-commutation rather than commutation relations as in bosons.

We begin with defining a vacuum state $|0\rangle$ where no electron exists. The creation of an electron with the quantum number α is given by $b_\alpha^\dagger\left|0\right\rangle$ with $b_\alpha^\dagger$ being the creation operator. If we want to introduce another electron with the quantum number β, we have to construct a state $b_\beta^\dagger b_\alpha^\dagger |0\rangle$. Noting that electron wavefunctions are anti-symmetric upon interchanging α and β, the operators must satisfy the condition:

$$\left\{b_\alpha^\dagger, b_\beta^\dagger\right\} \equiv b_\alpha^\dagger b_\beta^\dagger + b_\beta^\dagger b_\alpha^\dagger = 0 . \tag{II.189}$$

Upon taking the hermitean conjugation on EQ. (II.189), we have

$$\left\{b_\alpha, b_\beta\right\} \equiv b_\alpha b_\beta + b_\beta b_\alpha = 0 . \tag{II.190}$$

Both relations indicate no double occupancy of the same state. In addition, the operators satisfy

$$\left\{b_\alpha, b_\beta^\dagger\right\} \equiv b_\alpha b_\beta^\dagger + b_\beta^\dagger b_\alpha = \delta_{\alpha\beta} . \tag{II.191}$$

The number operator for the electrons is given $N = \sum_\alpha b_\alpha^\dagger b_\alpha$.

Recall that the free Dirac Lagrangian takes the form:

$$\mathcal{L} = \overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi .$$

The momentum conjugation to Ψ is:

$$\pi_\alpha = \delta\mathcal{L}/\delta\left(\partial_t \Psi_\alpha\right) = \overline{\Psi}_\alpha\left(i\gamma^0\right) = \left(\Psi_\alpha^\dagger \gamma^0\right)\left(i\gamma^0\right) = i\Psi_\alpha^\dagger . \tag{II.192}$$

Given these conditions, we anticipate that the correct canonical procedure leads to the following anti-commutation relation:

$$\left\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta^\dagger(0,t)\right\} = \delta^{(3)}(\mathbf{x})\,\delta_{\alpha\beta} . \tag{II.193}$$

We shall derive EQ. (II.193) in the following.

The Dirac field satisfies $\left(i\partial\!\!\!/ - m\right)\Psi = 0$, which can take on plane-wave solutions $u(p,s)e^{-ipx}$ and $v(p,s)e^{ipx}$ for Ψ, with $u(p,s)$ and $v(p,s)$ satisfying the following two equations:

$$(p\!\!\!/ - m)\,u(p,s) = 0 , \tag{II.194}$$

$$(p\!\!\!/ + m)\,v(p,s) = 0 . \tag{II.195}$$

The index $s = \pm 1$ indicates that each of the two equations has two solutions, one with spin up and the other with spin down. We further note that the two spinors u and v transform in the same way as the Dirac spinor Ψ under a Lorentz transformation. Thus, we can define

$$\overline{u} \equiv u^\dagger \gamma^0 \quad \text{and} \quad \overline{v} \equiv v^\dagger \gamma^0 , \tag{II.196}$$

so that $\overline{u}u$ and $\overline{v}v$ are Lorentz scalars. In this context, we can normalize the u and v in the rest frame, and the normalization condition thus imposed will hold in any frame because $\overline{u}u$ and $\overline{v}v$ are Lorentz scalars.

In the rest frame the two equations for u and v in EQs. (II.194) and (II.195) can be simplified into

$$(\gamma^0 - 1)\,u(p,s) = 0 , \tag{II.197}$$

$$(\gamma^0 + 1)\,v(p,s) = 0 . \tag{II.198}$$

If we take the Dirac basis, where $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, the independent spinors *u* and *v* acquire the following forms

$$u: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \qquad v: \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{II.199}$$

In EQ. (II.199) we have implicitly chosen the renormalization conditions $\bar{u}u = 1$ and $\bar{v}v = -1$. In addition, we have the orthogonality condition $\bar{u}v = 0$ and $\bar{v}u = 0$. These relations should hold in general because of Lorentz invariance and basis independence.

Specifically, in the rest frame we can derive the following conditions using EQ. (II.199) and that $u\bar{u} = uu^\dagger\gamma^0$ and $v\bar{v} = vv^\dagger\gamma^0$:

$$\begin{aligned} \sum_{\pm s} u_\alpha(p,s)\bar{u}_\beta(p,s) &= \sum_{\pm s} u_\alpha(p,s)u_\beta(p,s)^\dagger\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} = \frac{1}{2}\left(\gamma^0+1\right)_{\alpha\beta}, \\ \sum_{\pm s} v_\alpha(p,s)\bar{v}_\beta(p,s) &= \sum_{\pm s} v_\alpha(p,s)v_\beta(p,s)^\dagger\gamma^0 = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix}_{\alpha\beta} = \frac{1}{2}\left(\gamma^0-1\right)_{\alpha\beta}. \end{aligned} \tag{II.200}$$

To generalize the results in EQ. (II.200) to any frame, we apply a Lorentz boost $\boldsymbol{K}$ to the spinors, so that $u \to \exp\left(-i\boldsymbol{\theta}\boldsymbol{K}\right)u$, $v \to \exp\left(-i\boldsymbol{\theta}\boldsymbol{K}\right)v$, and $\left(\gamma^\mu p_\mu/m\right) = \left(\not{p}/m\right) = e^{-i\boldsymbol{\theta}\boldsymbol{K}}\gamma^0 e^{i\boldsymbol{\theta}\boldsymbol{K}}$. Thus, we obtain

$$\begin{aligned} \sum_{\pm s} u(p,s)\bar{u}(p,s) &= \frac{1}{2}\left(\frac{\not{p}}{m}+1\right)_{\alpha\beta}, \\ \sum_{\pm s} v(p,s)\bar{v}(p,s) &= \frac{1}{2}\left(\frac{\not{p}}{m}-1\right)_{\alpha\beta}. \end{aligned} \tag{II.201}$$

In Problem Set 2, you'll be asked to find the explicit expressions for *u* and *v* under a Lorentz boost.

Now we can promote the Dirac spinor $\Psi(x)$ to an operator using $u(p,s)$ and $v(p,s)$, similar to what has been done in EQ. (II.13) of Part II.1:

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}(E_p/m)^{1/2}} \sum_s \left[b(p,s)u(p,s)e^{-ipx} + d^\dagger(p,s)v(p,s)e^{ipx} \right], \tag{II.202}$$

where $E_p = p_0 = \sqrt{\boldsymbol{p}^2+m^2}$, $px = p_\mu x^\mu$, $b^\dagger(p,s)$ and $b(p,s)$ are the creation and annihilation operators for an electron of momentum *p* and spin *s*, whereas $d^\dagger(p,s)$ and $d(p,s)$ are the creation and annihilation operators for a positron of momentum *p* and spin *s*. These operators satisfy the following relations:

$$\begin{aligned} &\left\{b(p,s), b^\dagger(p',s')\right\} = \delta^{(3)}\left(\vec{p}-\vec{p}\,'\right)\delta_{ss'} && \left\{d(p,s), d^\dagger(p',s')\right\} = \delta^{(3)}\left(\vec{p}-\vec{p}\,'\right)\delta_{ss'}, \\ &\left\{b^\dagger(p,s), b^\dagger(p',s')\right\} = 0 && \left\{d^\dagger(p,s), d^\dagger(p',s')\right\} = 0, \\ &\left\{b(p,s), b(p',s')\right\} = 0 && \left\{d(p,s), d(p',s')\right\} = 0. \end{aligned} \tag{II.203}$$

We can obtain the anti-commutation relation in EQ. (II.193) through the following expressions:

$$\bar{\Psi}(0)=\int\frac{d^3p'}{(2\pi)^{3/2}(E_{p'}/m)^{1/2}}\sum_{s'}\left[b^\dagger(p',s')\bar{u}(p',s')+d(p',s')\bar{v}(p',s')\right],$$

$$\Psi(\boldsymbol{x},0)=\int\frac{d^3p}{(2\pi)^{3/2}(E_p/m)^{1/2}}\sum_{s}\left[b(p,s)u(p,s)e^{-ip\cdot x}+d^\dagger(p,s)v(p,s)e^{ip\cdot x}\right], \tag{II.204}$$

so that from EQs. (II.203) and (II.204), we have

$$\left\{\Psi(\boldsymbol{x},0),\bar{\Psi}(0)\right\}=\int\frac{d^3p}{(2\pi)^3(E_p/m)}\sum_{s}\left[u(p,s)\bar{u}(p,s)e^{-ip\cdot x}+v(p,s)\bar{v}(p,s)e^{ip\cdot x}\right]. \tag{II.205}$$

Inserting EQ. (II.201) to EQ. (II.205), we obtain

$$\left\{\Psi(\boldsymbol{x},0),\bar{\Psi}(0)\right\}=\int\frac{d^3p}{(2\pi)^3(2E_p)}\left[\left(\not{p}+m\right)e^{-ip\cdot x}+\left(\not{p}-m\right)e^{ip\cdot x}\right], \tag{II.206}$$

which can be further simplified by noting that the terms involving m inside the integrand of EQ. (II.205) vanish, so do the terms involving $\gamma^i p_i$. Therefore EQ. (II.206) becomes (under the implicit condition $s=s'$)

$$\left\{\Psi(\boldsymbol{x},0),\bar{\Psi}(0)\right\}=\int\frac{d^3p}{(2\pi)^3(2E_p)}\left[\gamma^0p_0e^{-ip\cdot x}+\gamma^0p_0e^{ip\cdot x}\right]=\int\frac{d^3p}{(2\pi)^3(2E_p)}\left(2\gamma^0p_0e^{-ip\cdot x}\right)=\gamma^0\delta^{(3)}(\boldsymbol{x}), \tag{II.207}$$

or, equivalently,

$$\left\{\Psi_\alpha(x,0),\Psi^\dagger_\beta(0)\right\}=\delta^{(3)}(\boldsymbol{x})\,\delta_{\alpha\beta},$$

which is indeed identical to EQ. (II.193), with α and β referring to the spin states. Similarly, it can be shown that

$$\left\{\Psi^\dagger,\Psi^\dagger\right\}=\left\{\Psi,\Psi\right\}=0.$$

Having quantized the Dirac field, we can now calculate the Dirac Hamiltonian ($\mathcal{H}$). We start with the Hamiltonian density (H):

$$\begin{aligned}H&=\pi\frac{\partial\Psi}{\partial t}-L=\left(i\Psi^\dagger\right)\partial_0\Psi-\bar{\Psi}\left(i\gamma^\mu\partial_\mu-m\right)\Psi=\left(i\bar{\Psi}\right)\gamma^0\partial_0\Psi-\bar{\Psi}\left(i\gamma^\mu\partial_\mu-m\right)\Psi\\&=\bar{\Psi}\left(i\vec{\gamma}\bullet\vec{\partial}+m\right)\Psi.\end{aligned} \tag{II.208}$$

Inserting the explicit operator expression in EQ. (II.202) for the Dirac field into EQ. (II.208), the Hamiltonian can be calculated by performing the volume integration:

$$\begin{aligned}\mathcal{H}&=\int d^3x\,H=\int d^3x\,\bar{\Psi}\left(i\gamma^i\partial_i+m\right)\Psi=\int d^3x\,\bar{\Psi}\left(i\gamma^0\partial_0\right)\Psi=\int d^3x\left(i\Psi^\dagger\right)\frac{\partial}{\partial t}\Psi\\&=\int d^3p\sum_s E_p\left[b^\dagger(p,s)b(p,s)-d(p,s)d^\dagger(p,s)\right]\\&=\left\{\int d^3p\sum_s E_p\left[b^\dagger(p,s)b(p,s)+d^\dagger(p,s)d(p,s)\right]\right\}-\left\{\delta^{(3)}(\vec{0})\int d^3p\sum_s E_p\right\}\\&\equiv\left\{\int d^3p\sum_s E_p\left[b^\dagger(p,s)b(p,s)+d^\dagger(p,s)d(p,s)\right]\right\}+E_0,\end{aligned} \tag{II.209}$$

where we have defined the vacuum energy E_0 as:

$$E_0 \equiv -\left\{\delta^{(3)}(\vec{0})\int d^3p\sum_s E_p\right\} = -\frac{1}{(2\pi)^3}\int d^3x\int d^3p\sum_s E_p \ . \tag{III.210}$$

Equation (II.210) effectively describes an energy (– ½ E_p) in each unit cell phase space for each spin. This infinite additive term is the analog of the zero-point energy of the harmonic oscillator, except for the *negative* sign! The peculiar negative energy associated with the fermions in contrast to the positive zero-point energy associated with the bosons is in fact of crucial importance to the development of supersymmetry that attempts to unify bosons and fermions.

Finally, we want to obtain the fermion propagator through space-time, which will be crucial for our consideration of the Feynman diagrams involving fermions. In analogy to our discussions of bosonic propagators in Part II.1, EQ. (II.16), the propagator for the fermion can be given via the time-ordering operator as:

$$i\mathcal{D}_{\alpha\beta}(x) \equiv \left\langle 0\left|\hat{T}\left[\Psi_\alpha(x)\overline{\Psi}_\beta(0)\right]\right|0\right\rangle , \tag{II.211}$$

where the argument of $\overline{\Psi}_\beta$ can be set to 0 because of translational invariance. We shall show that the anti-commutating character of the spinor field requires the time-ordered product be defined with a minus sign. That is,

$$T\Psi(x)\overline{\Psi}(0) \equiv \theta(x^0)\Psi(x)\overline{\Psi}(0) - \theta(-x^0)\overline{\Psi}(0)\Psi(x) \ . \tag{II.212}$$

Using EQs. (II.201) and (II.204), we find that for $x^0 > 0$,

$$i\mathcal{D}_{\alpha\beta}(x) \equiv \left\langle 0\left|\Psi_\alpha(x)\overline{\Psi}_\beta(0)\right|0\right\rangle = \int\frac{d^3p}{(2\pi)^3(E_p/m)}\sum_s u_\alpha(p,s)\overline{u}_\beta(p,s)e^{-ipx} = \int\frac{d^3p}{(2\pi)^3(E_p/m)}\left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta}e^{-ipx} ,$$

whereas for $x^0 < 0$, we have

$$i\mathcal{D}_{\alpha\beta}(x) \equiv -\left\langle 0\left|\overline{\Psi}_\beta(0)\Psi_\alpha(x)\right|0\right\rangle = -\int\frac{d^3p}{(2\pi)^3(E_p/m)}\sum_s \overline{v}_\beta(p,s)v_\alpha(p,s)e^{ipx} = -\int\frac{d^3p}{(2\pi)^3(E_p/m)}\left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta}e^{ipx} .$$

Therefore we find

$$i\mathcal{D}(x) = \int\frac{d^3p}{(2\pi)^3(E_p/m)}\left[\theta(x^0)\left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta}e^{-ipx} - \theta(-x^0)\left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta}e^{+ipx}\right]. \tag{II.213}$$

In fact, the above result can be more elegantly expressed as a 4-dimensional integral as follows:

$$i\mathcal{D}(x) = \int\frac{d^4p}{(2\pi)^4}e^{-ipx}\left(\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\right) = \int\frac{d^4p}{(2\pi)^4}e^{-ipx}\left(\frac{i}{\not{p}-m+i\varepsilon}\right). \tag{II.214}$$

To prove that EQ. (II.214) is indeed equivalent to EQ. (II.213), we first perform integration over p^0. Apparently the complex plane of p^0 has poles at $p^0 = \pm\sqrt{\boldsymbol{p}^2+m^2-i\varepsilon} \simeq \pm\left(E_p - i\varepsilon\right)$. For $x^0 > 0$, we need to go over the pole $+\left(E_p - i\varepsilon\right)$ at the lower plane, which yields the following:

$$i\mathcal{D}(x)=\int\frac{d^3p}{(2\pi)^3}e^{-ipx}\left(\frac{\not{p}+m}{2E_p}\right)=\int\frac{d^3p}{(2\pi)^3}e^{-iE_px^0+i\boldsymbol{p}\cdot\boldsymbol{x}}\left(\frac{\gamma^0E_p-\boldsymbol{\gamma}\,\boldsymbol{p}+m}{2E_p}\right); \qquad \text{(II.215)}$$

whereas for $x^0 < 0$, we need to go over the pole $-\left(E_p - i\varepsilon\right)$ at the upper plane, which yields

$$i\mathcal{D}(x)=-\int\frac{d^3p}{(2\pi)^3}e^{-ipx}\left(\frac{\not{p}-m}{-2E_p}\right)=-\int\frac{d^3p}{(2\pi)^3}e^{iE_px^0+i\boldsymbol{p}\cdot\boldsymbol{x}}\left(\frac{-\gamma^0E_p-\boldsymbol{\gamma}\,\boldsymbol{p}-m}{-2E_p}\right). \qquad \text{(II.216)}$$

We can invert $\vec{p}$ so that EQ. (II.216) becomes

$$i\mathcal{D}(x)=-\int\frac{d^3p}{(2\pi)^3}e^{ipx}\left(\frac{\not{p}-m}{2E_p}\right). \qquad \text{(II.217)}$$

Hence, the combination of EQ. (II.215) and EQ. (II.217) is indeed equal to EQ. (II.213).

The above derivation concludes that in momentum space the fermion propagator has an elegant form

$$i\mathcal{D}(p)=\left(\frac{i}{\not{p}-m+i\varepsilon}\right), \qquad \text{(II.218)}$$

which is to be compared with the scalar boson propagator $D(k) = 1/(k^2 - m^2 + i\varepsilon)$ discussed earlier. The expression for the fermion propagator essential for our subsequent relativistic discussion of the Feynman diagrams involving fermions.

II.5. Grassmann Integrals and Feynman Diagrams for Fermions

Having derived the Feynman propagator for spin ½ particles, we are one step closer to dealing with the path integrals of fermions. However, the anti-commutating character of fermions in fact imposes strange properties on the path integral. A new set of mathematics, known as the Grassmann mathematics, must be introduced to deal with the sign of the fermions.

[Grassmann mathematics]

As we have seen previously, the path integral that concerns us generally involves the Gaussian integration formula

$$\int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 ... \int_{-\infty}^{+\infty} dx_N \ \exp\left[-\frac{1}{2}(x \bullet A \bullet x)\right] = \left(\frac{(2\pi)^N}{\det[A]}\right)^{1/2} = C \exp\left(-\frac{1}{2}\mathrm{Tr}\left\{\log A\right\}\right), \tag{II.219}$$

where A is a $N \times N$ matrix, and we have replace the numerical factor with a constant C and have also used the following identity:

$$\det[A] = \exp\left(\mathrm{Tr}\left\{\log A\right\}\right). \tag{II.220}$$

For simplicity, we first consider the case that the matrix A is replaced by a constant a, and ask how we should develop the mathematics for a Gaussian integral that involves anti-commutating numbers rather than ordinary numbers. In fact, the mathematics has been developed by Grassmann, and these strange anti-commutating numbers are known as the Grassmann numbers. Let's denote η and ξ as Grassmann numbers, so that $\eta\xi = -\xi\eta$. In particular, we have $\eta^2 = 0$. If we assume that any well-behaved function can be expanded in a Taylor series, the most general function of η would be $f(\eta) = a + b\eta$ (with a and b being ordinary numbers) because $\eta^2 = 0$.

Next we consider the integration over η. Noting that in ordinary integrals we can shift the dummy integration variable so that $\int_{-\infty}^{+\infty} dx\, f(x+c) = \int_{-\infty}^{+\infty} dx\, f(x)$, we can also insist that the Grassmann integral obeys the same rule. Hence, $\int d\eta\, f(\eta+\xi) = \int d\eta\, f(\eta)$, where ξ is any arbitrary Grassmann number, and we obtain $\int d\eta\,(b\xi) = 0$. Since ξ is arbitrary, we must have $\int d\eta\, b = 0$ for any ordinary number b. Specifically, we can have $b = 1$ and therefore $\int d\eta \equiv \int d\eta\, 1 = 0$.

Given three Grassmann numbers χ, η, and ξ, we find that $\chi(\eta\xi) = -\eta\chi\xi = (\eta\xi)\chi$. In other words, the product of two Grassmann numbers commutes with any Grassmann number χ; or, more generally, any product of two anti-commuting numbers is an ordinary number. Therefore the integral $\int d\eta\,\eta$ must be an ordinary number, which we can normalize to 1. That is, the expression $\int d\eta\,\eta = 1$ defines the normalization of $d\eta$. Thus, we have two simple rules that define Grassmann integration:

$$\int d\eta = 0, \tag{II.221}$$

and

$$\int d\eta\,\eta = 1. \tag{II.222}$$

Now we can integrate any function of η as follows:

$$\int d\eta\, f(\eta) = \int d\eta\left(a + b\eta\right) = b \tag{II.223}$$

if b is an ordinary number so that $f(\eta)$ is Grassmannian, and

$$\int d\eta\, f(\eta) = \int d\eta \left(a + b\eta\right) = -b \tag{II.224}$$

if b is Grassmannian so that $f(\eta)$ is an ordinary number. It should be emphasized that the concept of a range of integration does not exist in Grassmann mathematics, so that it is in fact much easier to master Grassmann integration than ordinary integration.

Next, we consider a double integration involving two independent Grassmann numbers η, $\overline{\eta}$, and an ordinary number a. The Grassmannian analog of the Gaussian integral is therefore given by:

$$\int d\eta \int d\overline{\eta}\; e^{\overline{\eta} a\eta} = \int d\eta \int d\overline{\eta}\; (1 + \overline{\eta} a\eta) = \int d\eta\; (a\eta) = a = e^{+\log a}\,. \tag{II.225}$$

This expression can be easily generalized. We can define $\eta = (\eta_1, \eta_2, ..., \eta_N)$ by N Grassmann numbers, and similarly for $\overline{\eta}$. Thus, we find

$$\int d\eta \int d\overline{\eta}\; e^{\overline{\eta} A\eta} = \det[A] \tag{II.226}$$

for A being an antisymmetric $N \times N$ matrix. It should be noted that the inverse of A need not exist, contrary to the bosonic case that we have dealt with before.

Now we have all the mathematical tools to perform the Grassmann path integral. In analogy with the following generating functional for the scalar field:

$$Z = \int D\varphi\, e^{iS(\varphi)} = \int D\varphi \exp\left\{ i\int d^4x\, \frac{1}{2}\left[\left(\partial\varphi\right)^2 - \left(m^2 - i\varepsilon\right)\varphi^2 \right] \right\},$$

we can write the generating functional for the spinor field as:

$$Z = \int D\psi \int D\overline{\psi}\; e^{iS(\psi,\overline{\psi})} = \int D\psi \int D\overline{\psi}\; \exp\left\{ i\int d^4x\, \overline{\psi}\left(i\partial\!\!\!/ - m + i\varepsilon\right)\psi \right\}. \tag{II.227}$$

By treating both integration variables ψ and $\overline{\psi}$ as Grassmann-valued Dirac spinors, we obtain

$$Z = \int D\psi \int D\overline{\psi}\; \exp\left\{ i\int d^4x\, \overline{\psi}\left(i\partial\!\!\!/ - m + i\varepsilon\right)\psi \right\} = C' \det\left[i\partial\!\!\!/ - m + i\varepsilon \right] = C' e^{\mathrm{Tr}\left\{\log\left(i\partial\!\!\!/ - m + i\varepsilon\right)\right\}}, \tag{II.228}$$

where C' is some multiplicative constant. We also note that

$$\begin{aligned}\mathrm{Tr}\left\{\log\left(i\partial\!\!\!/ - m\right)\right\} &= \mathrm{Tr}\left\{\log \gamma^5\left(i\partial\!\!\!/ - m\right)\gamma^5\right\} = \mathrm{Tr}\left\{\log\left(-i\partial\!\!\!/ - m\right)\right\} = \frac{1}{2}\left[\mathrm{Tr}\left\{\log\left(i\partial\!\!\!/ - m\right)\right\} + \mathrm{Tr}\left\{\log\left(-i\partial\!\!\!/ - m\right)\right\} \right] \\ &= \frac{1}{2}\mathrm{Tr}\left\{\log\left(\partial^2 + m^2\right)\right\}.\end{aligned} \tag{II.229}$$

Consequently, we find that

$$Z = C' \exp\left[\frac{1}{2}\mathrm{Tr}\left\{\log\left(\partial^2 + m^2 - i\varepsilon\right)\right\} \right]. \tag{II.230}$$

It is important to note the opposite signs between the exponential part of EQ. (II.219) and that of EQ. (II.230). This implies opposite signs in the vacuum energy for bosons and fermions, as discussed previously using the canonical formalism. Clearly, had we not used Grassmann variables for ψ and $\bar{\psi}$, we would have obtained:

$$Z \sim \exp\left[-\mathrm{Tr}\left\{\log\left(i\partial\!\!\!/ - m\right)\right\}\right],$$

which would have given rise to the wrong sign for vacuum energy.

Let's now examine the introduction of Grassmannian spinor sources η and $\bar{\eta}$ to the Dirac field:

$$Z(\eta,\bar{\eta}) = \int D\psi \int D\bar{\psi}\, \exp\left\{i\int d^4x \left[\bar{\psi}\left(i\partial\!\!\!/ - m + i\varepsilon\right)\psi + \bar{\eta}\,\psi + \bar{\psi}\eta\right]\right\}. \tag{II.231}$$

If we denote $\left(i\partial\!\!\!/ - m + i\varepsilon\right)$ by $\mathcal{D}^{-1}$, we find that

$$\bar{\psi}\mathcal{D}^{-1}\psi + \bar{\eta}\,\psi + \bar{\psi}\eta = \left(\bar{\psi} + \bar{\eta}\,\mathcal{D}\right)\mathcal{D}^{-1}\left(\psi + \mathcal{D}\eta\right) - \bar{\eta}\,\mathcal{D}\eta,$$

and therefore

$$Z(\eta,\bar{\eta}) = C'\exp\left[\frac{1}{2}\mathrm{Tr}\left\{\log\left(\partial^2 + m^2 - i\varepsilon\right)\right\}\right]\exp\left\{-i\bar{\eta}\left(i\partial\!\!\!/ - m + i\varepsilon\right)^{-1}\eta\right\} \equiv C''\exp\left\{-i\bar{\eta}\left(i\partial\!\!\!/ - m + i\varepsilon\right)^{-1}\eta\right\}. \tag{II.232}$$

The fermion propagator $\mathcal{D}(x)$ for the Dirac field is associated with $\left(i\partial\!\!\!/ - m\right)^{-1}$, so that

$$\left(i\partial\!\!\!/ - m\right)\mathcal{D}(x) = \delta^{(4)}(x). \tag{II.233}$$

It is easily verifiable that the solution is

$$i\,\mathcal{D}(x) = \int\frac{d^4p}{(2\pi)^4}\left(\frac{i}{p\!\!\!/ - m + i\varepsilon}\right)e^{-ipx} \equiv \int\frac{d^4p}{(2\pi)^4}\,i\,\mathcal{D}(p)\,e^{-ipx}, \tag{II.234}$$

which is consistent with the result in EQ. (II.218) of Part II.4, and is clearly a much simpler derivation than the canonical formalism! You may also compare these results with Part II.2, EQs. (II.47) and (II.48) for bosons.

[Feynman rules for fermions]

With the knowledge of fermion propagators and Grassmann mathematics, we can now derive the Feynman rules for fermions. For instance, for a scalar field interacting with a Dirac field, we have the Yukawa theory:

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + \frac{1}{2}\left[(\partial\varphi)^2 - \mu^2\varphi^2\right] - \lambda\varphi^4 + f\varphi\bar{\psi}\psi, \tag{II.235}$$

where λ and f are coupling coefficients, and the corresponding generating functional becomes

$$Z(\eta,\bar{\eta},J) = \int D\psi\int D\bar{\psi}\int D\varphi\,\exp\left\{iS\left(\psi,\bar{\psi},\varphi\right)\right\}\,\exp\left\{i\int d^4x\left[J\varphi + \bar{\eta}\,\psi + \bar{\psi}\eta\right]\right\}, \tag{II.236}$$

which can be evaluated as a double series in the coupling coefficients λ and f. On the other hand, for a massive vector field interacting with the Dirac field, the Lagrangian becomes:

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu, \tag{II.237}$$

and the vector boson propagator must be explicitly included in the Feynman diagrams. Therefore we can list the Feynman rules for fermions (in addition to those already given for bosons) as follows:

1. Draw a diagram with straight lines for the fermions, dashed lines for the scalar bosons, and wiggled lines for the vector bosons. Label each fermion or scalar boson line with a momentum, and each vector boson line with two indices besides the momentum. (See Rule 7 for vector bosons).

2. Associate with each fermion line the fermion propagator

$$\left(\frac{i}{\not{p}-m+i\varepsilon}\right)=i\left(\frac{\not{p}+m}{\not{p}^2-m^2+i\varepsilon}\right).$$

3. For each interaction vertex with a scalar boson field, associate with the amplitude the coupling factor (*if*) and the factor $(2\pi)^4\delta^{(4)}\left(\sum_{in}p-\sum_{out}p\right)$ that expresses momentum conservation for the total incoming and outgoing momenta.

4. Momenta associated with each internal fermion line of momentum p is to be integrated over with the measure $\int\left[d^4p/(2\pi)^4\right]$.

5. External lines are to be amputated. Associate an incoming fermion line with the coefficient $u(p,s)$ and an outgoing fermion line with $\bar{u}(p',s)$. Similarly, for anti-fermions associate with an incoming line with $v(p,s)$ and an outgoing line with $\bar{v}(p',s)$. The spin polarization of the fermion being produced and absorbed at the sources and sinks must be recognized.

6. A factor of (–1) is to be associated with each closed fermion line.

7. For a massive vector field interacting with a Dirac field, the vector boson propagator is given by

$$\frac{i}{k^2-\mu^2}\left(\frac{k_\mu k_\nu}{\mu^2}-\eta_{\mu\nu}\right),$$

so that each vector boson line must be associated with a momentum k and two indices μ and ν at two vertices, and each vertex of interaction is associated with a factor $ie\gamma^\mu$ (or $ie\gamma^\nu$).

As an example, consider the following Feynman diagram in Fig. II.5.1:

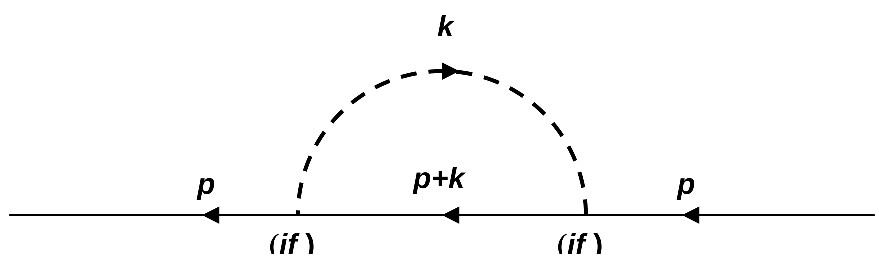

Figure II.5.1 Diagram for a fermion interacting with a scalar boson.

The amplitude for this Feynman diagram can be written as:

$$(if)^2 \bar{u}(p,s)\int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{k^2-\mu^2+i\varepsilon}\right)\left(\frac{i}{(\not{p}+\not{k})-m+i\varepsilon}\right)u(p,s),$$

where μ denotes the mass of the scalar boson field and m denotes the mass of the fermion.

Similarly, we can consider a Feynman diagram involving the interaction of a fermion with a vector boson, as shown in Fig. II.5.2:

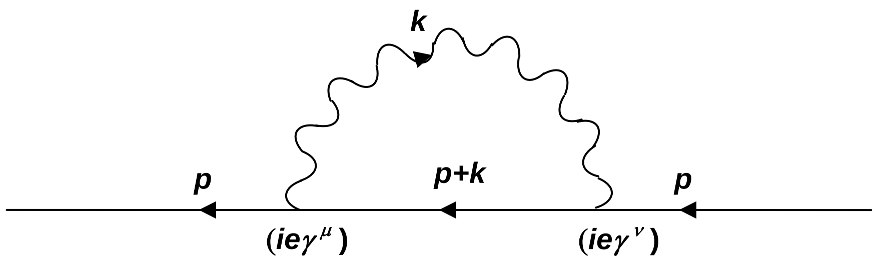

Figure II.5.2 Diagram for a fermion interacting with a vector boson.

In this case, the amplitude for the diagram is given by

$$(ie)^2\int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{k^2-\mu^2+i\varepsilon}\right)\left(\frac{k_\mu k_\nu}{\mu^2}-\eta_{\mu\nu}\right)\bar{u}(p,s)\,\gamma^\nu\left(\frac{i}{(\not{p}+\not{k})-m+i\varepsilon}\right)\gamma^\mu\, u(p,s). \tag{II.238}$$

Now we can begin to calculate realistic physical problems such as electron scattering using the Feynman diagrams for fermions. In particular, we want to examine how the "dangerous" term involving $(k_\mu k_\nu/\mu^2)$ in EQ. (II.238) behaves in the $\mu \to 0$ limit. Consider two electrons scattering off each other through the exchange of a photon of momentum k, as shown in Fig. II.5.3. Applying the Feynman rules, the amplitude for the diagram in Fig. II.5.3(a) is given by:

$$A(P_1,P_2) = (-ie)^2\left(\frac{i}{(P_1-p_1)^2-\mu^2}\right)\left(\frac{k_\mu k_\nu}{\mu^2}-\eta_{\mu\nu}\right)\bar{u}(P_1)\gamma^\mu u(p_1)\,\bar{u}(P_2)\gamma^\nu u(p_2). \tag{II.239}$$

Noting that $(\not{p}-m)u(p)=0$ and $\bar{u}(p)(\not{p}-m)=0$, we have

$$k_\mu \bar{u}(P_1)\gamma^\mu u(p_1) = (P_1-p_1)_\mu \bar{u}(P_1)\gamma^\mu u(p_1) = \bar{u}(P_1)(\not{P}_1-\not{p}_1)u(p_1) = \bar{u}(P_1)(m-m)u(p_1)=0,$$

so that the dangerous term vanishes even in $\mu \to 0$ limit, and EQ. (III.239) is simplified into

$$A(P_1,P_2) = \frac{ie^2}{(P_1-p_1)^2}\bar{u}(P_1)\gamma^\mu u(p_1)\,\bar{u}(P_2)\gamma_\mu u(p_2). \tag{II.240}$$

Next, comparing the diagram in Fig. II.5.3(b) with II.5.3(a), we may denote the amplitude in Fig. III.5.3(b) as $-A(P_2, P_1)$, where the minus sign comes from exchanging two fermions. Thus, the total amplitude for two electrons of momenta p_1 and p_2 to scatter into two electrons with momenta P_1 and P_2 to the order of $(ie)^2$ is:

$$M = A(P_1,P_2) + \left[-A(P_2,P_1)\right]. \tag{II.241}$$

The cross section of scattering is proportional to $|M|^2 = M^*M$:

$$\left|M\right|^2 = \left|A(P_1,P_2)\right|^2 + \left|A(P_2,P_1)\right|^2 - 2\,\mathrm{Re}\left\{A(P_2,P_1)^* A(P_1,P_2)\right\}. \tag{II.242}$$

Using EQ. (II.240), we have

$$\left|A(P_1,P_2)\right|^2 = \frac{e^4}{(P_1-p_1)^4}\left[\bar{u}(P_1)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu u(P_1)\right]\left[\bar{u}(P_2)\gamma_\mu u(p_2)\bar{u}(p_2)\gamma_\nu u(P_2)\right]. \tag{II.243}$$

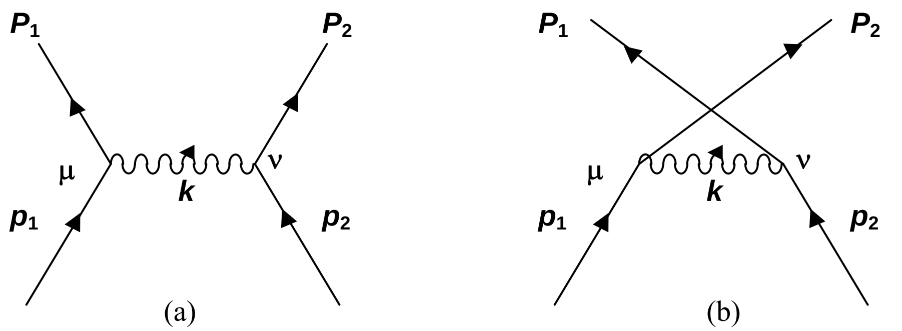

Figure II.5.3 Diagrams for two-electron scattering via the exchange of a photons of momenta k.

To compute $|A(P_2, P_1)|^2$ explicitly, we note that EQ. (II.243) factorizes into one factor carrying subscript 1 and the other carrying subscript 2, so that we can simply consider one of the factors $\bar{u}(P_1)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu u(P_1)$ averaged over spin polarizations. Using EQ. (II.201) in Part II.4:

$$\sum_{\pm s} u(p,s)\bar{u}(p,s) = \frac{1}{2}\left(\frac{\not{p}}{m}+1\right),$$

we obtain the following (with spin indices restored explicitly):

$$\begin{aligned}\tau^{\mu\nu}\left(P_1,p_1\right) &\equiv \sum_s\sum_S \bar{u}(P_1,S)\gamma^\mu u(p_1,s)\bar{u}(p_1,s)\gamma^\nu u(P_1,S) = \frac{1}{(2m)^2}\mathrm{Tr}\left\{\left(\not{P}_1+m\right)\gamma^\mu\left(\not{p}_1+m\right)\gamma^\nu\right\}\\ &= \frac{1}{(2m)^2}\left[\mathrm{Tr}\left\{\not{P}_1\gamma^\mu\not{p}_1\gamma^\nu\right\} + m^2\,\mathrm{Tr}\left\{\gamma^\mu\gamma^\nu\right\}\right],\end{aligned} \tag{II.244}$$

where we have used the fact that the trace of the product of an odd number of γ-matrices vanishes. In addition, we use the following identities:

$$\begin{aligned}&\mathrm{Tr}\left\{\gamma^\mu\gamma^\nu\right\} = 4\eta^{\mu\nu},\\ &\mathrm{Tr}\left\{\gamma^\rho\gamma^\mu\gamma^\lambda\gamma^\nu\right\} = 4\left(\eta^{\rho\mu}\eta^{\lambda\nu} - \eta^{\rho\lambda}\eta^{\mu\nu} + \eta^{\rho\nu}\eta^{\mu\lambda}\right),\end{aligned} \tag{II.245}$$

and the fact that

$$\mathrm{Tr}\left\{\not{P}_1\gamma^\mu\not{p}_1\gamma^\nu\right\} = P_{1\rho}p_{1\lambda}\,\mathrm{Tr}\left\{\gamma^\rho\gamma^\mu\gamma^\lambda\gamma^\nu\right\},$$

so that EQ. (II.244) can be rewritten as

$$\tau^{\mu\nu}\left(P_1,p_1\right) = \frac{4}{(2m)^2}\left(P_1^\mu p_1^{\ \nu} - \eta^{\mu\nu}P_1\cdot p_1 + P_1^\nu p_1^{\ \mu} + m^2\eta^{\mu\nu}\right). \tag{II.246}$$

Next, we consider the term associated with $A(P_2,P_1)^* A(P_1,P_2)$, which does not factorize as $|A(P_1,P_2)|^2$ and involves the following (with spin indices suppressed for brevity):

$$\kappa \equiv \sum\sum\sum\sum \bar{u}(P_1)\gamma^\mu u(p_1)\bar{u}(P_2)\gamma_\mu u(p_2)\bar{u}(p_1)\gamma^\nu u(P_2)\bar{u}(p_2)\gamma_\nu u(P_1). \qquad \text{(II.247)}$$

Equation (II.247) can be simplified if we restrict our consideration to the relativistic limit, such that:

$$\begin{aligned}\kappa &\equiv \frac{1}{(2m)^4}\mathrm{Tr}\left\{\not{P}_1\gamma^\mu \not{p}_1\gamma^\nu \not{P}_2\gamma_\mu \not{p}_2\gamma_\nu\right\} = \frac{1}{(2m)^4}\mathrm{Tr}\left\{\not{P}_1\gamma^\mu \not{p}_1\left(-2\not{p}_2\gamma_\mu \not{P}_2\right)\right\} = \frac{(-2)}{(2m)^4}\mathrm{Tr}\left\{\not{P}_1\gamma^\mu \not{p}_1\not{p}_2\gamma_\mu \not{P}_2\right\} \\ &= \frac{(-2)}{(2m)^4}Tr\left\{\not{P}_1\left(4p_1\cdot p_2\right)\not{P}_2\right\} = \frac{(-8)}{(2m)^4}\left(p_1\cdot p_2\right)Tr\left\{\gamma^\mu P_{1\mu}\gamma^\nu P_{2\nu}\right\} = \frac{(-32)}{(2m)^4}\left(p_1\cdot p_2\right)\left(P_1\cdot P_2\right),\end{aligned} \qquad \text{(II.248)}$$

where we have used the following identities:

$$\begin{aligned}&\mathrm{Tr}\left\{\gamma^\mu \not{p}\,\not{q}\,\not{r}\,\gamma_\mu\right\} = -2\not{r}\,\not{q}\,\not{p}\,, \\ &\mathrm{Tr}\left\{\gamma^\mu \not{p}\,\not{q}\,\gamma_\mu\right\} = 4\left(p\cdot q\right).\end{aligned} \qquad \text{(II.249)}$$

Similarly, in the same relativistic limit, we may drop the term involving m in EQ. (II.246), and find that

$$\begin{aligned}\tau^{\mu\nu}\left(P_1,p_1\right)\tau_{\mu\nu}\left(P_2,p_2\right) &= \frac{16}{(2m)^4}\left(P_1^\mu p_1^\nu - \eta^{\mu\nu}P_1\cdot p_1 + P_1^\nu p_1^\mu\right)\left(P_{2\mu}p_{2\nu} - \eta_{\mu\nu}P_2\cdot p_2 + P_{2\nu}p_{2\mu}\right) \\ &= \frac{16\cdot 2}{(2m)^4}\left[\left(p_1\cdot p_2\right)\left(P_1\cdot P_2\right) + \left(p_1\cdot P_2\right)\left(P_1\cdot p_2\right)\right].\end{aligned} \qquad \text{(II.250)}$$

If we further take the center-of-mass frame in the relativistic limit, we can express p_1, p_2, P_1, P_2 explicitly in terms of the total energy E and scattering angle θ as follows:

$$p_1 = E(1,0,0,1);\quad p_2 = E(1,0,0,-1);\quad P_1 = E(1,\sin\theta,0,\cos\theta);\quad P_2 = E(1,-\sin\theta,0,-\cos\theta).$$

Therefore we have

$$\begin{aligned}&\left(p_1\cdot p_2\right) = \left(P_1\cdot P_2\right) = 2E^2;\quad \left(p_1\cdot P_1\right) = \left(p_2\cdot P_2\right) = 2E^2\sin^2\left(\theta/2\right);\quad \left(p_1\cdot P_2\right) = \left(p_2\cdot P_1\right) = 2E^2\cos^2\left(\theta/2\right), \\ &\left(P_1 - p_1\right)^4 = \left(-2p_1\cdot P_1\right)^2 = 16E^4\sin^4\left(\theta/2\right);\quad \left(P_2 - p_1\right)^4 = \left(-2p_1\cdot P_2\right)^2 = 16E^4\cos^4\left(\theta/2\right).\end{aligned} \qquad \text{(II.251)}$$

Putting together EQs. (II.242), (II.243), (II.248), (II.250) and (II.251), we obtain the scattering probability:

$$\frac{1}{2}\sum_s\sum_S |M|^2 = \left(\frac{e^4}{4m^4}\right)\left[\frac{1+\cos^4\left(\theta/2\right)}{\sin^4\left(\theta/2\right)} + \frac{2}{\sin^2\left(\theta/2\right)\cos^2\left(\theta/2\right)} + \frac{1+\sin^4\left(\theta/2\right)}{\cos^4\left(\theta/2\right)}\right] \equiv \left(\frac{e^4}{4m^4}\right)f(\theta). \qquad \text{(II.252)}$$

The factor of ½ in EQ. (II.252) follows Feynman rules for symmetric diagrams as shown in Fig. II.5.3. We also note that the first and third terms are consistent with symmetric scattering under $\theta \to (\pi - \theta)$, whereas the second term is associated with quantum interference. Finally, from EQ. (II.252) we find that the differential cross section is given by:

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi}\right)^2\left(\frac{1}{8E^2}\right)f(\theta). \qquad \text{(II.253)}$$

[Quantum electrodynamics]

Having seen an explicit example of using Feynman diagrams to calculate real measurable quantities in experiments, let's return to an important issue regarding the interaction of fermions with massless vector bosons such as photons. As discussed earlier, the Lagrangian for QED in the limit of zero photon mass ($\mu \to 0$) is

$$\mathcal{L} = \bar{\Psi}\left[i\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right) - m\right]\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad . \tag{II.254}$$

It turns out that the Lagrangian $\mathcal{L}$ is invariant by the following gauge transformation:

$$\Psi(x) \to e^{i\Lambda(x)}\Psi(x), \tag{II.255}$$

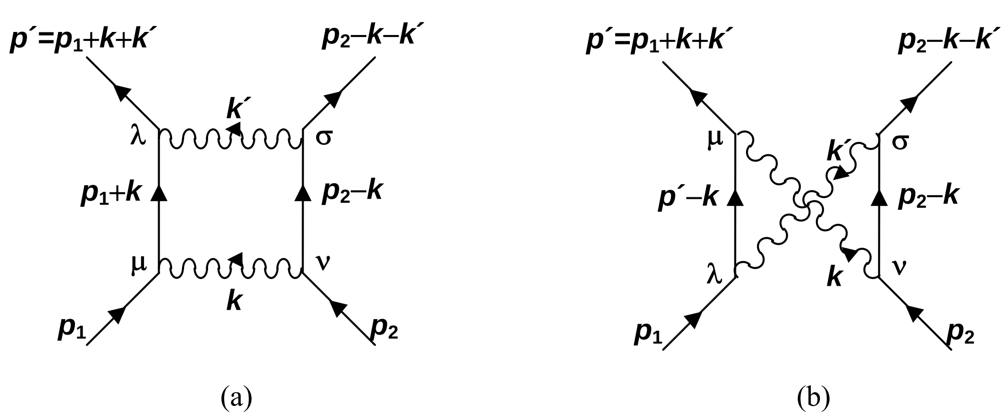

Figure II.5.4 Diagrams for two-electron scattering via exchange of photons of momenta k and k'.

and
$$A_{\mu}(x) \to A_{\mu}(x) + \frac{1}{ie}e^{-i\Lambda(x)}\partial_{\mu}e^{i\Lambda(x)} = A_{\mu}(x) + \frac{1}{e}\partial_{\mu}\Lambda(x)\,. \tag{II.256}$$

Clearly $F_{\mu\nu}(x)$ remains invariant under such gauge transformation.

Recall that the diagrammatic contribution of a massive vector boson takes on the form

$$iD_{\mu\nu} = i\left(\frac{1}{k^2-\mu^2}\right)\left(\frac{k_{\mu}k_{\nu}}{\mu^2} - \eta_{\mu\nu}\right) \xrightarrow{(\mu\to 0)} \frac{i}{k^2}\left(\frac{k_{\mu}k_{\nu}}{\mu^2} - \eta_{\mu\nu}\right). \tag{II.257}$$

We shall prove diagrammatically in the following that the "dangerous" term ($k_{\mu}k_{\nu}/\mu^2$) disappears in the case of massless photons. Moreover, EQ. (II.257) can actually be generalized into

$$iD_{\mu\nu} = \frac{i}{k^2}\left(\left(1-\xi\right)\frac{k_{\mu}k_{\nu}}{k^2} - \eta_{\mu\nu}\right), \tag{II.258}$$

where ξ can be any number that simplify our calculations, and the choice of ξ does not change the physics, indicating gauge invariance.

We first consider electron-electron scattering to order (e^4), as exemplified below in Fig. II.5.4. The total amplitude for these two Feynman diagrams is

$$\bar{u}(p')\left(\gamma^{\lambda}\frac{1}{\not{p}_1+\not{k}-m}\gamma^{\mu}+\gamma^{\mu}\frac{1}{\not{p}'-\not{k}-m}\gamma^{\lambda}\right)u(p_1)\frac{i}{k^2}\left(\frac{k_\mu k_\nu}{\mu^2}-\eta_{\mu\nu}\right)\Gamma_{\lambda}^{\ \nu}, \qquad \text{(II.259)}$$

where we have only explicitly expressed terms associated with the "left electron" and lumped all the rest into the factor $\Gamma_{\lambda\nu}$. (You can verify that by writing out the complete amplitudes for the two Feynman diagrams in Fig. II.5.4 and by explicitly considering the second photon of momentum k', you obtain the same results as we have asserted here.) For now, we are interested in the dangerous term of one of the photons, so we rewrite the dangerous term in EQ. (II.259) into:

$$\begin{aligned}
&\bar{u}(p')\left(\gamma^{\lambda}\frac{1}{\not{p}_1+\not{k}-m}\not{k}+\not{k}\frac{1}{\not{p}'-\not{k}-m}\gamma^{\lambda}\right)u(p_1)\frac{i}{k^2}\left(\frac{k_\nu}{\mu^2}\right)\Gamma_{\lambda}^{\ \nu},\\
&=\bar{u}(p')\left(\gamma^{\lambda}\frac{(\not{p}_1+\not{k}-m)-(\not{p}_1-m)}{\not{p}_1+\not{k}-m}+\frac{(\not{p}'-m)-(\not{p}'-\not{k}-m)}{\not{p}'-\not{k}-m}\gamma^{\lambda}\right)u(p_1)\frac{i}{k^2}\left(\frac{k_\nu}{\mu^2}\right)\Gamma_{\lambda}^{\ \nu},\\
&=\bar{u}(p')\left(-\gamma^{\lambda}\frac{(\not{p}_1-m)}{\not{p}_1+\not{k}-m}+\frac{(\not{p}'-m)}{\not{p}'-\not{k}-m}\gamma^{\lambda}\right)u(p_1)\frac{i}{k^2}\left(\frac{k_\nu}{\mu^2}\right)\Gamma_{\lambda}^{\ \nu}.
\end{aligned} \qquad \text{(II.260)}$$

Noting that $(\not{p}_1-m)u(p_1)=0$ and $\bar{u}(p')(\not{p}'-m)=0$, we find that EQ. (II.260) vanishes. In other words, the dangerous term does not contribute to the amplitude of the Feynman diagrams, even if we do not specify the factor $\Gamma_{\lambda}^{\ \nu}$. In fact, we can generalize Fig. II.5.4 into Fig. II.5.5, where arbitrarily complicated processes could take place under the shaded blob. In fact, we can further generalize the diagrams to Fig. II.5.6, where multiple photons besides the one with momentum k that we focused on are also attached to the "left electron" line. These photons are in fact "spectators" just like the photon with momentum k' in Fig. II.5.4.

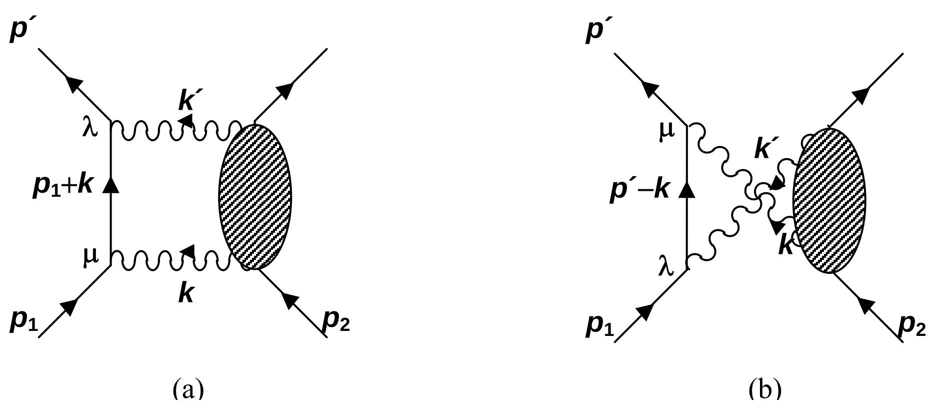

Figure II.5.5 Generalized diagrams for two-electron scattering via exchange of two photons.

The examples given in Figs. II.5.4 – II.5.6 all have the photon lines landed on an external electron line. What if the photon line in question lands on an internal line of a Feynman diagram? Let us consider yet another example in Fig. II.5.7 with electron-electron scattering in order e^8, where we deal with three distinct diagrams: we consider the "left electron" emitting three photons and the "right electron" emitting one photon, with all photon lines linking to an internal fermion loop, as shown in Fig. II.5.7. We want to consider whether the dangerous term in the photon propagator vanishes or not. For convenience, let's focus specifically on the photon propagator emitted by the electron on the right, which is given by

$i\left(-\eta_{\mu\rho}+k_\mu k_\rho/\mu^2\right)/k^2$. The relevant integrations *A*, *B* and *C* associated with the term $\gamma^\mu k_\mu = \not{k}$ for the diagrams in Fig. II.5.7 are given below, where we have defined $p_1 = p+q_1$ and $p_2 = p_1+q_2$:

$$A=\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2+\not{k}-m}\gamma^\sigma\frac{1}{\not{p}_1+\not{k}-m}\gamma^\lambda\frac{1}{\not{p}+\not{k}-m}\not{k}\frac{1}{\not{p}-m}\right\}, \tag{II.261}$$

$$B=\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2+\not{k}-m}\gamma^\sigma\frac{1}{\not{p}_1+\not{k}-m}\not{k}\frac{1}{\not{p}_1-m}\gamma^\lambda\frac{1}{\not{p}-m}\right\}, \tag{II.262}$$

$$C=\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2+\not{k}-m}\not{k}\frac{1}{\not{p}_2-m}\gamma^\sigma\frac{1}{\not{p}_1-m}\gamma^\lambda\frac{1}{\not{p}-m}\right\}. \tag{II.263}$$

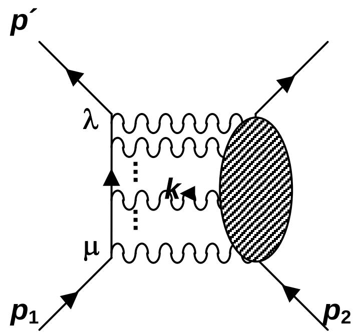

Figure II.5.6 Generalized diagrams for two-electron scattering via exchange of multiple photons.

Next, we write $\not{k}$ in the integral *A* as $\not{k}=\left(\not{p}+\not{k}-m\right)-\left(\not{p}-m\right)$, in *B* as $\not{k}=\left(\not{p}_1+\not{k}-m\right)-\left(\not{p}_1-m\right)$, and in *C* as $\not{k}=\left(\not{p}_2+\not{k}-m\right)-\left(\not{p}_2-m\right)$, so that EQs. (II.361) -- (II.363) become:

$$A=\int\frac{d^4p}{(2\pi)^4}\left[\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2+\not{k}-m}\gamma^\sigma\frac{1}{\not{p}_1+\not{k}-m}\gamma^\lambda\frac{1}{\not{p}-m}\right\}-\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2+\not{k}-m}\gamma^\sigma\frac{1}{\not{p}_1+\not{k}-m}\gamma^\lambda\frac{1}{\not{p}+\not{k}-m}\right\}\right]$$

$$B=\int\frac{d^4p}{(2\pi)^4}\left[\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2+\not{k}-m}\gamma^\sigma\frac{1}{\not{p}_1-m}\gamma^\lambda\frac{1}{\not{p}-m}\right\}-\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2+\not{k}-m}\gamma^\sigma\frac{1}{\not{p}_1+\not{k}-m}\gamma^\lambda\frac{1}{\not{p}-m}\right\}\right]$$

$$C=\int\frac{d^4p}{(2\pi)^4}\left[\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2-m}\gamma^\sigma\frac{1}{\not{p}_1-m}\gamma^\lambda\frac{1}{\not{p}-m}\right\}-\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2+\not{k}-m}\gamma^\sigma\frac{1}{\not{p}_1-m}\gamma^\lambda\frac{1}{\not{p}-m}\right\}\right], \tag{II.264}$$

so that

$$A+B+C=\int\frac{d^4p}{(2\pi)^4}\left[\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2-m}\gamma^\sigma\frac{1}{\not{p}_1-m}\gamma^\lambda\frac{1}{\not{p}-m}\right\}-\mathrm{Tr}\left\{\gamma^\nu\frac{1}{\not{p}_2+\not{k}-m}\gamma^\sigma\frac{1}{\not{p}_1+\not{k}-m}\gamma^\lambda\frac{1}{\not{p}+\not{k}-m}\right\}\right]$$

If we change the variable *p* to (*p* – *k*) in the second term above, which is allowed because the integral is only logarithmically divergent, we find that *A* + *B* + *C* = 0. Hence, the term $\left(k_\mu k_\rho/\mu^2\right)$ associated with the photon propagator indeed vanishes. You can also expect the same for the three photon propagators emitted by the electron on the left in Fig. II.5.7.

The above discussion can be generalized into what's known as a Ward-Takahashi identity. That is, for any physical amplitude $T^{\mu}\ldots(k, \ldots)$ with external electrons on shell, [*i.e.*, all necessary factors $u(p)$ and $\bar{u}(p)$ are included in $T^{\mu}\ldots(k, \ldots)$], if we describe a process with a photon carrying momentum k coming out of or going into a vertex labeled by the Lorentz index μ, we have the identity

$$k_\mu T^{\mu\cdots}(k,\cdots)=0\,. \tag{II.265}$$

Therefore we can in general discard the $\left(k_\mu k_\rho / \mu^2\right)$ term in the photon propagator and simply write the photon propagator as: $iD_{\mu\nu}=-i\eta_{\mu\nu}/k^2$. In fact, we can also add a term $\left(k_\mu k_\rho / k^2\right)$ with an arbitrary coefficient to the photon propagator

$$iD_{\mu\nu}=\frac{i}{k^2}\left[(1-\xi)\frac{k_\mu k_\nu}{k^2}-\eta_{\mu\nu}\right], \tag{II.266}$$

where the number ξ can be chosen to simplify our calculations. The choice of ξ effectively amounts to the choice of gauge for the electromagnetic field. In particular, $\xi = 1$ is known as the Feynman gauge, whereas $\xi = 0$ is known as the Landau gauge. The arbitrary choice of ξ in EQ. (II.266) is consistent with the gauge invariance of quantum electrodynamics.

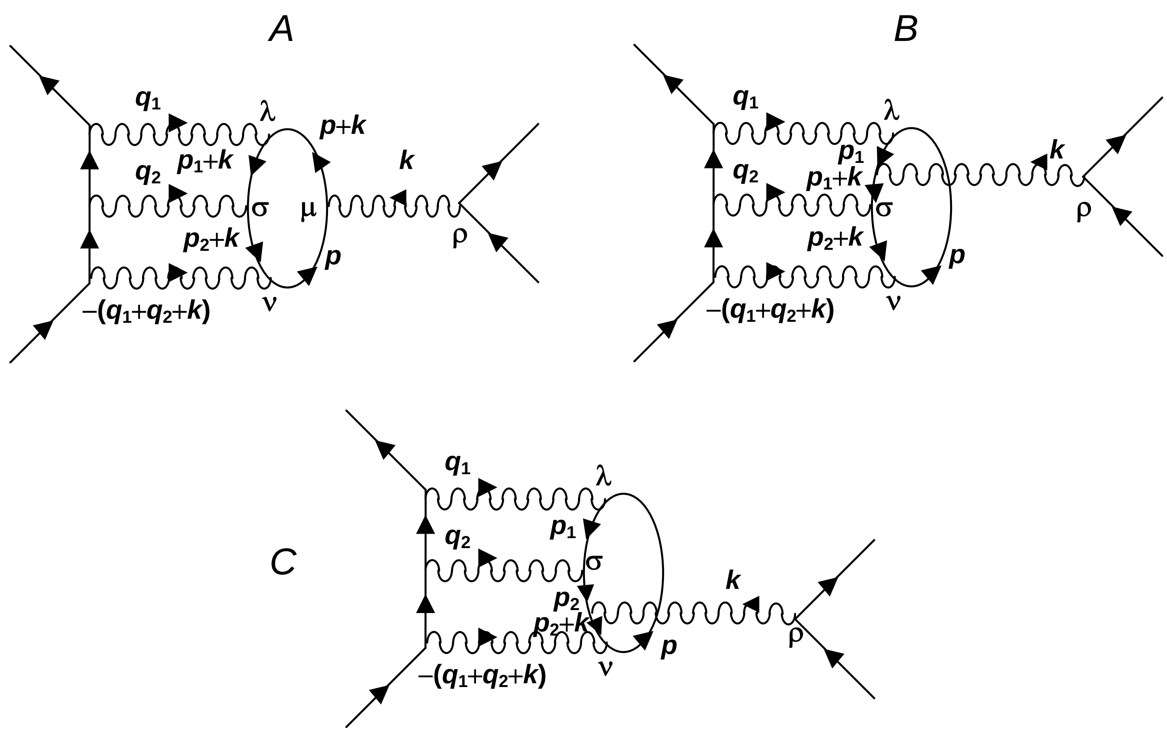

Figure II.5.7 Diagrams for two-electron scattering via exchange of 4 photons to order e^8.

II.6. From Relativistic to Non-Relativistic Quantum Field Theory

The relativistic quantum field theory is in fact inclusive of the non-relativistic quantum mechanics. To appreciate this point, let us consider the following two cases, one concerns scalar fields and the other spinor fields.

Consider the relativistic Klein-Gordon equation that describes the scalar field $\varphi(x)$ of a free particle with mass m in the spacetime dimension:

$$(\partial^2 + m^2)\varphi = 0,$$

where $\partial^2 = \partial_\mu\partial^\mu$, $\partial_\mu \equiv (\partial/\partial t, -\partial/\partial x^i)$, $\partial^\mu \equiv (\partial/\partial t, \partial/\partial x^i)$, and the superscript index i = 1, 2, 3 runs through three spatial dimensions. A mode φ with energy $\mathcal{E} = m + \varepsilon$ (where ε denotes the kinetic energy) would oscillate in time so that $\varphi(\mathbf{x},t) \sim \exp(-i\mathcal{E}t)$. In the non-relativistic limit, $\varepsilon << m$ so that we can express $\varphi(x)$ (with proper normalization) as $\varphi(\mathbf{x},t) \sim \exp(-imt)\phi(\mathbf{x},t)/(2m)^{1/2}$, with the field $\phi(\mathbf{x},t)$ varies much slower in time than the term e^{-imt}. Plugging the new expression of $\varphi(x)$ into the Klein-Gordon equation and dropping the term $(\partial^2/\partial t^2)\phi$, we obtain the time-dependent Schrödinger's equation for a free particle of mass m:

$$i\frac{\partial}{\partial t}\phi(\mathbf{x},t) = -\frac{\nabla^2}{2m}\phi(\mathbf{x},t). \qquad \text{(II.267)}$$

Thus, we have derived the standard Schrödinger's equation from the Klein-Gordon equation simply by taking the non-relativistic limit. An important difference between the Schrödinger's equation and the Klein-Gordon equation is noteworthy. That is, the former involves linear time derivative, whereas the latter is quadratic in time derivative. (Can you make sense out of this difference?)

Next we consider the Dirac equation for an electron in an external electromagnetic field:

$$\left[i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right]\Psi \equiv \left(i\gamma^\mu D_\mu - m\right)\Psi = 0. \qquad \text{(II.268)}$$

Multiplying EQ. (II.268) by $-\left(i\gamma^\nu D_\nu + m\right)$, we obtain

$$\begin{aligned}\left(\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2\right)\Psi &= \left(\frac{1}{2}\left(\left\{\gamma^\mu,\gamma^\nu\right\} + \left[\gamma^\mu,\gamma^\nu\right]\right)D_\mu D_\nu + m^2\right)\Psi = \left[\left(\eta^{\mu\nu} - i\sigma^{\mu\nu}\right)D_\mu D_\nu + m^2\right]\Psi \\ &= \left(D_\mu D^\mu - \frac{i}{2}\sigma^{\mu\nu}\left[D_\mu, D_\nu\right] + m^2\right)\Psi = \left(D_\mu D^\mu - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} + m^2\right)\Psi = 0. \qquad \text{(II.269)}\end{aligned}$$

On the other hand, if we were to consider a charged scalar field Φ interacting with an external electromagnetic field, we would have the following "generalized" Klein Gordon equation:

$$\left(D_\mu D^\mu + m^2\right)\Phi = 0. \qquad \text{(II.270)}$$

Comparing EQ. (II.269) with EQ. (II.270), we note that the additional term $-(e/2)\sigma^{\mu\nu}F_{\mu\nu}$ in EQ. (II.269) is associated with the contribution from the spin of the electron.

Now if we assume that a weak constant magnetic field $\boldsymbol{B}$ is applied along the x^3-direction so that terms quadratic in $(A_i)^2$ can be neglected, we can choose a convenient gauge (because of gauge invariance) that ensures $\left(\partial_1 A_2 - \partial_2 A_1\right) = \boldsymbol{B}$. The gauge of our choice is

$$A = \left(A_0, A_1, A_2, A_3\right) = \left(0, -\frac{1}{2}Bx^2, \frac{1}{2}Bx^1, 0\right). \tag{II.271}$$

Inserting the gauge in EQ. (II.271) to EQ. (II.269), we obtain

$$\begin{aligned} D_\mu D^\mu &= \left(\partial_0\right)^2 - \left(\partial_i\right)^2 + ie\left(\partial_i A^i + A_i \partial^i\right) + O\left(A_i^2\right) = \left(\partial_0\right)^2 - \left(\partial_i\right)^2 + ie\left(\left(\partial_i A^i\right) + A^i\partial_i + A_i\partial^i\right) + O\left(A_i^2\right) \\ &= \left(\partial_0\right)^2 - \left(\partial_i\right)^2 + ieB\left(x^1\partial_2 - x^2\partial_1\right) + O\left(A_i^2\right) = \left(\partial_0\right)^2 - \nabla^2 - e\boldsymbol{B}\cdot\left(\boldsymbol{x}\times\boldsymbol{p}\right) + O\left(A_i^2\right). \end{aligned} \tag{II.272}$$

We note that in EQ. (II.272) the angular momentum appears, which is given by $\boldsymbol{L} = \left(\boldsymbol{x}\times\boldsymbol{p}\right)$. In addition, our choice of gauge yields:

$$\begin{aligned} -(e/2)\sigma^{\mu\nu}F_{\mu\nu} &= -(e/2)\sigma^{ij}F_{ij} = -(e/2)\varepsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}F_{ij} = (e/2)\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\left(F_{21} - F_{12}\right), \\ &= -eB\sigma^3\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix} = -2e\boldsymbol{B}\cdot\left(\frac{1}{2}\boldsymbol{\sigma}\right)\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix} = -2e\boldsymbol{B}\cdot\boldsymbol{S}\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}. \end{aligned} \tag{II.273}$$

We also note that in the non-relativistic limit the Dirac spinor can be approximated by:

$$\Psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix} \approx \begin{pmatrix}\phi\\ 0\end{pmatrix}, \tag{II.274}$$

where $\phi = e^{-imt}\psi$ with ψ oscillating much slower than e^{-imt} so that

$$\left(\partial_0{}^2 + m^2\right)\phi = \left(\partial_0{}^2 + m^2\right)e^{-imt}\psi \simeq e^{-imt}\left[-2im\left(\partial\psi/\partial t\right)\right]. \tag{II.275}$$

Thus, from EQs. (II.269) – (II.272) we obtain:

$$\left[-2im\left(\frac{\partial}{\partial t}\right) - \nabla^2 - e\boldsymbol{B}\cdot\left(\boldsymbol{L} + 2\boldsymbol{S}\right)\right]\psi = 0. \tag{II.276}$$

Equation (II.276) is equivalent to the familiar form of the Schrödinger's equation:

$$i\left(\frac{\partial}{\partial t}\right)\psi = H\psi = \left[\frac{-\nabla^2}{2m} - \frac{e}{2m}\boldsymbol{B}\cdot\left(\boldsymbol{L} + 2\boldsymbol{S}\right)\right]\psi, \tag{II.277}$$

and it successfully accounts for the experimental finding that a unit of spin angular momentum interacts with a magnetic field twice as much as a unit of orbital angular momentum. This result derived from non-relativistic Dirac equation is considered as a major triumph of the Dirac spinor field theory.

The above two examples illustrate the consistency between the relativistic and non-relativistic quantum field theory for both scalar and spinor fields. Starting from Part II.7, we shall develop Green's function technique in the non-relativistic limit for applications to general condensed matter physics problems.